國 立 政 治 大 學 應 用 數 學 系
碩士學位論文

On Two-Dimensional Smooth Tropical Toric Fano Varieties

二維平滑熱帶環面法諾曲體之研究

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中 華 民 國 一 百 零 一 年 六 月

誌 謝

這本論文能夠完成, 真的要非常感謝幾位老師: 陳天進老師、蔡炎龍老師、余屹正老師。首先感謝我的指導教授蔡炎龍老師, 他帶領我進入熱帶幾何的世界, 但也害老師必須忍受我每次在講台上胡說八道的講解, 不過這幾年下來也讓我獲得許多知識與經驗, 我未來也會繼續努力。而老師除了課業上的指導, 在生活上也對我非常照顧與教導, 尤其是老師跟我的共同興趣「咖啡」, 這幾年下來, 我們時常一起去咖啡店, 也時常在老師研究室玩咖啡, 甚至後來還一起弄了個咖啡社, 而老師對於咖啡的想法, 也是跟數學一樣, 凡事講求科學根據, 不要受一些毫無道理的咖啡知識所矇騙, 這也深深影響了我, 更讓我對許多事物改變了看法, 真的很感謝老師。

感謝陳天進老師, 這幾年來上過老師很多堂課, 也從中學到一些數學知識, 不過我這位不成才的學生, 還是沒能學的很精及融會貫通。還有每次在老師面前報告, 也都讓我相當緊張, 因為老師對數學的認知十分廣泛, 所以總是讓我在台上不知所云的亂回答, 也讓老師聽到快暈倒, 但也很感謝老師每次在我亂回答之後, 都會細心解說正確的觀念, 讓我獲得不少知識。而老師除了數學上的教導, 平時更是很關心且很照顧我們這些學生, 老師更是時常邀我們大家一起上餐館, 每次在餐桌上大家總是嘻嘻哈哈的聊天, 這真的是我在政大快樂回憶之一。因此真的很感謝陳天進老師這幾年的教導與照顧。

感謝余屹正老師, 記得幾年前有次在理論中心辦的活動上, 不小心聽到于靖老

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過去幫忙我一些課業問題, 還有當年跟我一起玩咖啡, 尤其當年因為你的關係讓我認識到水岸的郎叔 (郎毓彬), 也使我對咖啡的了解能夠更加深入。還有感謝其他學長姊、同學們、學弟妹, 這幾年跟大家相處, 讓我在政大有一段愉快的生活。

最後感謝最支持我的父母親與家人們, 這幾年來我總是讓他們操煩, 但他們也總是給我很多幫助跟支持, 希望我可以早日在專業領域上有所成就, 讓家人們可以不用再替我未來煩惱, 同時也很感謝父母親幫忙裝訂我的論文。

Abstract

In this thesis, we survey and study tropical toric varieties with focus on tropical toric Fano varieties. To construct tropical toric varieties, we start with fans, just like the situation in classical algebraic geometry. However, some constructions does not make sense in tropical settings. Therefore, we need to choose a reasonable definition which give an analogue of a classical toric variety. In the end of this paper, we use the definition we choose, and explicitly calculate all smooth two-dimensional tropical toric Fano varieties which we found are very similar to classical cases.
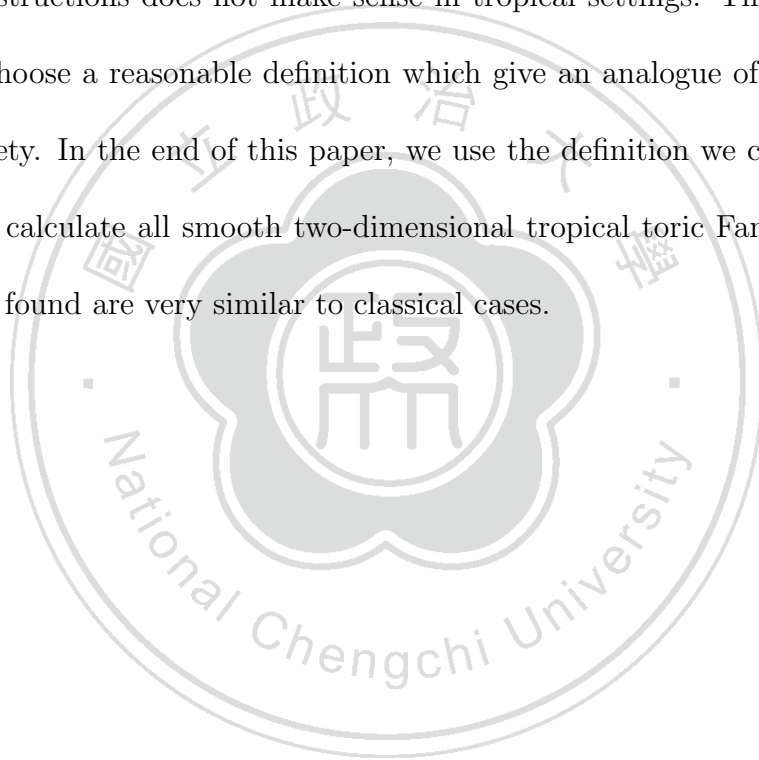
摘 要

這篇論文裡, 我們研究熱帶環面曲體, 尤其是熱帶環面法諾曲體。如同古典代數幾何裡的情況一樣, 要建構熱帶環面曲體, 我們先從扇型開始建構。然而在某些結構裡沒辦法有熱帶化的對應, 因此我們需要選一個適當的定義, 這個定義必需可看成是古典情況類推而來的。在我們的論文中, 使用我們認爲合適的定義, 計算所有平滑二維熱帶環面法諾曲體的情況, 結果也證實非常類似古典的情形。

Contents

Chapter 1

Introduction

The tropical geometry is a relatively recent subject in the field of mathematics. Why does it have an adjective "tropical"? In the 1980s, Imre Simon (August 14, 1943 - August 13, 2009) who is a Hungarian-born Brazilian mathematician and computer scientist pioneered the tropical geometry. The word "tropical" was coined by some French mathematicians in honor of Imre Simon, because they thought Brazil is a tropical country. Hence there is not any deeper meaning in an adjective "tropical", and this is why the tropical geometry is not called "temperate geometry" or "frigid geometry".

Why the mathematicians attend to the tropical geometry in recent years? Because Grigory Mikhalkin has proven that the number of simple tropical

curves (counted with appropriate multiplicities) of degree d and genus g that pass through $g + 3d - 1$ generic points in $\mathbb{R}^2$ is equal to the Gromov-Witten number $N_{g,d}$ of the complex projective plane $\mathbb{CP}^2$, so the theorem is called Mikhalkin's correspondence theorem, see [19].

The main reference is the paper [17] by Henning Meyer. Some difference are that we prove some propositions and provide more details with examples and figures. Moreover, this paper main discusses about the smooth tropical toric Fano varieties on two dimensional. The tropical toric variety and toric varieties have the similar properties, for example, the tropical toric variety $\mathbb{X}_\Delta(\mathbb{T}) \simeq \mathbb{TP}^2$ where $\mathbb{TP}$ is the tropical projective space in the Example 4.3.2, and the toric variety $\mathbb{X}_\Delta(\mathbb{T}) \simeq \mathbb{CP}^2$ where $\mathbb{CP}$ is the complex projective space in the Example 3.3.18.

The structure of the paper is as follows. In chapter 2, we recall the semigroup, semiring and semifield, and introduce amoebas and the tropical geometry where the tropical semifield $(\mathbb{T}, \oplus, \odot)$ is a semifield with two operations $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. Note that some papers or books may be defined the tropical sum by $a \oplus b := \min\{a, b\}$ (e.g. [15]), in fact, the algebraic structures of max-algebra and min-algebra are isomorphic. For more information see [13], [20], [10], [23] and [26].

In chapter 3, we review some basic concepts of polyhedral geometry and explain how they relate to toric varieties. For more details see [5], [22], [14], [12], [11], [4], [28], [24] and [16]. And we will give a brief introduction to Fano varieties and Fano polytopes. For careful statement see [14], [22] and [5].

In the first part of the chapter 4, we describe the relationship between $K(G, R, M)$, $hom(S, M)$, and explain the relationship between the algebraic structures of $K(G, R, M)$ and the algebraic structures of M. In the setion 2 and 3 of chapter 4, we explain the properties of tropical toric varieties, and calculate five types of the smooth tropical toric Fano varieties.

Chapter 2

Background

The chapter contains some basic definitions and propositions from tropical geometry.

2.1   Non-Archimedean amoebas

In this section, we recall the valuation and amoebas. And we set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ in this paper. Let K be an algebraic closed field (e.g. $\mathbb{C}$), and let $\mathbb{A}^n$ denote an affine n-space over K.

Firstly, we define a map $Log : (\mathbb{C}^*)^n \to \mathbb{R}^n$ by

$$Log(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|).$$

Let $u = (u_1, \ldots, u_n)$ be in $\mathbb{Z}^n$, then we said that $z^u := z_1^{u_1} z_2^{u_2} \cdots z_n^{u_n}$ is the Laurent monomial. Moreover, the Laurent polynomial f is a finite linear combination of Laurent monomials, that is, $f = \sum_{u \in \mathbb{Z}^n} a_u z^u$ where a_u is in a field F (as $\mathbb{C}$) and is only finitely many. Denoted by $F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is the ring of Laurent polynomials in n variables over F.

Definition 2.1.1. An affine algebraic variety is the common zero set of a collection $\{F_i\}_{i \in I}$ of complex polynomials. We write

$$V = \mathbb{V}(\{F_i\}_{i \in I}) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid F_i(x_1, \ldots, x_n) = 0 \ \forall \ i \in I\}$$

where $F_i = F_i(x_1, \ldots, x_n) \in \mathbb{C}^n[x_1, \ldots, x_n]$

Example 2.1.2. There are some trivial cases of algebraic varieties.

(1) $\mathbb{V}(0) = \mathbb{A}^n$.

(2) $\mathbb{V}(1) = \emptyset$.

(3) $\mathbb{V}(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\}$.

For any subset S of $\mathbb{C}^*[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, we denote by

$$Z(S) := \{z \in (\mathbb{C}^*)^n \mid f(z) = 0 \text{ for all } f \in S\},$$

that is, $Z(S)$ is the common zero set of a collection $\{f_i\}_{i \in I}$ of Laurent polynomials in a subset S of $\mathbb{C}^*[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$.

Definition 2.1.3. We define the amoeba of $Z(S)$ as $A(Z(S)) := Log(Z(S))$ which is a subset of $\mathbb{R}^n$.

Remark 2.1.4. Let V be an algebraic variety, then we can also define the amoeba of algebraic variety by $A(V) := Log(V)$.

Example 2.1.5. Let $f = \frac{1}{3}z_1 + \frac{5}{7}z_2 - 1$ in $\mathbb{C}^*[z_1, z_2]$. If $f = 0$, then $z_2 = -\frac{7}{15}z_1 + \frac{7}{5}$, and so $\mathbb{V}(f) = \{(t, -\frac{7}{15}t + \frac{7}{5}) | t \in \mathbb{C}\}$. Then $A(\mathbb{V}(f)) = Log(\mathbb{V}(f)) = (\log|t|, \log| -\frac{7}{15}t + \frac{7}{5}|)$.

For more information about the amoebas see [11] chapter 6 and [27]. The figure of the amoeba is used for GeoGebra ($Curve[ln(abs(t)), ln(abs(exp(-5/7) - t*exp(-8/21))), t, -100, 100]$) or we can also use for maple, for more details see [1].

Next, we define a map $Log_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$ by

$$Log_t(z_1, \ldots, z_n) = (\log_t |z_1|, \ldots, \log_t |z_n|)$$
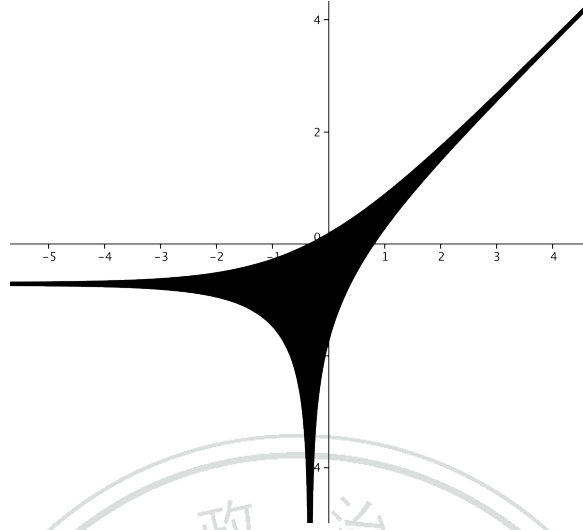
Figure 2.1: The amoeba $A(Z(f))$ for $f = \frac{1}{3}z_1 + \frac{5}{7}z_2 - 1$.

for small t in $\mathbb{R}$. And we denote by

$$Z_t := \{z \in (\mathbb{C}^*)^n \mid \sum_{u \in \mathbb{Z}^n} a_u(t) z^u = 0\}.$$

Let the amoeba of Z_t as $A_t(Z_t) := Log_t(Z_t)$ which is a subset of $\mathbb{R}^n$. Similarly, if V_t is an algebraic variety which depend on a parameter t, then we can also define the amoeba of algebraic variety by $A_t(V_t) := Log_t(V_t)$.

We recall that the Hausdorff distance. Let (M, d) be a metric space, and let A and B be two non-empty subsets of (M, d). Then we define the Hausdorff distance $d_H(A, B)$ between A and B by

$$d_H(A, B) = max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$

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On $\mathbb{R}^n$, the subsets A_t converges to A as $t \to \infty$ in the Hausdorff metric on compacts, that is, for any compact set D in $\mathbb{R}^n$, and there exists a neighborhood U of D such that $d_H(A_t \cap U, A \cap U) \to 0$ as $t \to \infty$ ([13], Prop. 1.2 and [18], Prop. 1.6).

Definition 2.1.6. The set $\mathbb{C}\{\{t\}\}$ is called the field of Puiseux series with complex coefficients if $\mathbb{C}\{\{t\}\}$ is the set of all formal power series $a(t) = \sum_{q \in \mathbb{Q}} a_q t^q$ where a_q is in $\mathbb{C}^*$ and $\{q\}$ is bounded below and has a finite set of denominators, that is,

$$\mathbb{C}\{\{t\}\} := \{\sum_{i=m}^{\infty} a_i t^{i/n} \mid a_i \in \mathbb{C}^*, m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\}.$$

We set $\mathbb{C}\{\{t\}\} = K$ from now on.

Definition 2.1.7. Let K be a field of Puiseux series. A non-Archimedean valuation on K is a function

$$val : K \to \mathbb{R} \cup \{-\infty\}$$

satisfying the properties:

(i) $val(a) = -\infty$ if and only if $a = 0$,

(ii) $val(ab) = val(a) + val(b)$,

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(iii) $val(a + b) \leq max\{val(a), val(b)\}$.

For each a in K with $a \neq 0$, we define the valuation of a by $val(a) = min\{q \mid a_q \neq 0\}$ (since $\{q\}$ is bounded below). And we define a norm by $|a|_{val} := exp(val(a))$. Let V_K be an algebraic variety on $(K^*)^n$, we define a map $Val : (K^*)^n \to \mathbb{R}^n$ by

$$Val(z_1, \ldots, z_n) = (\log |z_1|_{val}, \ldots, \log |z_n|_{val}) = (val(z_1), \ldots, val(z_n)).$$

Then we can also define the amoeba of algebraic variety V_K on $(K^*)^n$ by $A(V_K) := Val(V_t)$.

Theorem 2.1.8 (a version of Viro patchworking). Let V_t be an algebraic variety for small t in $\mathbb{R}$, and let V_K be an algebraic variety on $(K^*)^n$, then the non-Archimedean amoeba $A(V_K)$ is the limit of the amoebas $A_t(V_t)$ as $t \to \infty$ with respect to the Hausdorff metric on compacts.

2.2　Semifield

In this section, we will introduce the semigroup, semiring, and semifield.

Definition 2.2.1. Let G be a nonempty set. A binary operation in G is a function $* : G \times G \to G$. We denote the element $f(a, b)$ of G by $a * b$ for all

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$(a, b) \in G$. The set G is said to be closed under the binary operation $*$ and denoted by $(G, *)$.

The usual addition and multiplicative are two binary operations on $\mathbb{R}$.

Definition 2.2.2. A semigroup is a nonempty set G together with a binary operation $*$ which satisfies associative, that is, $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

A semigroup $(G, *)$ is called a commutative if $a * b = b * a$ for all $a, b \in G$. A semigroup $(G, *)$ is called idempotent if $a * b \in \{a, b\}$ for all $a, b \in G$.

Definition 2.2.3. A monoid is a semigroup G with an identity element e, that is, $a * e = e * a = a$ for all $a \in G$.

Proposition 2.2.4. Every group is a monoid.

Proof. Let G be a group. According to the definition of group, G is closed under a binary operation $*$, and G satisfies associative and has an identity element e, hence G is a monoid.

Example 2.2.5. Let $(2\mathbb{Z}_{>0}, *)$ be the set of the positive even integers under the usual multiplication of real numbers. Suppose that x, y, and z belong to $2\mathbb{Z}_{>0}$, then $x * (y * z) = (x * y) * z$, so $2\mathbb{Z}_{>0}$ satisfies associative. But 1 is not

even, that is, 1 is not in $2\mathbb{Z}_{>0}$, so $2\mathbb{Z}_{>0}$ does not have an identity element. Hence $(2\mathbb{Z}_{>0}, *)$ is a semigroup which is not a monoid.

Definition 2.2.6. Let S_1 and S_2 be semigroup. A map $\psi : S_1 \to S_2$ is a morphism of semigroup if $\psi(xy) = \psi(x)\psi(y)$ for all x, y in S_1.

Example 2.2.7. Define a map $\phi : (\mathbb{Z}_{>0}, +) \to (\mathbb{Z}_4, +)$ by $\phi(x) = \bar{x}$, for all x in $\mathbb{Z}_{>0}$. For all x, y in $\mathbb{Z}_{>0}$, $\phi(x + y) = \overline{x + y} = \bar{x} + \bar{y} = \phi(x) + \phi(y)$. Hence ϕ is a morphism of semigroup.

Definition 2.2.8. A semiring is a nonempty R together with two binary operations

$$\oplus : R \times R \to R \ and \otimes : R \times R \to R$$

such that $(R, \oplus)$ is a commutative monoid with identity element 0_R, $(R, \otimes)$ is a semigroup, and the operation $\otimes$ distributes over $\oplus$, that is, $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$ where a, b, c are in R.

Note that, according to the definition of ring, every ring is a semiring.

Definition 2.2.9. A semifield is a semiring $(R, \oplus, \otimes)$ together with $(R \backslash \{0_R\}, \otimes)$ is an abelian group where 0_R is an identity element for the binary operation $\oplus$.

Note that, by the definition of ring, every field is a semifield.

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Example 2.2.10. Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. We define two operations on $\mathbb{T}$ by $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. Suppose that a and b, and c are in $\mathbb{T}$. Without loss of generality, assume that $a \geq b$. Becuase $a \oplus b = \max\{a, b\} = a$ is in $\mathbb{T}$, so $\mathbb{T}$ is closed under a binary operation $\oplus$. If a and b are in $\mathbb{R}$, then $a \odot b = a + b$ is in $\mathbb{R}$; if one of a and b is $-\infty$, then $a \odot b = -\infty$, and so $\mathbb{T}$ is closed under a binary operation $\odot$. Suppose that a, b, and c are in $\mathbb{T}$. Without loss of generality, assume that $a \geq b \geq c$. Because $a \oplus (b \oplus c) = \max\{a, \max\{b, c\}\} = \max\{a, b\} = a$ and $(a \oplus b) \oplus c = \max\{\max\{a, b\}, c\} = \max\{a, c\} = a$, so $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, i.e. $(\mathbb{T}, \oplus)$ is a semigroup. Since $-\infty \oplus a = a \oplus -\infty = a$, $-\infty$ is an identity element, and so $a \oplus b = b \oplus a = \max\{a, b\}$, that is, $(\mathbb{T}, \oplus)$ is a monoid with identity element $-\infty$. Moreover, $a \oplus b = \max\{a, b\} = b \oplus a$, so $(\mathbb{T}, \oplus)$ is a commutative monoid. Since $(\mathbb{T} \backslash \{-\infty\}, \odot) = (\mathbb{R}, +)$ is abelian group, $(\mathbb{T}, \oplus, \odot)$ is a semifield.

The above $\mathbb{T}$ will be discussed in more details in the next section.

2.3   Tropical Semifields

Definition 2.3.1. Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. The tropical semifield $(\mathbb{T}, \oplus, \odot)$ is the semifield with operations $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. (c.f. Example 2.2.10)

Remark 2.3.2. Since $(\mathbb{T}\backslash\{-\infty\}, \odot)$ is an abelian group, we can define the tropical division by

$$x \oslash y := x - y,$$

for all x and y in $\mathbb{T}\backslash\{-\infty\}$.

Proposition 2.3.3. (a) Both addition and multiplication are *commutative*: $x \oplus y = y \oplus x$ and $x \odot y = y \odot x$. (b) The *distributive law* holds for tropical addition and tropical multiplication: $x \odot (y \oplus z) = x \odot y \oplus x \odot z$.

Proof. (a)   (i) $x \oplus y = \max\{x, y\} = \max\{y, x\} = y \oplus x$.

(ii) $x \odot y = x + y = y + x = y \odot x$.

(b) $x \odot (y \oplus z) = x + (y \oplus z) = x + \max\{y, z\} = \max\{x + y, x + z\} = (x + y) \oplus (x + z) = x \odot y \oplus x \odot z$.

Remark 2.3.4. For all integer n and all x in $\mathbb{T}$, we define

$$x^{\odot n} := x \odot \cdots \odot x = \sum_{i=1}^{n} x = nx.$$

Note that $-\infty$ is the additive identity and zero is the multiplicative unit, that is, $x \oplus (-\infty) = x$ and $x \odot 0 = x$.

Definition 2.3.5. The $\mathbb{R}^n$ is a module over the tropical semiring

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$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, with the operations of coordinatewise tropical addition

$$(a_1, \cdots, a_n) \oplus (b_1, \cdots, b_n) = (\max\{a_1, b_1\}, \cdots, \max\{a_n, b_n\})$$

and tropical scalar multiplication

$$\lambda \odot (a_1, \cdots, a_n) = (\lambda + a_1, \cdots, \lambda + a_n)$$

Definition 2.3.6. Let $(M, \oplus_M)$ be a commutative monoid over tropical semi-field $\mathbb{T}$. Then M is called a tropical module if there exists a scalar multiplication $\odot_M : \mathbb{T} \times M \to M$ denoted by $\odot_M(t, m) = t \odot_M x$ for all t in $\mathbb{T}$ and x in M, such that for all t_1, t_2 in $\mathbb{T}$ and x, y in M,

(i) $t_1 \odot_M (x \oplus_M y) = (t_1 \odot_M x) \oplus_M (t_1 \odot_M y)$;

(ii) $t_1 \odot_M (t_2 \odot_M x) = (t_1 \odot t_2) \odot_M x$;

(iii) $1_{\mathbb{T}} \odot_M x = x$ where $1_{\mathbb{T}} = 0$ is the multiplicative identity of $\mathbb{T}$;

(iv) if $t_1 \odot_M x = t_2 \odot_M x$ then either $t_1 = t_2$ or $x = -\infty$.

For careful statements, we refer the reader to [21].

Definition 2.3.7. A $\mathbb{T}$-vector space or tropical vector space M over $\mathbb{T}$ consists of a commutative monoid $(M, \oplus_M)$ and $\odot_M : \mathbb{T} \times M \to M$ such that

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for all t_1, t_2 in $\mathbb{T}$, x, y in M, we have:

(i) $t_1 \odot_M (x \oplus_M y) = (t_1 \odot_M x) \oplus_M (t_1 \odot_M y)$;

(ii) $t_1 \odot_M (t_2 \odot_M x) = (t_1 \odot t_2) \odot_M x$;

(iii) $1_{\mathbb{T}} \odot_M x = x$ for the tropical multiplicative identity $1_{\mathbb{T}}$.

(iv) $(t_1 \oplus t_2) \odot_M x = (t_1 \odot_M x) \oplus_M (t_2 \odot_M x)$;

Definition 2.3.8. The tropical projective n-space, denoted by $\mathbb{TP}^n$, is defined as the quotient

$$\left(\mathbb{T}^{n+1} \setminus (-\infty, \ldots, -\infty)\right)\Big/_{\sim},$$

where $\sim$ denotes the equivalence relation, $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if and only if there exists a λ in $\mathbb{T}^*$ such that $(y_0, \ldots, y_n) = (\lambda \odot x_0, \ldots, \lambda \odot x_n) = (\lambda + x_0, \ldots, \lambda + x_n)$.

Definition 2.3.9. Fix a weight vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$. The weight of the variable x_i is ω_i. The weight of a term $p(t) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is the real number

$$order(p(t)) + \alpha_1 \omega_1 + \cdots + \alpha_n \omega_n.$$

Definition 2.3.10. The tropical monomial is defined to be an expression of

the form

$$c \odot x_1^{a_1} \odot \cdots \odot x_n^{a_n}$$

where $a_1, \cdots, a_n \in \mathbb{Z}_{\geq 0}$ and c is a constant.

Definition 2.3.11. The finite linear combination of tropical monomials is called a tropical polynomial. Namely, $f = c_1 \odot x_1^{a_{11}} \odot \cdots \odot x_n^{a_{1n}} \oplus \cdots \oplus c_k \odot x_1^{a_{k1}} \odot \cdots \odot x_n^{a_{kn}}$

Definition 2.3.12. Consider a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and a vector $\omega \in \mathbb{R}^n$, the initial form $in_\omega(f)$ is the sum of all terms in f of smallest ω-weight.

Definition 2.3.13. The tropical hypersurface of f is the set

$$\mathcal{T}(f) = \{\omega \in \mathbb{R}^n \mid in_\omega(f) \text{ is not a monomial}\}.$$

Remark 2.3.14. All of points ω of the $\mathcal{T}(f)$ are attained by at least two of the linear functions. Note that $\mathcal{T}(f)$ is invariant under dilation, so we can say $\mathcal{T}(f)$ by giving its intersection with the unit sphere. (See [2] and the references therein)

Chapter 3

Toric variety and Fano variety

We begin by recalling the some basic definitions and notations which are necessary for study tropical toric varieties.

3.1 Polyhedral Geometry

In this section, we will recall the polyhedral geometry since they relate to affine toric varieties and tropical toric varieties.

Definition 3.1.1. Let R be a ring. A right R-module M over R is an abelian group, usually written additively, and an operation $M \times R \to M$ (denoted $(m, r) \mapsto mr$) such that for all $r, \ s$ in R, $x, \ y$ in M, we have:

(i) $(x + y)r = xr + yr$.

(ii) $x(r + s) = xr + xs$.

(iii) $(xs)r = x(sr)$.

(iv) $x1_R = x$ if R has multiplicative identity 1_R.

Similarly, we can define a left R-module via an operation $R \times M \rightarrow M$ denoted $(m, r) \mapsto rm$ and satisfy the above conditions. If R is a ring with identity, then a right R-module is also called a unitary right R-module. If R is a commutative ring, then a right R-modules are the same as left R-modules with $mr = rm$ for all m in M, r in R and are called R-modules.

If R is a field, then a R-module M is called a vector space.

Definition 3.1.2. An abelian group F is called a free abelian group if it has a basis.

Example 3.1.3. The trivial group $\{0\}$ is the free abelian group on the empty basis.

Definition 3.1.4. Let R be a ring. Let M be a right module and N be a left module over R. Let F be the free abelian group on $M \times N$. Let K be the subgroup of F generated by all elements of the forms

(i) $(a + b, c) - (a, c) - (b, c)$;

(ii) $(a, c + d) - (a, c) - (a, d)$;

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(iii) $(ar, c) - (a, rc)$,

for all $a, b \in M$; $c, d \in N$; $r \in R$. The quotient group F/K is called the tensor product of M and N, and we write $M \otimes_R N$ or simply $M \otimes N$ for F/K. The element (a, c) in F/K is denoted by $a \otimes c$.

We denote by $N \simeq \mathbb{Z}^n$ the free abelian group and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the associated real vector space; moreover, we denote by $M := hom(N, \mathbb{Z})$ the dual lattice of N and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 3.1.5. The polyhedron P is the intersection of finitely many halfspaces in $N_{\mathbb{R}}$, that is, a set of the form

$$P = \{X \in N_{\mathbb{R}} \mid AX \geq b\}$$

where $A \in (N_{\mathbb{R}}^{\vee})^d$ and $b \in \mathbb{R}^d$.

If $A \in (N^{\vee})^d$, $b \in \mathbb{Z}^d$, then P is called a rational polyhedron.

If $A \in (N^{\vee})^d$, $b \in \mathbb{R}^d$, then P is called polyhedron with rational slopes.

Example 3.1.6. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then

$$P = \{X \in \mathbb{R}^2 \mid AX \geq b\} = \{(x, y) \in \mathbb{R}^2 \mid x + 2y \geq 1, 3x + 4y \geq 2\}$$
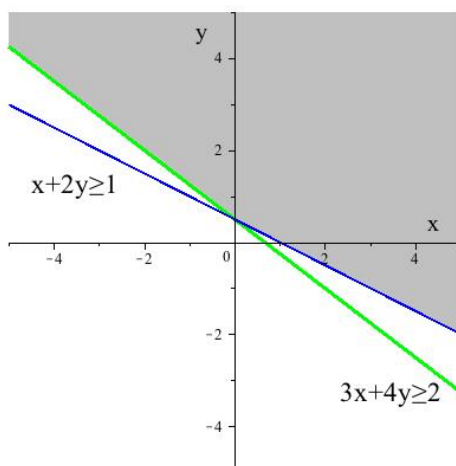
Figure 3.1: rational polyhedron

Definition 3.1.7. For every finite set $S \subseteq \mathbb{R}^d$, if a set S is not convex set, the convex hull of S is the smallest convex set containing it, which we denote it by $conv(S)$, that is,

$$conv(S) := \bigcap \{K \subseteq \mathbb{R}^d \mid S \subseteq K, \mathrm{K} \text{ is a convex set}\}$$

.

Proposition 3.1.8. Let S be a finite subset of $\mathbb{R}^n$. Then

$$conv(S) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m \mid x_1, \ldots, x_m \in S, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}$$

Proof. For any finite set $\{x_1, \ldots, x_m\} \subseteq S$, $\lambda_i \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$.

We have $\lambda_1 x_1 + \cdots + \lambda_m x_m = (1 - \lambda_m) \left(\sum_{i=1}^{m} \frac{\lambda_i x_i}{1-\lambda_m} \right) + \lambda_m x_m$ for $\lambda_m <$

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1. Therefore, $\sum_{i=1}^{m} \lambda_i x_i \in conv(S)$. Conversely, for any finite set $S_0 = \{x_1, \ldots, x_m\} \subseteq S$, $conv(S_0) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m \mid x_1, \ldots, x_m \in S_0, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\} \subseteq conv(S)$. So $conv(S) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m \mid \{x_1, \ldots, x_m\} \subseteq S, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}$. Hence if $S = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^n$ is a finite set, then we have $conv(S) = \{\lambda_1 x_1 + \cdots + \lambda_m x_m \mid x_1, \ldots, x_m \in S, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}$.

Definition 3.1.9. For every finite set S in a real vector space, the positive hull or conical hull of S is denoted by $pos(S)$ and is the set

$$pos(S) = \{\sum_{i \in I} \lambda_i m_i \mid \{m_i\}_{i \in I} \subseteq S, \lambda_i \geq 0\}.$$

Note that if $S = \emptyset$, then $pos(\emptyset) = \{0\}$.

Definition 3.1.10. The Minkowski sum of two sets X and Y in a vector space, defined by $X + Y$, is the set $\{x + y \mid x \in X, y \in Y\}$

Definition 3.1.11. A set σ is called a polyhedral cone (or simply a cone later) if

$$\sigma = pos(S) = \{\sum_{i \in I} \lambda_i m_i \mid \{m_i\}_{i \in I} \subseteq S, \lambda_i \geq 0\}$$

where $S \subseteq N_{\mathbb{R}}$ is finite.

By the Minkowski-Weyl theorem for cones, the cone σ is a finitely generated if and only if $\sigma = \{X \in N_{\mathbb{R}} \mid AX \geq 0\}$ where $A \in (M_{\mathbb{R}})^d$ and $b \in \mathbb{R}^d$.

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(i.e. σ is a polyhedron). For more details see [28] Section 1.3, [4] Theorem 1.15, and [16] p. 88.

Example 3.1.12. Consider the cone

$$
\begin{aligned}
\sigma \ &= pos\{(1,0),(1,1)\} \\
&= \{\lambda_1(1,0) + \lambda_2(1,1) \mid \lambda_1, \lambda_2 \geq 0\} \\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

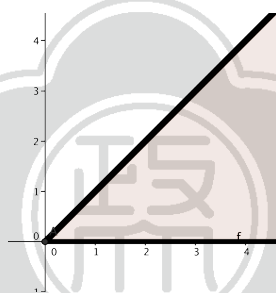in $\mathbb{R}^2$, then we can see its picture below:

Figure 3.2: the cone σ

Definition 3.1.13. Let σ be a cone. We have $u^{\perp} := \{v \in N_{\mathbb{R}} \mid \langle u, v \rangle = 0\}$ for a dual vector u in $M_{\mathbb{R}}$. Moreover, we define a face τ of the cone σ by

$$
\tau := \sigma \cap u^{\perp} = \{v \in \sigma \mid \langle u, v \rangle = 0\}.
$$

Definition 3.1.14. Let τ and σ be nonempty polyhedra. τ is called a *facet* of σ if τ is a face of σ and $dim(\tau) + 1 = dim(\sigma)$ (denoted by $\tau \prec \sigma$), that is, a facet τ is a face of codimension 1.

Definition 3.1.15. A polyhedral cone σ is said a pointed cone if the origin is a face of σ. Otherwise, the polyhedral cone is called a blunt.

Example 3.1.16. In $\mathbb{R}$, $C_1 = \{x \in \mathbb{R} \mid x \geq 0\}$ is a pointed cone, and $C_2 = \{x \in \mathbb{R} \mid x > 0\}$ is a blunt.

Definition 3.1.17. A cone σ is called simplicial if it is generated by a linearly independent subset of the lattice N, that is, $\sigma = pos(C)$ is called simplicial cone if C is linearly independent.

Definition 3.1.18. A simplicial cone σ is called unimodular if it is generated by a subset of a basis of the lattice N.

Example 3.1.19. Let $N = \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$. We consider the cone $\sigma = pos\{(1,0),(3,2)\}$ in N. Then $\{(1,0),(3,2)\}$ is linearly independent, but $(2,1)$ is not in $\mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(3,2)\}$. Hence the cone σ is simplicial.

Example 3.1.20. Let $N = \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$. Given the cone $\sigma = pos\{(1,0),(1,1)\}$ in N. Then $\{(1,0),(1,1)\}$ is a linearly independent set, and $\mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(1,1)\}$ can generate all of integer vectors in the cone σ. Hence the cone σ is unimodular.

Definition 3.1.21. The set $P = conv(S) = \{\sum_{i \in I} \lambda_i m_i \mid \{m_i\}_{i \in I} \subseteq S, \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1\}$ is said a polytope in $N_{\mathbb{R}}$ where $S \subseteq N_{\mathbb{R}}$ is finite.

The $P = conv(S) + pos(V)$ for some finite sets S, V in $N_\mathbb{R}$ if and only if P is a polyhedron, that is, $P = \{X \in N_\mathbb{R} \mid AX \geq b\}$ where $A \in (M_\mathbb{R})^d$ and $b \in \mathbb{R}^d$. For more details see [28] Theorem 1.2 and Section 1.2.

Example 3.1.22. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \\ -1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ -6 \\ -4 \end{pmatrix}$, then

$P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 1, -2x - y \geq -6, -x - y \geq -4\}$

Therefore, $\emptyset, V_1, \ldots, V_4, e_1, \ldots, e_4,$ or P are faces of P where $V_1, \ldots, V_4$ and $e_1, \ldots, e_4$ are vertices and edges of P respectively.
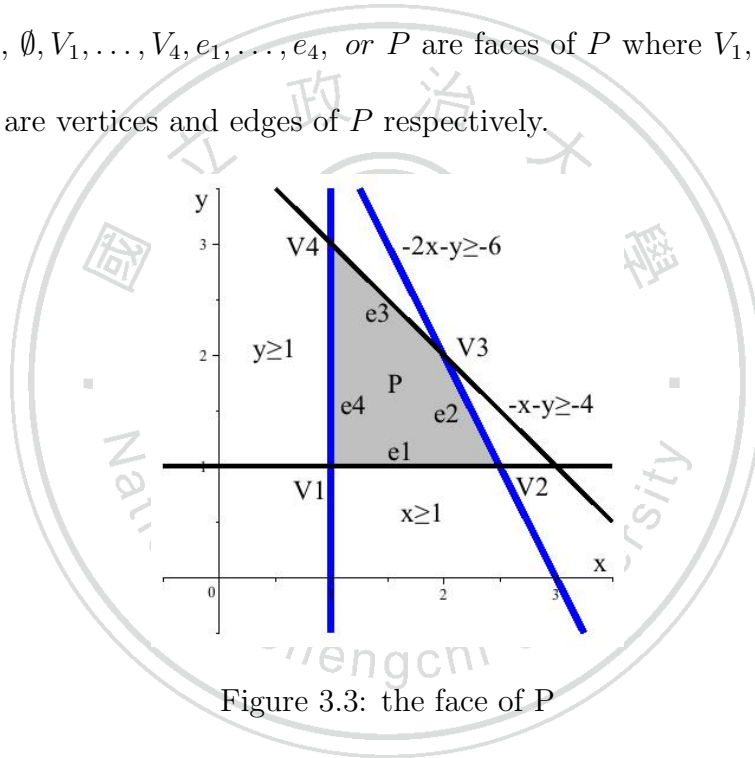

Figure 3.3: the face of P

Definition 3.1.23. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$, and write $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha x^\alpha$. The Newton polytope of f, denoted $NP(f)$ or $New(f)$, is the lattice polytope

$$New(f) = conv(\{\alpha \in \mathbb{Z}_{\geq 0} \mid c_\alpha \neq 0\}).$$
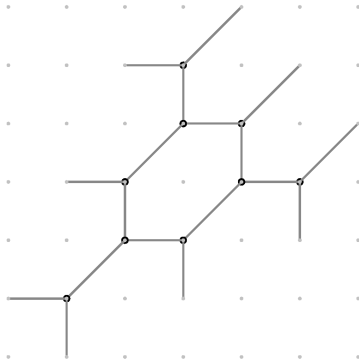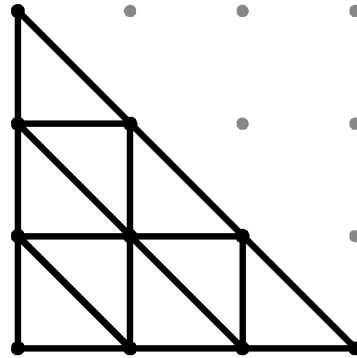
Figure 3.4: The tropical curve $Trop(f)$

Figure 3.5: The Newton subdivision of $Trop(f)$

Example 3.1.24. The tropicalisation of

$$f = t^2 \cdot x^3 + x^2 y + xy^2 + t^2 \cdot y^3 + x^2 + \tfrac{1}{t} \cdot xy + y^2 + x + y + t^2$$

is the tropical curve (as illustrated in Figure 3.4)

$$Trop(f) = 2 \odot x^{\odot 3} \oplus x^{\odot 2} \odot y \oplus x \odot y^{\odot 2} \oplus 2 \cdot y^{\odot 3} \oplus x^{\odot 2} \oplus x \odot y \odot (-1) \oplus y^{\odot 2}$$

$$\oplus\, x \oplus y \oplus 2$$

$$= \max\{2+3x, 2x+y, x+2y, 2+3y, 2x, x+y-1, 2y, x, y, 2\}$$

The vertices of the tropical curve are:

$$(2,0),\ (1,1),\ (1,0),\ (0,2),\ (0,1),\ (0,-1),\ (-1,0),\ (-1,-1),\ (-2,-2)$$

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The Newton subdivision of the tropical curve $Trop(f)$ is

$$New(Trop(f)) = conv\{(0,0),(1,0),(2,0),(3,0),(0,1),(0,2),$$

$$(0,3),(1,1),(1,2),(2,1)\}.$$

We illustrate the Newton subdivision of in Figure 3.5.

Definition 3.1.25. A polyhedral complex Δ is a collection of polyhedra such that the following the two conditions are satisfied: if $U \in \Delta$ and F is a face of U, then $F \in \Delta$; if $U, V \in \Delta$, then $U \cap V$ is a face of U and V.

The empty set is in the polyhedral complex Δ, i.e. a polyhedral complex Δ contains empty face.

Definition 3.1.26. F is a polyhedral fan if F is a polyhedral complex and each σ in F is a cone.

Note that we consider the fan is collection of non-empty polyhedral cones in this paper.

Example 3.1.27. Suppose that $\tau_1 = pos\{(-1,0)\}$, $\tau_2 = pos\{(1,1)\}$, $\tau_3 = pos\{(0,-1)\}$, $\sigma_1 = pos\{(-1,0),(0,-1)\}$, $\sigma_2 = pos\{(-1,0),(1,1)\}$, and $\sigma_3 = pos\{(0,-1),(1,1)\}$ Let $F = \{(0,0),\tau_1,\tau_2,\tau_3,\sigma_1,\sigma_2,\sigma_3\}\}$. Then $(0,0)$ is the face of other elements of F and $(0,0) = \tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3$, $\tau_1 = \sigma_1 \cap \sigma_2$

is the face of σ_1 and σ_2, $\tau_2 = \sigma_2 \cap \sigma_3$ is the face of σ_2 and σ_3, $\tau_3 = \sigma_1 \cap \sigma_3$ is the face of σ_1 and σ_3. Hence F is a fan.
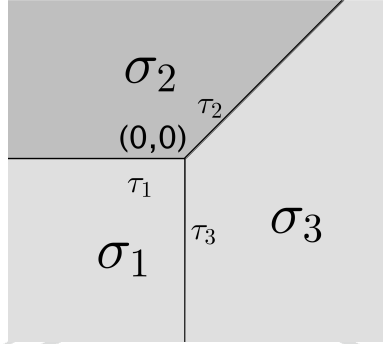

Figure 3.6: The fan F

Definition 3.1.28. Let F be a polyhedral complex. We define the following two notations:

- $F^{(k)}$ is a collection of k-dimendional polyhedra of F.
- $|F| = \bigcup_{U \in F} U$ is said the support of F.

Example 3.1.29. Recall from Example 3.1.27 that

$$F = \{\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3, (0,0)\}.$$

$$F^{(0)} = \{(0,0)\}, \ F^{(1)} = \{\tau_1, \tau_2, \tau_3\}, \ F^{(2)} = \{\sigma_1, \sigma_2, \sigma_3\}.$$

$$|F| = \bigcup_{U \in F} U = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau_1 \cup \tau_2 \cup \tau_3 \cup (0,0).$$

Definition 3.1.30. A polyhedral fan F is a rational fan if all cones in F are rational polyhedra.

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Definition 3.1.31. A polyhedral fan F in a vector space $N_{\mathbb{R}}$ is complete if the support of F is $N_{\mathbb{R}}$, i.e. $|F| = N_{\mathbb{R}}$.

Example 3.1.32. Recall from Example 3.1.27 that $F = \{\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3, (0,0)\}$. Then $|F| = \bigcup_{U \in F} U = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau_1 \cup \tau_2 \cup \tau_3 \cup (0,0) = \mathbb{R}^2$.

Definition 3.1.33. Let σ be a pointed rational cone in $N_{\mathbb{R}}$. The dual cone

$$\sigma^{\vee} := \{v \in M_{\mathbb{R}} \mid \langle v, u \rangle \geq 0, \forall u \in \sigma\}.$$

Example 3.1.34. Let $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $e_1 = (1,0)$ and $e_2 = (0,1)$ are the standard basis vectors, and let $\sigma = \{0\}$. Then $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, and $M = Hom(N, \mathbb{Z}) = \mathbb{Z}e_1^{\vee} \oplus \mathbb{Z}e_2^{\vee}$, thus $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^2$. Since $v \cdot 0 \geq 0$ for all v in $M_{\mathbb{R}}$, we have

$$\sigma^{\vee} = \{v \in M_{\mathbb{R}} \mid \langle v, 0 \rangle \geq 0\}$$
$$= pos\{(1,0), (-1,0), (0,1), (0,-1)\}.$$

Definition 3.1.35. Let P be a polytope in $N_{\mathbb{R}}$. We define the dual polytope

$$P^{\vee} := \{v \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } u \in P\}.$$

Theorem 3.1.36 (Farkas' Theorem)**.** Let σ be a polyhedral cone in $N_{\mathbb{R}}$,

then the dual cone $\sigma^\vee$ is a polyhedral cone in $M_\mathbb{R}$.

Proof. See [24] Corollary 22.3.1, [8] P.11 and [28] § 1.4.

Definition 3.1.37. A rational polyhedral cone is called strongly convex if it contains non-zero linear subspaces, namely, it does not contain line through the origin.

Proposition 3.1.38. Let σ lie in $N_\mathbb{R} \simeq \mathbb{R}^n$ be a polyhedral cone. Then the following conditions are equivalent:

(i) σ is strongly convex;

(ii) $\{0\}$ is a face of σ;

(iii) $\sigma \cap (-\sigma) = \{0\}$;

(iv) n is the dimension of $\sigma^\vee$;

(v) σ contains no positive-dimensional subspace of $N_\mathbb{R}$.

Lemma 3.1.39 (Separation Lemma)**.** Let Δ be a fan in $N_\mathbb{R}$, and let σ_1 and σ_2 be polyhedral cones in Δ. Let $\tau = \sigma_1 \cap \sigma_2$ be a common face of σ_1 and σ_2. Then there exists u in $\sigma_1^\vee \cap \sigma_2^\vee$ such that

$$\tau = \sigma_1 \cap u^\perp = \sigma_2 \cap u^\perp.$$

Proposition 3.1.40. Let σ be a unimodular cone. Let the set $\{u_1, \ldots, u_n\}$ be a basis of N. If $\sigma = pos\{u_1, \ldots, u_k\}$, then we have

$$\sigma^\vee = pos\{u_1^\vee, \ldots, u_k^\vee, \pm u_{k+1}^\vee, \ldots, \pm u_n^\vee\}.$$

Proof. For all $j = 1, \ldots, k$. Let u be in σ. By the definition of then cone, suppose that $u = \sum_{i=1}^{k} \lambda_i u_i$ where $\lambda_1, \ldots, \lambda_k \geq 0$. Then we get that

$$u_j^\vee \cdot (\sum_{i=1}^{k} \lambda_i u_i) = \lambda_j \geq 0.$$

For $j = k+1, \ldots, n$, then we have

$$u_j^\vee \cdot (\sum_{i=1}^{k} \lambda_i u_i) = \lambda_j = 0.$$

Conversely, let v be in $\sigma^\vee$, then $v = \sum_{j=1}^{m} \eta_j u_j^\vee = \sum_{j=1}^{k} \eta_j u_j^\vee + \sum_{j=k+1}^{m} \eta_j u_j^\vee$. For $j = 1, \ldots, k$, we have $v \cdot u = \lambda_j \geq 0$ where $u = \sum_{j=1}^{k} \lambda_j u_i$ is in σ. For $j = k+1, \ldots, m$, since $\sum_{j=k+1}^{m} \eta_j u_j^\vee = \sum_{j=k+1}^{m} \eta_j^+ u_j^\vee + \sum_{j=k+1}^{m} \eta_j^- (-u_j^\vee)$, v is in $pos\{u_1^\vee, \ldots, u_k^\vee, \pm u_{k+1}^\vee, \ldots, \pm u_n^\vee\}$.

Theorem 3.1.41 (Duality Theorem)**.** If σ is a convex polyhedral cone in $N_\mathbb{R}$, then $(\sigma^\vee)^\vee = \sigma$.

Proof. This is well known result. For careful information see [12] P.47 and

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[24] Theorem 14.1.

Proposition 3.1.42. Let the set $\{u_1, \ldots, u_n\}$ be a basis of N. Let $\sigma = pos\{u_1, \ldots, u_k\}$ be a unimodular cone in $N_{\mathbb{R}}$, then $(\sigma^\vee)^\vee = \sigma$.

Proof. According to the above Proposition 3.1.40, we have

$$\sigma^\vee = pos\{u_1^\vee, \ldots, u_k^\vee, \pm u_{k+1}^\vee, \ldots, \pm u_n^\vee\}$$

since σ be a unimodular cone. So we get that

$$\sigma^{\vee\vee} = \{\omega \in N_{\mathbb{R}} \mid < \omega, v > \geq 0, \ \forall v \in \sigma^\vee\}.$$

Let u belong to a cone $\sigma = pos\{u_1, \ldots, u_k\}$, then $u = \sum_{i=1}^{k} \lambda_i u_i$, and so $u \cdot v = \lambda_i \geq 0$ for some i.

Conversely, let ω be in $\sigma^{\vee\vee}$, then we write $\omega = \sum_{i=1}^{n} c_i u_i$. Then

$$\omega = \sum_{i=1}^{k} c_i u_i + \sum_{i=k+1}^{n} c_i^+ u_i + \sum_{i=k+1}^{n} c_i^- (-u_i).$$

For $i = 1, \ldots, k$, $0 \leq \omega \cdot u_i^\vee = c_i$. For $i = k+1, \ldots, n$, $0 \leq \omega \cdot u_i^\vee = c_i$ and $0 \leq \omega \cdot (-u_i^\vee) = c_i$, so we get that $c_i = 0$ for $i = k+1, \ldots, n$.

Lemma 3.1.43 (Gordon's Lemma). Let σ be a rational convex polyhedral

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cone in $N_{\mathbb{R}}$, then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup where M is a daul lattice of N.

Proof. By the Farkas' Theorem, the dual cone $\sigma^\vee$ is a polyhedral cone in $M_{\mathbb{R}} \simeq \mathbb{R}^n$. Let $\sigma^\vee = pos\{U\}$ where $U = \{u_1, \ldots, u_m\}$ is a finite subset of M. Take $K = \{\sum_{i=1}^m t_i u_i \mid 0 \leq t_i \leq 1\}$, then K is a bounded region of $M_{\mathbb{R}}$, and so K is compact. Since $M \simeq \mathbb{Z}^n$, $K \cap M$ is finite. We claim that $U \cup (K \cap M)$ generate the semigroup $S_\sigma = \sigma^\vee \cap M$. If u is in S_σ, then we write $u = \sum_{i=1}^m r_i u_i$ where r_i is nonnegative real number for all $i = 1, \ldots, m$. Because $r_i = \lfloor r_i \rfloor + t_i$ where $\lfloor r_i \rfloor$ which denotes the floor of r_i is a nonnegative integer and $0 \leq t_i \leq 1$ for all $i = 1, \ldots, m$, $u = \sum_{i=1}^m \lfloor r_i \rfloor u_i + \sum_{i=1}^m t_i u_i$. Then $\sum_{i=1}^m t_i u_i$ is in $K \cap M$. Therefore, u is a nonnegative integer combination of elements of $U \cup (K \cap M)$.

The Gordon's lemma is well known result. For more information also see [8], [22] or [28].

Example 3.1.44. Let N=$\mathbb{Z}(-1,0) \oplus \mathbb{Z}(0,-1)$. Take $\sigma = pos\{(-1,0)\}$. If $(x_1, x_2) \cdot (-1, 0) = -x_1 \geq 0$, then the dual cone

$$\sigma^\vee = \{v \in M_{\mathbb{R}} \mid v \cdot (-1, 0) \geq 0\}$$

$$= pos\{(-1, 0), (0, 1), (0, -1)\}$$

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So the corresponding semigroup $S_\sigma = \sigma^\vee \cap M = \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(0, -1)$.

Proposition 3.1.45. Let Δ be a fan in $N_\mathbb{R}$, and let τ is a face of σ in Δ, then

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$$

for some $-u$ in the dual lattice M.

Proposition 3.1.46. Take a fan Δ in $N_\mathbb{R}$. Let σ_1 and σ_2 in Δ, and let $\tau = \sigma_1 \cap \sigma_2$, then

$$S_\tau = S_{\sigma_1} + S_{\sigma_2}$$

3.2 Fiber products of affine varieties

In the section, our references are from definitions in [9], [6], [8] and [5].

Given two affine varieties $V_1 = \mathbb{V}(f_1, \ldots, f_s)$ and $V_2 = \mathbb{V}(g_1, \ldots, g_t)$ where $f_1, \ldots, f_s$ are in $\mathbb{C}[x_1, \ldots, x_m]$ and $g_1, \ldots, g_t$ are in $\mathbb{C}[y_1, \ldots, y_n]$, then we have

$$V_1 \times V_2 := \mathbb{V}(f_1, \ldots, f_s, g_1, \ldots, g_t).$$

Let V_1 and V_2 be algebraic variety in $\mathbb{A}^n$ and $\mathbb{A}^m$, respectively. A map

$f : V_1 \to V_2$ is a morphism of algebraic variety if there is a map $\tilde{f} : \mathbb{A}^n \to \mathbb{A}^m$ with $\tilde{f}(x) = (\tilde{f}_1(x), \ldots, \tilde{f}_m(x))$, that is, $f = \tilde{f} \mid_{V_1} : V_1 \to V_2$

Definition 3.2.1. For any subset V of $\mathbb{A}^n$, we define the ideal of V to be

$$I(V) = \{f \in K[x_1, \ldots, x_n] \mid f(x) = 0 \ for \ all \ x \in V\}$$

Definition 3.2.2. Let X be nonempty set ,and let $\mathscr{T}$ be a collection of subsets of X. $\mathscr{T}$ is a topology on X if it satisfies the following properties:

(1) $\emptyset$ and X are in $\mathscr{T}$.

(2) If U_i is in $\mathscr{T}$ for all i in index I, then $\bigcup_{i \in I} U_i$ is in $\mathscr{T}$.

(3) If $U_1, \ldots, U_n$ are in $\mathscr{T}$, then $\bigcap_{i=1}^{n} U_i$ is in $\mathscr{T}$.

The members in $\mathscr{T}$ are called the open sets in X. Moreover, the complements of the open sets is called closed sets in X.

The algebraic varieties are closed sets on $\mathbb{A}^n$. Therefore, we will show a topology on $\mathbb{A}^n$.

Proposition 3.2.3. If $\mathscr{T}$ is a collection of the algebraic varieties on $\mathbb{A}^n$, i.e. $\mathscr{T} = \{V \subseteq \mathbb{A}^n \mid V \text{ is an affine algebraic variety}\}$, then $\mathscr{T}$ is a topology.

Proof. To start with, since $\emptyset = \mathbb{V}(1)$ and $\mathbb{A}^n = \mathbb{V}(0)$, $\emptyset$ and $\mathbb{A}^n$ are in $\mathscr{T}$.

Next, suppose that $V_1 = \mathbb{V}(\{f_i\}_{i \in I})$ and $V_2 = \mathbb{V}(\{f_j\}_{j \in J})$ where f_i and f_j

are in $K[x_1, \ldots, x_n]$ for all i and j. We claim that $V_1 \cap V_2 = \mathbb{V}(\{f_i\}_{i \in I \cup J})$. A

point p is in $V_1 \cap V_2$ if and only if p is in $\mathbb{V}(\{f_i\}_{i \in I})$ and $\mathbb{V}(\{f_j\}_{j \in j})$,which is

$f_i(p) = 0$ for all i in $I \cup J$, so a point p is in $\mathbb{V}(\{f_i\}_{i \in I \cup J})$.

Finally, we only need to prove that $\mathbb{V}(f_1) \cup \mathbb{V}(f_2) = \mathbb{V}(f_1 f_2)$ where f_1 and f_2

are in $K[x_1, \ldots, x_n]$. If p belongs to $\mathbb{V}(f_1) \cup \mathbb{V}(f_2)$, then p belongs to $\mathbb{V}(f_1)$

or $\mathbb{V}(f_2)$, and so $f_1(p) = 0$ and $f_2(p) = 0$. Then $f_1(p)f_2(p) = 0$. Hence

p belongs to $\mathbb{V}(f_1 f_2)$. Conversely, if p belongs to $\mathbb{V}(f_1 f_2)$. If p belongs to

$\mathbb{V}(f_1)$, then we are done, and if not, then $f_1(p) \neq 0$. Since p belongs to

$\mathbb{V}(f_1 f_2)$, that is, $f_1(p)f(p) = 0$, $f_2(p) = 0$. So p belongs to $\mathbb{V}(f_2)$. Hence p

belongs to $\mathbb{V}(f_1) \cup \mathbb{V}(f_2)$.

We call the above topology $\mathscr{T}$ the Zariski topology on $\mathbb{A}^n$.

Definition 3.2.4. The maximal spectrum $maxSpec(R)$ (or $Specm(R)$) of a

ring R is the set of all maximal ideals of R, that is,

$$Specm(R) = \{m \subset R \mid \text{m is a maximal ideal of R }\}.$$

If we have a ring homomorphism $f : R \to S$, then we might not have

a map $maxSpec(S) \to maxSpec(R)$ since $f^{-1}(m)$ is not always a maximal

ideal in R for m is a maximal ideal in S. For example, a ring homomorphism $f : \mathbb{Z} \to \mathbb{Q}$, (0) is a maximal ideal in $\mathbb{Q}$, but $f^{-1}(0) = (0)$ is not a maximal ideal in $\mathbb{Z}$ since $(0) \subset (2) \subset \mathbb{Z}$.

Definition 3.2.5. The spectrum $Spec(R)$ of a ring R is the set of all prime ideals of R, that is,

$$Spec(R) = \{ p \subset R \mid \text{p is a prime ideal of R} \}.$$

Proposition 3.2.6. Let $f : R \to S$ be a ring homomorphism. If P be a prime ideal in S, then $f^{-1}(P)$ is a prime ideal in R.

Proof. Assume that P is a prime ideal in S. Let xy belong to $f^{-1}(P)$, then $f(xy) = f(x)f(y)$ is in P. Since P is a prime ideal, $f(x)$ is in P or $f(y)$ is in P. Then x is in $f^{-1}(P)$ or y is in $f^{-1}(P)$. Hence $f^{-1}(P)$ is a prime ideal in R.

By the above proposition, if a ring homomorphism $f : R \to S$, then we define a map $\phi : Spec(S) \to Spec(R)$ by $\phi(P) = f^{-1}(P)$, which is a well-defined.

Definition 3.2.7. Given two sets X and Y over a third set S, that is, the

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mappings of sets $f : X \to S$ and $g : Y \to S$, then the fiber product

$$X \times_S Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

Let $f : X \to S$ and $g : Y \to S$ be two morphisms of schemes. The fiber product of X and Y is a scheme $X \times_S Y$ together with projection $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ such that whenever we have morphisms $\phi_1 : W \to X$ and $\phi_2 : W \to Y$ for any scheme W. There exists a unique morphism $\pi : W \to X \times Y$ making this diagram commute
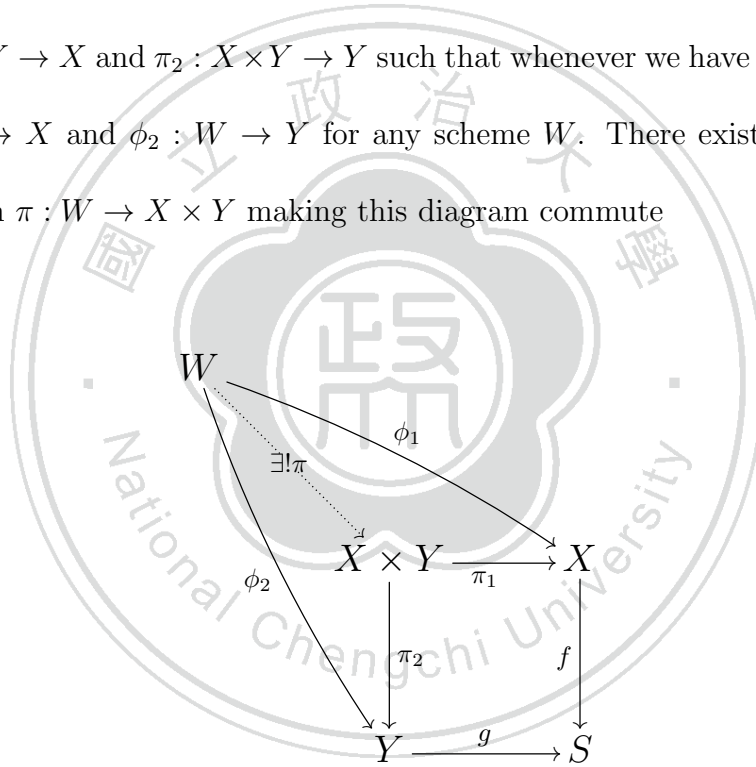
Definition 3.2.8. Let K be a commutative ring with identity. A ring R is called a K-algebra if the additive group $(R, +)$ is a unitary K-module, and $k(ab) = (ka)b = a(kb)$ for all k in K and $a,\ b$ in R.

Definition 3.2.9. Let V be an affine variety in $\mathbb{A}^n$. We define the coordinate

ring of V to be the quotient of the polynomial ring by the ideal, that is,

$$K[V] \simeq K[x_1, \ldots, x_n] \Big/ I(V).$$

For coordinate rings whenever we have a diagram with $\mathbb{C}$-algebra homomorphism $\phi_i^* : \mathbb{C}[V_i] \to \mathbb{C}[W]$, there should be a unique $\mathbb{C}$-algebra homomorphism $\pi^* : \mathbb{C}[V_1 \times V_2] \to \mathbb{C}[W]$ that make this diagram commute

$$
\begin{array}{ccc}
\mathbb{C}[W] & \xleftarrow{\phi_1^*} & \mathbb{C}[V_1] \\
\phi_2^* \uparrow & \exists! \pi^* \nwarrow & \downarrow \pi_1^* \\
\mathbb{C}[V_2] & \xrightarrow{\pi_2^*} & \mathbb{C}[V_1 \times V_2]
\end{array}
$$

3.3 Toric Varieties

Definition 3.3.1. Let $(K, +, \times)$ be a semifield and let $K^* = K \setminus \{0_+\}$ where 0_+ is the identity element for the binary operation $+$. The set $(K^*)^n$ is called the n-dimensional algebraic torus over K.

Example 3.3.2. The $S^1 = \{z \in \mathbb{C}^* \mid z\bar{z} = 1\} \simeq \mathbb{C}^*$, so S^1 is the one-dimensional algebraic torus over $\mathbb{C}$

Definition 3.3.3. A one-parameter subgroup of a tours T is a morphism $\lambda : \mathbb{C}^* \to T$ which is a group homomorphism.

Remark 3.3.4. The group $hom(\mathbb{C}^*, T)$ of one-parameter subgroup is a lattice where $hom(\mathbb{C}^*, T) = \{\lambda : \mathbb{C}^* \to T \mid \lambda$ is a morphism of variety and a group homomorphism$\}$.

Definition 3.3.5. A character of a tours T is a morphism $\chi : T \to \mathbb{C}^*$ which is a group homomorphism.

Remark 3.3.6. The character group $hom(T, \mathbb{C}^*)$ of a torus T is a lattice where $hom(T, \mathbb{C}^*) = \{\chi : T \to \mathbb{C}^* \mid \chi$ is a morphism of variety and a group homomorphism$\}$.

In fact, we can show that the following propositions. Let $v = (v_1, \ldots, v_n)$ be in $\mathbb{Z}^n$. Let χ and λ be in $hom((\mathbb{C}^*)^n$ and $hom(\mathbb{C}^*, (\mathbb{C}^*)^n)$ respectively. Then we have

$$\chi^v(x_1, \ldots, x_n) = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}, \text{ and } \lambda^v(t) = (t^{v_1}, \ldots, t_{v_n}).$$

Definition 3.3.7. Let $(S, *)$ be a semigroup and $(X, \cdot)$ be nonempty. A map $S \times X \to X$ given by $(s, x) \mapsto s \cdot x$ is called an action of S on X if $s_1 \cdot (s_2 \cdot x) = (s_1 * s_2) \cdot x$ where for all s_1, s_2 in S, and x in X. Moreover, S acts on X, X is called S-set.

If S is a monoid (or group) with identity e, then S acts on X if there exists a map which satisfies the above conditions and $e \cdot x = x$ for all x in X.

Definition 3.3.8. Let X be a S-set. For any x in X, $S \cdot x = \{s \cdot x \mid g \in S\}$ is called the S-orbit of x.

Definition 3.3.9. Let σ be a cone, and let S_σ be the corresponding semi-group. We define the affine toric variety U_σ corresponding to a cone σ by

$$U_\sigma := hom(S_\sigma, \mathbb{C})$$

where $hom(S_\sigma, \mathbb{C})$ denotes the semigroup homomorphism $S_\sigma \to \mathbb{C}$ and $\mathbb{C}$ is considered as a semigroup under multiplication.

Example 3.3.10. Let $N \simeq \mathbb{Z}^2$ be a lattice with associated vector space $N_\mathbb{R} \simeq \mathbb{R}^2$, and let M be a dual lattice of N with associated vector space $M_\mathbb{R} \simeq \mathbb{R}^2$. Given a cone $\sigma = pos\{0\}$ in $N_\mathbb{R}$ where 0 denotes $(0,0)$, then its dual cone $\{0\}^\vee = pos\{(1,0), (-1,0), (0,1), (0,-1)\}$, then $S_{\{0\}} = \{0\}^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1)$. Let $\chi^{(1,0)} = x_1$, $\chi^{(-1,0)} = x_2$, $\chi^{(0,1)} = x_3$, and $\chi^{(0,-1)} = x_4$. Then $1 = \chi^{(0,0)} = \chi^{(1,0)}\chi^{(-1,0)} = x_1 x_2$ and $1 = \chi^{(0,0)} = \chi^{(0,1)}\chi^{(0,-1)} = x_3 x_4$, and so $x_2 = x_1^{-1}$ and $x_4 = x_3^{-1}$. So $\mathbb{C}[S_{\{0\}}] = \mathbb{C}[\{0\}^\vee \cap M] = \mathbb{C}[x_1, x_1^{-1}, x_3, x_3^{-1}] \simeq \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_2 x_3 x_4 - 1)$.

Hence we have

$$U_{\{0\}} = Spec(\mathbb{C}[S_{\{0\}}])$$

$$= Spec(\mathbb{C}[x_1, x_1^{-1}, x_3, x_3^{-1}])$$

$$\simeq Spec(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_2 x_3 x_4 - 1))$$

$$\simeq (\mathbb{C}^*)^2.$$

So $U_{\{0\}}$ is the 2-dimensional algebraic torus over $\mathbb{C}$.

Similarly, given a lattice $N \simeq \mathbb{Z}^n$, then we can also show that $U_{\{0\}}$ is the n-dimensional algebraic torus over $\mathbb{C}$.

Proposition 3.3.11. Let σ be a cone, and let S_σ be the corresponding semigroup. Then there is bijective correspondence between $hom(S_\sigma, \mathbb{C})$ and $Spec(\mathbb{C}[S_\sigma])$.

Proof. See [5] Proposition 1.3.1.

Remark 3.3.12. A toric variety is an irreducible variety X containing an algebraic torus T as a Zariski open subset of X such that these exists an open T-orbit of X isomorphic to T.

In $\mathbb{C}$ (or field), the semigroup homomorphism $hom(S_\sigma, \mathbb{C})$ is isomorphic to $Spec(\mathbb{C}[S_\sigma])$. Moreover, if t is in the algebraic torus T and f is in the

semigroup algebra $\mathbb{C}[S_\sigma]$, then $t \cdot f$ lies in $\mathbb{C}[S_\sigma]$ is defined by $s \mapsto f(t^{-1} \cdot s)$

for all s in T. (See [5] chapter 1 and chapter 5)

Example 3.3.13. Let X be the multiplicative group of nonzero complex

numbers $\mathbb{C}^*$ and let T be the a 1-dimensional algebraic torus $\mathbb{C}^*$. Since $\{0\}$

is an affine algebraic variety, $T = \mathbb{C}\backslash\{0\}$ is a Zariski open subset of X.

Define a map $f : T \times X \to X$ given by $(t, x) \mapsto t \cdot x$. Take $x = 1 \in \mathbb{C}^*$,

$\mathbb{C}^* \cdot 1 = \{t \cdot 1 \mid t \in \mathbb{C}^*\}$ is an open $\mathbb{C}^*$-orbit of x and is isomorphic to $\mathbb{C}^*$.

Example 3.3.14. Let X be the multiplicative group of nonzero complex

numbers $\mathbb{P}^n$ and let T be the a n-dimensional algebraic torus $(\mathbb{C}^*)^n$. Since

$\{0\}$ is an affine algebraic variety, $T = \mathbb{P}^n\backslash V(x_0 x_1 \cdots x_n)$ is a Zariski open

subset of X. Define a map $f : T \times X \to X$ given by $(t, x) \mapsto t \cdot x$. Take

$x = [1, \cdots, 1] \in \mathbb{P}^n$, then $(\mathbb{C}^*)^n \cdot x = \{[1 : t_1 : \cdots : t_n] \mid (t_1, \cdots, t_n) \in (\mathbb{C}^*)^n\}$

is an open $(\mathbb{C}^*)^n$-orbit of x and is isomorphic to $(\mathbb{C}^*)^n$.

Proposition 3.3.15. Let σ be a polyhedral cone, and let τ be a face of σ,

then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.

Remark 3.3.16. Because $\{0\}$ is a face of all polyhedral cone σ, the torus

$U_{\{0\}}$ is a principal open subset of all U_σ.

Let Δ be a fan, and let σ_1 and σ_2 be in Δ. Then $\sigma_1 \cap \sigma_2$ is a face

of σ_1 and σ in Δ. Moreover, U_{σ_1} U_{σ_2} are the corresponding affine toric

varieties. According to Proposition 3.3.15, we have two embedding maps $h_1 : U_{\sigma_1 \cap \sigma_2} \hookrightarrow U_{\sigma_1}$ and $h_2 : U_{\sigma_1 \cap \sigma_2} \hookrightarrow U_{\sigma_2}$. We define an equivalence relation by $A \sim B$ where A and B are in U_{σ_1} and U_{σ_2} respectively if and only if $h_1 \circ h_2^{-1}(B) = A$. Note that we have the following commutative diagram:

$$
\begin{array}{ccc}
U_{\sigma_1 \cap \sigma_2} & \xhookrightarrow{\;h_2\;} & U_{\sigma_2} \\
{\scriptstyle h_1} \downarrow & {\scriptstyle h_1 \circ h_2^{-1}} \;\; {\scriptstyle h_2 \circ h_1^{-1}} & \\
U_{\sigma_1} & &
\end{array}
$$

Definition 3.3.17. Given a fan Δ in $N_{\mathbb{R}}$. We define the toric variety by the quotient space

$$
\mathbb{X}_{\Delta} := \left(\coprod_{\sigma \in \Delta} U_{\sigma} \right) \Big/ {\sim},
$$

that is the disjoint union of the affine toric varieties, and $\sim$ is the above equivalence relation.

Next, the following example of the toric variety is over $\mathbb{C}$. Albeit we will discuss the same case in the Example 4.3.2, it is over tropical semifield $\mathbb{T}$. And we will know the difference between toric varieties and tropical toric varieties later.

Example 3.3.18. Given the lattice $N \simeq \mathbb{Z}^2$, then $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the dual lattice $M \simeq \mathbb{Z}^2$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let the fan Δ in $N_{\mathbb{R}}$. Suppose that
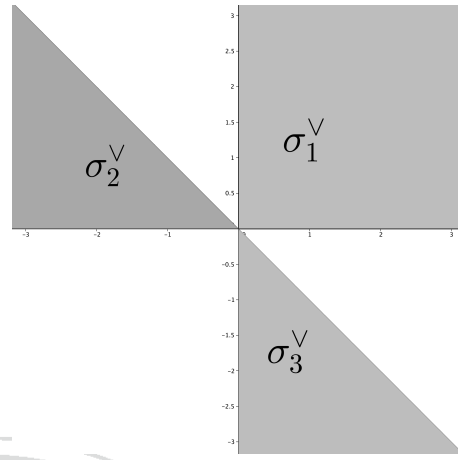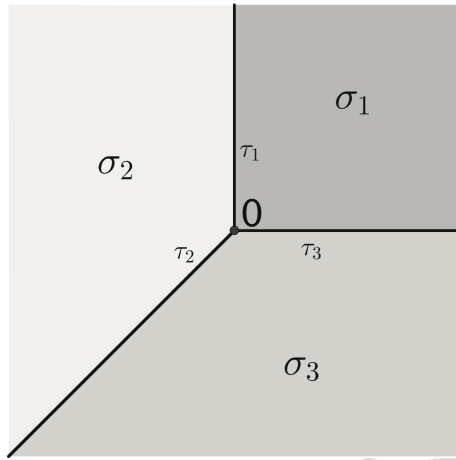
Figure 3.7: the fan Δ Figure 3.8: the dual cones $\sigma_1^\vee, \sigma_2^\vee, \sigma_3^\vee$

the fan Δ has

$$\sigma_1 = pos\{(1,0),(0,1)\}, \ \sigma_2 = pos\{(-1,-1),(0,1)\}, \ \sigma_3 = pos\{(1,0),(-1,-1)\},$$

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(0,1)\}, \ \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(-1,-1)\},$$

$$\tau_3 = \sigma_3 \cap \sigma_1 = pos\{(1,0)\}, \ \text{and the origin.}$$

Then the dual cones

$$\sigma_1^\vee = pos\{(1,0),(0,1)\}, \ \sigma_2^\vee = pos\{(-1,0),(-1,1)\}, \ \sigma_3^\vee = pos\{(1,-1),(0,-1)\}.$$

Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\sigma_2} = \sigma_2^\vee \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\sigma_3} = \sigma_3^\vee \cap M = \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1) \oplus \mathbb{Z}_{\geq 0}(-1,-1),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1).$$

Let $x_1 := \chi^{(1,0)}$, $x_2 := \chi^{(-1,0)}$, $x_3 := \chi^{(0,1)}$ and $x_4 := \chi^{(0,-1)}$. Then $x_1 x_2 = 1$ and $x_3 x_4 = 1$, and we have

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x_1, x_3], \ \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[x_2, x_2 x_3], \ \mathbb{C}[S_{\sigma_3}] = \mathbb{C}[x_1 x_4, x_4],$$

$$\mathbb{C}[S_{\tau_1}] = \mathbb{C}[x_1, x_2, x_3], \mathbb{C}[S_{\tau_2}] = \mathbb{C}[x_1 x_4, x_2 x_3, x_2 x_4], \text{ and } \mathbb{C}[S_{\tau_3}] = \mathbb{C}[x_1, x_3, x_4].$$

Therefore, the affine toric variety

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{C}) \simeq Spec\mathbb{C}[S_{\sigma_1}] = Spec\mathbb{C}[x_1, x_3] \simeq \mathbb{C} \times \mathbb{C},$$

$$U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{C}) \simeq Spec\mathbb{C}[S_{\sigma_2}] = Spec\mathbb{C}[x_2, x_2 x_3] \simeq \mathbb{C} \times \mathbb{C},$$

$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{C}) \simeq Spec\mathbb{C}[S_{\sigma_3}] = Spec\mathbb{C}[x_1 x_4, x_4] \simeq \mathbb{C} \times \mathbb{C},$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{C}) = \mathbb{C}^* \times \mathbb{C},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}, \ U_{\{0\}} = hom(S_{\{0\}}, \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*.$$

The gluing of the affine toric varieties U_{σ_1} and U_{σ_2} along their common subset U_{τ_1} gives $\mathbb{CP}^2$ with coordinates $(z_0 : z_1 : z_2)$ where $x_1 = z_1/z_0$ and $x_3 = z_2/z_0$. The gluing of the affine toric varieties U_{σ_2} and U_{σ_3} along their common subset U_{τ_2} gives $\mathbb{CP}^2$ with coordinates $(z_0 : z_1 : z_2)$ where $x_2 x_3 = z_1/z_0$ and $x_2 = z_2/z_0$. The gluing of the affine toric varieties U_{σ_3} and U_{σ_4} along their common subset U_{τ_3} gives $\mathbb{CP}^2$ with coordinates $(z_0 : z_1 : z_2)$ where $x_3 = z_1/z_0$ and $x_1 = z_2/z_0$.

The following commutative diagram:

$$U_{\sigma_1} \longleftarrow U_{\tau_1} \longhookrightarrow U_{\sigma_2}$$

$$U_{\tau_3} \qquad U_{\tau_2}$$

$$U_{\sigma_3}$$

Hence the gluing of these two gives the toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = \left(\coprod_{\sigma \in \Delta} U_\sigma\right)\Big/_\sim = \mathbb{CP}^2.$$

Theorem 3.3.19 (Hironaka's Theorem). Let V be a quasi-projective variety. Then there exists a smooth quasi-projective variety X and a projective birational morphism $\pi : X \to V$. Furthermore, π may be assumed to be an isomorphism on the smooth locus of V, and if V is a projective variety, then so is X.

Proof. See [25] p.106.

Let $X = \{(x, [p]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x \in [p]\}$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$. The $blow - up$ of $\mathbb{A}^n$ at a point $[p]$ is the map $\pi : X \to \mathbb{A}^n$ via $(x, [p]) \mapsto x$.

3.4 Fano varieties

In this section, we will introduce and outline the Fano variety and Fano polytope. For more information see [5], [22], [7], [8] and [14].

Definition 3.4.1. For all r in $\mathbb{Z}$, we define the Hirzebruch surface

$$H_r := \{([x_0 : x_1], [y_0 : y_1 : y_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid x_0^r y_0 = x_1^r y_1\}.$$

Since H_r is isomorphic to H_{-r} for all r in $\mathbb{Z}$, we sometimes assume $\mathbb{Z}_{\geq 0}$.

Theorem 3.4.2. Let P ba a polytope in $N_{\mathbb{R}}$. If all of facets of P are the convex hull of a basis of N if and only if X_P is a smooth Fano variety.

Proof. See [14] Proposition 3.6.7 and [7] Lemma 8.5.

The Fano varieties in two-dimension are also called a del Pezzo surface.

Theorem 3.4.3. There exist five distinct toric Fano varieties of two-dimension up to isomorphism,

1. $\mathbb{CP}^2$,

2. $\mathbb{CP}^1 \times \mathbb{CP}^1$,

3. the equivariant blowing-up of $\mathbb{CP}^2$ at one point (i.e. the Hirzebruch surface H_1),

4. the equivariant blowing-up of $\mathbb{CP}^2$ at two point,

5. the equivariant blowing-up of $\mathbb{CP}^2$ at three point.

Proof. See [22] Propsition 2.21.

Chapter 4

Tropical Toric Variety

4.1 $K(G, R, M)$

Definition 4.1.1. For all real number x, we define

$$x^+ := max(x, 0),$$

$$x^- := max(-x, 0),$$

and called positive part and negative part of x, respectively.

Remark 4.1.2. The x^+ and x^- are non-negative and $x = x^+ - x^-$.

Definition 4.1.3. For all extended real-valued function f, the positive part

of f is defined by $f^+(x) := \max\{f(x), 0\}$, and the negaive part of f is defined by $f^-(x) := \max\{-f(x), 0\}$. So we have $f = f^+ - f^-$.

Remark 4.1.4. Let $f := (f_1, \cdots, f_n) \in \mathbb{R}^{1 \times n}$, then $f^+ = (f_1^+, \cdots, f_n^+)$, $f^- = (f_1^-, \cdots, f_n^-)$, and $f = f^+ - f^-$.

Proposition 4.1.5. Let $f := (f_1, \cdots, f_n) \in \mathbb{R}^{1 \times n}$ and $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$, then the equations $f \cdot x = 0$ if and only if $f^+ \cdot x = f^- \cdot x$.

Proof. Suppose that $f \cdot x = 0$ where $f := (f_1, \cdots, f_n) \in \mathbb{R}^{1 \times n}$ and $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then $(f_1, \cdots, f_n) \cdot (x_1, \cdots, x_n) = 0$ implies $f_1 \times x_1 + \cdots f_n \times x_n = 0$. Since $f_i = f_i^+ - f_i^-$ for all $i = 1, \cdots, n$, we have $(f_1^+ - f_1^-) \times x_1 + \cdots + (f_n^+ - f_n^-) \times x_n = 0$. Therefore, $f_1^+ \times x_1 + \cdots + f_n^+ \times x_n = f_1^- \times x_1 + \cdots + f_n^- \times x_n$. Hence $f^+ \cdot x = f^- \cdot x$.

Conversely, assume that $f^+ \cdot x = f^- \cdot x$ where $f^+, f^- \in \mathbb{R}^{1 \times n}$ and $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$, then $f_1^+ \times x_1 + \cdots + f_n^+ \times x_n = f_1^- \times x_1 + \cdots + f_n^- \times x_n$. So we have $(f_1^+ - f_1^-) \times x_1 + \cdots + (f_n^+ - f_n^-) \times x_n = 0$. Hence $f \cdot x = 0$ since $f_i = f_i^+ - f_i^-$ for all $i = 1, \cdots, n$ and $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$.

Definition 4.1.6. Let S be a semigroup in $\mathbb{Z}^n$ and let $G = \{g_1, \cdots, g_m\}$ be a finite set of generators of S. Let $R = \{r_1, \cdots, r_k\} \subseteq \mathbb{Z}^m$ generate the integer relation between a set of G, that is, $Span_{\mathbb{Z}}(R) = \{z \in \mathbb{Z}^m \mid g_1 z_1 + \cdots + g_m z_m = 0\}$. Let M be another commutative semigroup. We

define

$$K(G, R, M) \coloneqq \left\{ x \in M^{|G|} \mid r^+ \cdot x = r^- \cdot x \; \forall r \in R \right\}$$

We will discuss about that $r \cdot x = 0$ is different from $r^+ \cdot x = r^- \cdot x$ in the tropical semifield $\mathbb{T}$.

Example 4.1.7. Let $S \subseteq \mathbb{Z}^2$ and let $G = \{(1,1),(4,4)\}$ be the generating set of S. Then $|G| = 2$, and we have $(1,1)z_1 + (4,4)z_2 = 0$ for all $z_1, z_2 \in \mathbb{Z}$. This implies $R = \{(-4,1)\}$ and $Span_{\mathbb{Z}}(R) = \{(z_1, z_2) \in \mathbb{Z}^2 \mid z_1 + 4z_2 = 0\}$ Let $M = \mathbb{T}$ and let $x = (x_1, x_2) \in \mathbb{T}^2$. Then $r \cdot x = 0_{\mathbb{T}}$ where $0_{\mathbb{T}}$ is the tropical additive identity. This implies $(-4 \odot x_1) \oplus (1 \odot x_2) = 0_{\mathbb{T}}$, so $max\{-4 + x_1, 1 + x_2\} = -\infty$. Hence $(x_1, x_2) = (-\infty, -\infty)$. However, $r^+ \cdot x = (0,1) \cdot (x_1, x_2) = (0 \odot x_1) \oplus (1 \odot x_2) = max\{0 + x_1, 1 + x_2\}$. Similarly $r^- \cdot x = max\{4 + x_1, 0 + x_2\}$. So $\{(x_1, x_2) \in \mathbb{T}^2 \mid r^+ \cdot x = r^- \cdot x \; \forall r \in R\} = \{-\infty, x_1 + 3 = x_2\}$. Then $K(G, R, \mathbb{T}) = \{-\infty, x_1 + 3 = x_2\}$ Hence $r \cdot x = 0$ is different from $r^+ \cdot x = r^- \cdot x$ in $\mathbb{T}$.

Proposition 4.1.8. Given $G = \{g_1, \ldots, g_m\}$. Let $R = \{r_1, \cdots, r_k\} \subseteq \mathbb{Z}^m$ Let M be a tropical semifield $\mathbb{T}$. Suppose that $K = K(G, R, \mathbb{T}) = \{x \in \mathbb{T}^m \mid r^+ \cdot x = r^- \cdot x \; \forall r \in R\}$. We define two operations $\oplus_K : K \times K \to K$ and $\otimes_K : \mathbb{T} \times K \to K$ by $x \oplus_K y = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$ and $t \otimes_K x = (t \odot x_1, \ldots, t \odot x_m)$, respectively. Then $(K, \oplus_K, \otimes_K)$ is a tropical vector space.

Proof. Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, and $v = (v_1, \ldots, v_m)$ be in K, and let t, t_1, and t_2 be in $\mathbb{T}$. Given $r = (z_1, \ldots, z_m)$ is in R. Then $r^+ \cdot x = r^- \cdot x$ and $r^+ \cdot y = r^- \cdot y$. We claim that K is closed under an operations $\oplus_K$.

$$
\begin{aligned}
r^+ \cdot (x \oplus_K y) &= (z_1^+, \ldots, z_m^+) \cdot (x_1 \oplus y_1, \ldots, x_m \oplus y_m) \\
&= (z_1^+ \odot (x_1 \oplus y_1)) \oplus \cdots \oplus (z_m^+ \odot (x_m \oplus y_m)) \\
&= (z_1^+ + \max\{x_1, y_1\}) \oplus \cdots \oplus (z_m^+ + \max\{x_m, y_m\}) \\
&= \max\{z_1^+ + x_1, z_1^+ + y_1\} \oplus \cdots \oplus \max\{z_m^+ + x_m, z_m^+ + y_m\} \\
&= ((z_1^+ \odot x_1) \oplus (z_1^+ \odot y_1)) \oplus \cdots \oplus ((z_m^+ \odot x_m) \oplus (z_m^+ \odot y_m)) \\
&= ((z_1^+ \odot x_1) \oplus \cdots \oplus (z_m^+ \odot x_m)) \oplus ((z_1^+ \odot y_1) \oplus \cdots \oplus (z_m^+ \odot y_m)) \\
&\qquad (\text{since } (\mathbb{T}, \oplus) \text{ is a commutative monoid.}) \\
&= ((z_1^+, \ldots, z_m^+) \cdot (x_1, \ldots, x_m)) \oplus ((z_1^+, \ldots, z_m^+) \cdot (y_1, \ldots, y_m)) \\
&= (r^+ \cdot x) \oplus (r^+ \cdot y) \\
&= (r^- \cdot x) \oplus (r^- \cdot y) \\
&= ((z_1^-, \ldots, z_m^-) \cdot (x_1, \ldots, x_m)) \oplus ((z_1^-, \ldots, z_m^-) \cdot (y_1, \ldots, y_m)) \\
&= ((z_1^- \odot x_1) \oplus \cdots \oplus (z_m^- \odot x_m)) \oplus ((z_1^- \odot y_1) \oplus \cdots \oplus (z_m^- \odot y_m)) \\
&= ((z_1^- \odot x_1) \oplus (z_1^- \odot y_1)) \oplus \cdots \oplus ((z_m^- \odot x_m) \oplus (z_m^- \odot y_m)) \\
&\qquad (\text{since } (\mathbb{T}, \oplus) \text{ is a commutative monoid.})
\end{aligned}
$$

$$= \max\{z_1^- + x_1, z_1^- + y_1\} \oplus \cdots \oplus \max\{z_m^- + x_m, z_m^- + y_m\}$$

$$= (z_1^- + \max\{x_1, y_1\}) \oplus \cdots \oplus (z_m^- + \max\{x_m, y_m\})$$

$$= (z_1^- \odot (x_1 \oplus y_1)) \oplus \cdots \oplus (z_m^- \odot (x_m \oplus y_m))$$

$$= (z_1^-, \ldots, z_m^-) \cdot (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$$

$$= r^- \cdot (x \oplus_K y)$$

Hence $x \oplus_K y$ is in K.

We claim that K is closed under an operations $\otimes_K$.

$$r^+ \cdot (t \otimes_K x) = (z_1^+, \ldots, z_m^+) \cdot (t \odot x_1, \ldots, t \odot x_m)$$

$$= (z_1^+ \odot (t \odot x_1)) \oplus \cdots \oplus (z_m^+ \odot (t \odot x_m))$$

$$= ((z_1^+ \odot t) \odot x_1) \oplus \cdots \oplus ((z_m^+ \odot t) \odot x_m) \text{ (since } (\mathbb{T}, \odot) \text{ satisfies associative.)}$$

$$= ((t \odot z_1^+) \odot x_1) \oplus \cdots \oplus ((t \odot z_m^+) \odot x_m) \text{ (since } (\mathbb{T} \setminus \{0_\mathbb{T}\}, \odot) \text{ is abelian.)}$$

$$= (t \odot (z_1^+ \odot x_1)) \oplus \cdots \oplus (t \odot (z_m^+ \odot x_m))$$

$$= t \odot ((z_1^+ \odot x_1) \oplus \cdots \oplus (z_m^+ \odot x_m)) \text{ (since } (\mathbb{T}, \oplus, \odot) \text{ is a semifield.)}$$

$$= t \odot (r^+ \cdot x)$$

$$= t \odot (r^- \cdot x)$$

$$= t \odot ((z_1^- \odot x_1) \oplus \cdots \oplus (z_m^- \odot x_m))$$

$$= (t \odot (z_1^- \odot x_1)) \oplus \cdots \oplus (t \odot (z_m^- \odot x_m)) \text{ (since } (\mathbb{T}, \oplus, \odot) \text{ is a semifield.)}$$

$$= ((t \odot z_1^-) \odot x_1) \oplus \cdots \oplus ((t \odot z_m^-) \odot x_m) \text{ (since } (\mathbb{T} \setminus \{0_{\mathbb{T}}\}, \odot) \text{ is abelian.)}$$

$$= ((z_1^- \odot t) \odot x_1) \oplus \cdots \oplus ((z_m^- \odot t) \odot x_m)$$

$$= (z_1^- \odot (t \odot x_1)) \oplus \cdots \oplus (z_m^- \odot (t \odot x_m)) \text{ (since } (\mathbb{T}, \odot) \text{ satisfies associative.)}$$

$$= (z_1^-, \ldots, z_m^-) \cdot (t \odot x_1, \ldots, t \odot x_m)$$

$$= r^- \cdot (t \otimes_K x)$$

Hence $t \otimes_K x$ is in K.

$$x \oplus_K (y \oplus_K v) = x \oplus_K (y_1 \oplus v_1, \ldots, y_m \oplus v_m)$$

$$= (x_1 \oplus (y_1 \oplus v_1), \ldots, x_m \oplus (y_m \oplus v_m))$$

$$= ((x_1 \oplus y_1) \oplus v_1), \ldots, (x_m \oplus y_m) \oplus v_m)$$

$$= (x_1 \oplus y_1, \ldots, x_m \oplus y_m) \oplus_K v$$

$$= (x \oplus_K y) \oplus_K v$$

$$(0_{\mathbb{T}}, \ldots, 0_{\mathbb{T}}) \oplus_K x = (-\infty, \ldots, -\infty) \oplus_K (x_1, \ldots, x_m)$$

$$= ((-\infty) \oplus x_1, \ldots, (-\infty) \oplus x_m)$$

$$= (x_1, \ldots, x_m)$$

$$(t_1 \odot t_2) \otimes_K x = ((t_1 \odot t_2) \odot x_1, \dots, (t_1 \odot t_2) \odot x_m)$$

$$= (t_1 \odot (t_2 \odot x_1), \dots, t_1(\odot t_2 \odot x_m))$$

$$= t_1 \otimes_K (t_2 \odot x_1, \dots, t_2 \odot x_m)$$

$$= t_1 \otimes_K (t_2 \otimes_K x)$$

$$(t_1 \oplus t_2) \otimes_K x = ((t_1 \oplus t_2) \odot x_1, \dots, (t_1 \oplus t_2) \odot x_m)$$

$$= ((t_1 \odot x_1) \oplus (t_2 \odot x_1), \dots, (t_1 \odot x_m) \oplus (t_2 \odot x_m))$$

$$= (t_1 \odot x_1, \dots, t_1 \odot x_m) \oplus_K (t_2 \odot x_2, \dots, t_1 \odot x_m)$$

$$= (t_1 \otimes_K x) \oplus_K (t_2 \otimes_K x)$$

$$t \otimes_K (x \oplus_K y) = t \otimes_K (x_1 \oplus y_1, \dots, x_m \oplus y_m)$$

$$= (t \odot (x_1 \oplus y_1), \dots, t \odot (x_m \oplus y_m))$$

$$= ((t \odot x_1) \oplus (t \odot y_1), \dots, (t \odot x_m) \oplus (t \odot y_m))$$

$$= (t \odot x_1, \dots, t \oplus x_m) \oplus_K (t \odot y_1, \dots, t \oplus y_m)$$

$$= (t \otimes_K x) \oplus_K (t \otimes_K y)$$

Hence $(K, \oplus_K, \otimes_K)$ is a tropical vector space.

Proposition 4.1.9. If M is a abelian group, then $K(G, R, M)$ is a abelian

group.

Proof. Let M be an abelian group with an operation $* : M \times M \to M$ by $*(a, b) = a * b$. Then $M^{|G|}$ is also an abelian group. Because G is finite set, so we can consider $|G| = k$. We want to show that $K(G, R, M)$ is a subgroup of M^k. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $K(G, R, M)$. We claim that $x * y^{-1} = (x_1 * y_1^{-1}, \ldots, x_k * y_k^{-1})$ in $K(G, R, M)$ where y^{-1} is the inverse for y. Since M is an abelian group, we have $x_i * y_i^{-1}$ in M for all $i = 1, \ldots, k$, thus $x * y^{-1}$ in M^k. Let $r = (z_1, \ldots, z_k)$ in $Span_{\mathbb{Z}} R$. Since M is an abelian, M is a $\mathbb{Z}$-module, thus $(z_i^+ \cdot (x_i * y_i^{-1})) = ((z_i^+ \cdot x_i) * (z_i^+ \cdot y_i^{-1}))$ and $(z_i^- \cdot (x_i * y_i^{-1})) = ((z_i^- \cdot x_i) * (z_i^- \cdot y_i^{-1}))$ for all $i = 1, \ldots, k$. Moreover, $r^+ \cdot x = r^- \cdot x$ and $r^+ \cdot y^{-1} = r^- \cdot y^{-1}$, because $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $K(G, R, M)$. So we have

$$
\begin{aligned}
r^+ \cdot (x * y^{-1}) &= (z_1^+, \ldots, z_k^+) \cdot (x_1 * y_1^{-1}, \ldots, x_k * y_k^{-1}) \\
&= (z_1^+ \cdot (x_1 * y_1^{-1})) * \cdots * (z_k^+ \cdot (x_k * y_k^{-1})) \\
&= ((z_1^+ \cdot x_1) * (z_1^+ \cdot y_1^{-1})) * \cdots * ((z_k^+ \cdot x_k) * (z_k^+ \cdot y_k^{-1})) \\
&= ((z_1^+ \cdot x_1) * \cdots * (z_k^+ \cdot x_k)) * ((z_1^+ \cdot y_1^{-1}) * \cdots * (z_k^+ \cdot y_k^{-1})) \\
&= r^+ \cdot x * r^+ \cdot y^{-1} \\
&= r^- \cdot x * r^- \cdot y^{-1}
\end{aligned}
$$

$$= ((z_1^- \cdot x_1) * \cdots * (z_k^- \cdot x_k)) * ((z_1^- \cdot y_1^{-1}) * \cdots * (z_k^- \cdot y_k^{-1}))$$

$$= ((z_1^- \cdot x_1) * (z_1^- \cdot y_1^{-1})) * \cdots * ((z_k^- \cdot x_k) * (z_k^- \cdot y_k^{-1}))$$

$$= (z_1^- \cdot (x_1 * y_1^{-1})) * \cdots * (z_k^- \cdot (x_k * y_k^{-1}))$$

$$= (z_1^-, \ldots, z_k^+) \cdot (x_1 * y_1^{-1}, \ldots, x_k * y_k^{-1})$$

$$= r^- \cdot (x * y^{-1}).$$

Hence we get $x * y^{-1}$ in $K(G, R, M)$, i.e. $K(G, R, M)$ is a group.

Next we claim that $K(G, R, M)$ is an abelian. Suppose that $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $K(G, R, M)$. Since M is an abelian, $x_i * y_i = y_i * x_i$ for all $i = 1, \ldots, k$. Then

$$\begin{aligned} x * y &= (x_1, \ldots, x_k) * (y_1, \ldots, y_k) \\ &= (x_1 * y_1, \ldots, x_k * y_k) \\ &= (y_1 * x_1, \ldots, y_k * x_k) \\ &= y * x \end{aligned}$$

Hence $K(G, R, M)$ is an abelian group.

By the above proposition, $K(G, R, \mathbb{T} \setminus \{-\infty\})$ is an abelian group since $(\mathbb{T} \setminus \{-\infty\}, \odot)$ is an abelian group.

Example 4.1.10. Let $S \subseteq \mathbb{Z}^2$ and let $G = \{(1,2),(2,4)\}$ be the generating

set of S. Then $|G| = 2$, and we have $(1,2)z_1 + (2,4)z_2 = 0$ for all $z_1, z_2 \in \mathbb{Z}$.

This implies $R = \{r = (-2,1)\}$. So $r^+ = (0,1)$ and $r^- = (2,0)$.

Given an abelian group $M = (\mathbb{Z}_4, +)$. Let $x = (x_1, x_2) \in \mathbb{Z}_4 \times \mathbb{Z}_4$. If $r^+ \cdot x =$

$r^- \cdot x$, then $x_2 = 2x_1$, and so $K(G, R, \mathbb{Z}_4) = \{(x_1, x_2) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \mid x_2 = x_1\}$

is subset of $\mathbb{Z}_4 \times \mathbb{Z}_4$.

We claim that $K(G, R, \mathbb{Z}_4)$ is a subgroup of $\mathbb{Z}_4^2$. Suppose that (x_1, x_2) and

(y_1, y_2) are in $K(G, R, \mathbb{Z}_4)$, then $x_2 = 2x_1$ and $y_2 = 2y_1$. Then $2(x_1 + (-y_1)) =$

$2x_1 + 2(-y_1) = 2x_1 + (-2y_1) = x_2 + (-y_2)$ where $(-y_1, -y_2)$ is the inverse

element of (y_1, y_2), so $(x_1 + (-y_1), x_2 + (-y_2))$ is in $K(G, R, \mathbb{Z}_4)$.

Since $x_i + y_i = y_i + x_i$ for all $i = 1, 2$, $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) =$

$(y_1 + x_1, y_2 + x_2) = (y_1, y_2) + (x_1, x_2)$

Hence $K(G, R, \mathbb{Z}_4)$ is abelian group.

Proposition 4.1.11. If M is a ring, then $K(G, R, M)$ is a M-module.

Proof. Suppose that M is a ring with binary operation $*$, together with

a second binary operation $\otimes$. Because $(M, *, \otimes)$ is a ring, so $(M, *)$ is an

abelian group, thus $K(G, R, M)$ is also an abelian group.

Define a operation $\ominus : K(G, R, M) \times M \to K(G, R, M)$ via $x \ominus m = (x_1 \otimes$

$m, \dots, x_k \otimes m)$. To check that it is well-defined. Suppose that r is in R

and $|G| = k$. Let m, n be in M and let $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k)$ in

$K(G, R, M)$. Then $r^+ \cdot x = r^- \cdot x$.

$$r^+ \cdot (x \ominus m) = (z_1^+, \ldots, z_k^+) \cdot (x_1 \otimes m, \ldots, x_k \otimes m)$$

$$= z_1^+ \otimes (x_1 \otimes m) * \cdots * z_k^+ \otimes (x_k \otimes m)$$

$$= (z_1^+ \otimes x_1) \otimes m * \cdots * (z_k^+ \otimes x_k) \otimes m$$

$$= (z_1^+ \otimes x_1, \ldots, z_k^+ \otimes x_k) \ominus m$$

$$= (r^+ \cdot x) \ominus m$$

$$= (r^- \cdot x) \ominus m$$

$$= (z_1^- \otimes x_1, \ldots, z_k^- \otimes x_k) \ominus m$$

$$= (z_1^- \otimes x_1) \otimes m * \cdots * (z_k^- \otimes x_k) \otimes m$$

$$= z_1^- \otimes (x_1 \otimes m) * \cdots * z_k^- \otimes (x_k \otimes m)$$

$$= (z_1^-, \ldots, z_k^-) \cdot (x_1 \otimes m, \ldots, x_k \otimes m)$$

$$= r^- \cdot (x \ominus m)$$

So $x \ominus m$ is in $K(G, R, M)$.

Suppose that $x = y$ (i.e. $x_i = y_i$ for all $i = 1, \ldots, k$) and $m = n$, then $x \ominus m = (x_1 \otimes m, \ldots, x_k \otimes m) = (y_1 \otimes n, \ldots, y_k \otimes n) = y \ominus n$. Hence it is well-defined.

$$(x * y) \ominus m = (x_1 * y_1, \ldots, x_k * y_k) \ominus m$$

$$= ((x_1 * y_1) \otimes m, \ldots, (x_k * y_k) \otimes m)$$

$$= ((x_1 \otimes m) * (y_1 \otimes m), \ldots, (x_k \otimes m) * (y_k \otimes m))$$

$$= (x_1 \otimes m, \ldots, x_k \otimes m) * (y_1 \otimes m, \ldots, y_k \otimes m)$$

$$= (x \ominus m) * (y \ominus m)$$

$$x \ominus (m * n) = (x_1, \ldots, x_k) \ominus (m * n)$$

$$= (x_1 \otimes (m * n), \ldots, x_k \otimes (m * n))$$

$$= ((x_1 \otimes m) * (x_1 \otimes n), \ldots, (x_k \otimes m) * (x_k \otimes n))$$

$$= (x_1 \otimes m, \ldots, x_k \otimes m) * (x_1 \otimes n, \ldots, x_k \otimes n)$$

$$= (x \ominus m) * (x \ominus n)$$

$$(x \ominus m) \ominus n = (x_1 \otimes m, \ldots, x_k \otimes m) \ominus n$$

$$= ((x_1 \otimes m) \otimes n, \ldots, (x_k \otimes m) \otimes n)$$

$$= (x_1 \otimes (m \otimes n), \ldots, x_k \otimes (m \otimes n))$$

$$= x \ominus (m \otimes n)$$

If M has an identity 1_M, that is $m \otimes 1_M = m$ for all $m \in M$. Then

$$x \ominus 1_M = (x_1 \otimes 1_M, \ldots, x_k \otimes 1_M) = (x_1, \ldots, x_k) = x.$$

Remark 4.1.12. If M is a field, then $K(G, R, M)$ is a vector space.

Example 4.1.13. Let $M = GF(4)$ be a Galois field. Let $S \subseteq \mathbb{Z}^2$ and let $G = \{(1, 2), (4, 8)\}$ be the generating set of S. Then $|G| = 2$, and we have $(1, 2)z_1 + (4, 8)z_2 = 0$ for all $z_1, z_2 \in \mathbb{Z}$. This implies $R = \{r = (-4, 1)\}$. So $r^+ = (0, 1)$ and $r^- = (4, 0)$. If $r^+ \cdot x = r^- \cdot x$ where $x = (x_1, x_2)$ is in $GF(4)^2$, then $x_2 = 4x_1 = 0$, and so $K(G, R, GF(4)) = \{(x_1, x_2) \in GF(4)^2 \mid x_2 = 0\}$ is a subset of $GF(4)^2$. We claim that $K(G, R, GF(4))$ is a vector space, in fact, we just need to show that $K(G, R, GF(4))$ is a subspace of $GF(4)^2$. To start with, it is clearly that $(0, 0)$ is in $K(G, R, GF(4))$. Next, suppose that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $K(G, R, GF(4))$, then $x_2 = 0$ and $y_2 = 0$, then $x_2 + y_2 = 0$, and so $x + y$ is in $K(G, R, GF(4))$. Finally, let c be in $GF(4)$, and let $x = (x_1, x_2)$ be in $K(G, R, GF(4))$, then $x_2 = 0$ and $cx = (cx_1, cx_2)$, then $cx_2 = 0$, and so cx is in $K(G, R, GF(4))$. Hence $K(G, R, GF(4))$ is a subspace of $GF(4)^2$.

Note that the Galois field $GF(4)$ is isomorphic to $GF(2) \big/ (x^2 + x + 1)$, in fact $GF(2) \simeq \mathbb{Z}_2$ since $GF(p) \simeq \mathbb{Z}_p$ for all prime p.

Theorem 4.1.14. Let S be a finitely generated semigroup on $\mathbb{Z}^n$. Let $G = \{g_1, \cdots, g_l\}$ be a set of generaters with relations generated by $R = \{r_1, \cdots, r_k\}$. Let M be an additive semigroup. Then there is a bijection between $hom(S, M)$ and $K(G, R, M)$.

Proof. We define a map $\phi : hom(S, M) \to K(G, R, M)$. Define a function $f : S \to M$ by $f(g_i) = x_i$. We claim that f is well-defined. Suppose that $s = \sum_{i=1}^{l} a_i g_i$ and $t = \sum_{i=1}^{l} b_i g_i$ are in S. Let $s = t$. If $a_i = b_i$ for all $i = 1, \ldots, l$, then

$$f(s) = \sum_{i=1}^{l} a_i x_i = \sum_{i=1}^{l} b_i x_i = f(t).$$

If $a_i = b_i + z_i$ for all $i = 1, \ldots, l$, where $z_1, \ldots, z_l$ are in $Span_{\mathbb{Z}}(R)$. Then

$$f(s) = \sum_{i=1}^{l} a_i x_i = \sum_{i=1}^{l} (b_i + z_i) x_i = \sum_{i=1}^{l} b_i x_i + \sum_{i=1}^{l} z_i x_i = \sum_{i=1}^{l} b_i x_i = f(t).$$

We claim that f is in $hom(S, M)$, i.e. to show that f is a semigroup homomorphism. Let s and t are in S, then $s = \sum_{i=1}^{l} a_i g_i$ and $t = \sum_{i=1}^{l} b_i g_i$. So $s + t = \sum_{i=1}^{l} (a_i + b_i) g_i$. This implies $f(s + t) = \sum_{i=1}^{l} (a_i + b_i) x_i = \sum_{i=1}^{l} a_i x_i + \sum_{i=1}^{l} b_i x_i = f(s) + f(t)$. Hence f is in $hom(S, M)$.

Since $R = \{r_1, \cdots, r_k\} \subseteq \mathbb{Z}^m$ generates the integer relation between a set of G, for any $r = (z_1, \ldots, z_l)$ in R, then $\sum_{i=1}^{l} z_i^+ g_i = \sum_{i=1}^{l} z_i^- g_i$. Because f is in $hom(S, M)$, so $r^+ \cdot (x_1, \ldots, x_l) = \sum_{i=1}^{l} z_i^+ x_i = \sum_{i=1}^{l} z_i^+ f(g_i) = f(\sum_{i=1}^{l} z_i^+ g_i) = f(\sum_{i=1}^{l} z_i^- g_i) = \sum_{i=1}^{l} z_i^- f(g_i) = r^- \cdot (x_1, \ldots, x_l)$. So $(x_1, \ldots, x_l)$ is in $K(G, R, M)$.

Define a function $\psi : K(G, R, M) \to hom(S, M)$ by $\psi(x) = f$ for all x in

$K(G, R, M)$. To check that ϕ is surjective, and ψ is injective, i.e. $\phi \circ \psi$ identity function.

$$\phi \circ \psi(x) = \phi(\psi(x)) = (f(g_1), \ldots, f(g_l)) = (x_1, \ldots, x_l) = x.$$

Proposition 4.1.15. Let S be a finitely generated semigroup on $\mathbb{Z}^n$. Let $G = \{g_1, \cdots, g_l\}$ be a set of generaters with relations generated by $R = \{r_1, \cdots, r_k\}$. Then there are bijective between $Spec(\mathbb{C}[S])$, $hom(S, \mathbb{C})$ and $K(G, R, \mathbb{C})$.

Proof. The correspondence between $Spec(\mathbb{C}[S_\sigma])$ and $hom(S, \mathbb{C})$ is immediate from Proposition 3.3.11. The correspondence between $hom(S, \mathbb{C})$ and $K(G, R, \mathbb{C})$ is immediate from the Theorem 4.1.14.

Theorem 4.1.16. Let S be a finitely generated semigroup on $\mathbb{Z}^n$. Let $G = \{g_1, \cdots, g_l\}$ and $H = \{h_1, \cdots, h_d\}$ be the different sets of the generators of S with relations generated by $R = \{r_1, \cdots, r_k\}$ and $P = \{p_1, \cdots, p_m\}$, respectively. Then there is a linear isomorphism $\phi : K(G, R, \mathbb{R}) \to K(H, P, \mathbb{R})$.

Proof. Since $G = \{g_1, \cdots, g_l\}$ and $H = \{h_1, \cdots, h_d\}$ are the different sets of the generators of S, let $g_i = \sum_{j=1}^{d} \lambda_{ij} h_j$ with nonnegative integer λ_{ij} for all $i = 1, \ldots, l$, and let $h_i = \sum_{i=1}^{l} \mu_{ji} g_i$ with nonnegative integer μ_{ji} for all $j = 1, \ldots, d$. Since $hom(S, \mathbb{R})$ is in bijection with $K(G, R, \mathbb{R})$, let $x_i = f(g_i)$

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for all $i = 1, \ldots, l$ where f is in $hom(S, \mathbb{R})$. Then $\sum_{i=1}^{l} z_i^+ x_i = \sum_{i=1}^{l} z_i^- x_i$

for all $(z_1, \ldots, z_l)$ is in $Span_{\mathbb{Z}} R$ since $(x_1, \ldots, x_l)$ is in $K(G, R, \mathbb{R})$. Let $y_j = \sum_{i=1}^{l} \mu_{ji} x_i$ for all $j = 1, \ldots, d$. We claim that

$$\sum_{j=1}^{d} t_j^+ y_j = \sum_{j=1}^{d} t_j^- y_j$$

for all $(t_1, \ldots, t_d)$ is in $Span_{\mathbb{Z}} P = \{(t_i, \ldots, t_d) \in \mathbb{Z}^d \mid \sum_{j=1}^{d} t_j^+ h_j = \sum_{j=1}^{d} t_j^- h_j\}$.

Because we have $h_i = \sum_{i=1}^{l} \mu_{ji} g_i$ with nonnegative integer μ_{ji} for all $j = 1, \ldots, d$, so this implies

$$\begin{aligned}
\sum_{j=1}^{d} t_j^+ h_j &= \sum_{j=1}^{d} t_j^+ (\sum_{i=1}^{l} \mu_{ji} g_i) \\
&= \sum_{j=1}^{d} (\sum_{i=1}^{l} t_j^+ \mu_{ji}) g_i \\
&= \sum_{j=1}^{d} (\sum_{i=1}^{l} t_j^- \mu_{ji}) g_i \\
&= \sum_{j=1}^{d} t_j^- h_j.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{d} t_j^+ y_j &= \sum_{j=1}^{d} t_j^+ (\sum_{i=1}^{l} \mu_{ji} x_i) \\
&= \sum_{j=1}^{d} (\sum_{i=1}^{l} t_j^+ \mu_{ji}) x_i
\end{aligned}$$

$$= \sum_{j=1}^{d}(\sum_{i=1}^{l} t_j^- \mu_{ji})x_i$$

$$= \sum_{j=1}^{d} t_j^- y_j.$$

Define a function $\phi : K(G,R,\mathbb{R}) \to K(H,P,\mathbb{R})$ via $\phi(x_1,\ldots,x_l) = (y_1,\ldots,y_d) = (\sum_{i=1}^{l} \mu_{1i}x_i,\ldots,\sum_{i=1}^{l} \mu_{di}x_i)$. Then the function is well-defined by the above discussion.

Because $\sum_{i=1}^{l} \mu_{1i}x_i,\ldots,$ and $\sum_{i=1}^{l} \mu_{di}x_i$ are linear , ϕ is linear map.

Example 4.1.17. Let $G = \{(1,1),(4,4)\}$ and $S = \{c(1,1) \mid c \in \mathbb{Z}_{\geq 0}\}$. By example 4.1.7, we have $R = \{(-4,1)\}$. Thus, $K(G,R,\mathbb{R}) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = 4x_1\}$. If $H = \{(1,1)\}$, then $Span_{\mathbb{Z}}P = \{x \in \mathbb{Z} \mid x^+ \cdot (1,1) = x^- \cdot (1,1)\} = \{0\}$, this implies $P =$. So we have

$$K(H,P,\mathbb{R}) = \{y \in \mathbb{R} \mid x^+ \cdot y = x^- \cdot y, x \in P\} = \mathbb{R}.$$

Then we can obtain a linear isomorphism $\phi : K(G,R,\mathbb{R}) \to K(H,P,\mathbb{R})$ via $\phi(x_1,4x_1) = x_1$.

Example 4.1.18. Let $G = \{(1,1),(4,4)\}$ and $S = \{c(1,1) \mid c \in \mathbb{Z}_{\geq 0}\}$. By example 4.1.7, we have $R = \{(-4,1)\}$. Thus, $K(G,R,\mathbb{R}) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = 4x_1\}$. If $H = \{(5,0),(2,1),(1,-2)\}$, then $Span_{\mathbb{Z}}P = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 \mid x_1(5,1)+x_2(2,1)+x_3(1,-2) = 0\} = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 \mid x_2 = -2x_1, x_3 =$

$-x_1\}$, thus $S = \{(1, -2, -1)\}$. So we have

$$K(H, P, \mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (1, 0, 0) \cdot (x_1, x_2, x_3) = (0, 2, 1) \cdot (x_1, x_2, x_3)\}$$

$$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 2x_2 + x_3\}.$$

Then the function $\phi : K(G, R, \mathbb{R}) \to K(H, P, \mathbb{R})$ via $\phi(x_1, x_2) = (x_1 + 2x_2, x_2, x_1)$ is a linear isomophism.

4.2 Tropical Toric Variety

Definition 4.2.1. Let $\mathbb{T}$ be a tropical semifield. Let σ be a rational polyhedral cone with the semigroup S_σ. Then the affine toric variety is $U_\sigma := hom(S_\sigma, \mathbb{T})$ where $hom(S_\sigma, \mathbb{T})$ is the semigroup homomorphisms $S_\sigma \to \mathbb{T}$.

Note that we set $\mathbb{T}^* = \mathbb{T} \setminus \{-\infty\}$.

Example 4.2.2. Let $N \simeq \mathbb{Z}^2$ be a lattice with associated vector space $N_\mathbb{R} \simeq \mathbb{R}^2$, and let M be a dual lattice of N with associated vector space $M_\mathbb{R} \simeq \mathbb{R}^2$. Given a cone $\sigma = pos\{0\}$ in $N_\mathbb{R}$, then its dual cone $\{0\}^\vee = pos\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$, then $S_{\{0\}} = \{0\}^\vee \cap M = \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(0, -1)$. Because $U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T})$. Define a homomorphism $f : S_{\{0\}} \to \mathbb{T}$ by $f(1, 0) = x_1$, $f(-1, 0) = x_2$, $f(0, 1) = x_3$, and

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$f(0, -1) = x_4$. Then $x_1 + x_2 = 0$ and $x_3 + x_4 = 0$, and so x_1, x_2, x_3, and x_4 don't equal $-\infty$. So $U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) \simeq (\mathbb{T}^*)^2$. Hence $(\mathbb{T}^*)^2$ is the 2-dimensional algebraic torus over $\mathbb{T}$.

Example 4.2.3. Let $\sigma = pos\{(0, -1)\}$ in $N_{\mathbb{R}} \simeq \mathbb{R}^2$, then the dual cone $\sigma^{\vee} = pos\{(1, 0), (-1, 0), (0, -1)\}$, and the corresponding semigroup $S_{\sigma} = \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(0, -1)$. Let f be in $hom(S_{\sigma}, \mathbb{T})$. Suppose that $f(1, 0) = x_1$, $f(-1, 0) = x_2$, and $f(0, -1) = x_3$, then $0 = f(0, 0) = f(0, 1) + f(0, -1) = f(0, 1) \otimes f(0, -1) = x_1 \otimes x_2 = x_1 + x_2$. Hence the corresponding affine toric variety is $U_{\sigma} = hom(S_{\sigma}, \mathbb{T}) = \mathbb{R} \times \mathbb{R}$.

Proposition 4.2.4. If $\tau \subset \sigma$ is a face of a cone σ, then we obtain the embedding $hom(S_{\tau}, \mathbb{T}) = U_{\tau} \hookrightarrow U_{\sigma} = hom(S_{\sigma}, \mathbb{T})$.

Proof. Because $\tau \subset \sigma$ is a face of a cone σ, so $\sigma^{\vee}$ is a subset of $\tau^{\vee}$. Since $S_{\sigma} = \sigma^{\vee} \cap M$ is a subset of $S_{\tau} = \tau^{\vee} \cap M$, we have the embedding $S_{\sigma} \hookrightarrow S_{\tau}$. Because $U_{\tau} = hom(S_{\tau}, \mathbb{T})$ and $U_{\sigma} = hom(S_{\sigma}, \mathbb{T})$, so $hom(S_{\tau}, \mathbb{T}) = U_{\tau} \hookrightarrow U_{\sigma} = hom(S_{\sigma}, \mathbb{T})$. Note that we have the following commutative diagram:

$$S_{\sigma} \overset{h}{\hookrightarrow} S_{\tau} \overset{f}{\longrightarrow} \mathbb{T}$$
$$\underset{f \circ h}{\longrightarrow}$$

Definition 4.2.5. Let F be a rational fan. Then the tropical toric variety

$\mathbb{X}_F(\mathbb{T})$ is defined as the quotient

$$\mathbb{X}_F(\mathbb{T}) := \coprod_{\sigma \in F} U_\sigma / \sim$$

Let τ and σ be in F. Suppose that A and B are in U_τ and U_σ respectively. Since $\tau \cap \sigma$ is a face of τ and σ (because F is a fan), we have two embedding $h_1 : U_{\tau \cap \sigma} \hookrightarrow U_\tau$ and $h_2 : U_{\tau \cap \sigma} \hookrightarrow U_\sigma$. $A \sim B$ if and only if $h_1 \circ h_2^{-1}(B) = A$. Note that we have the following commutative diagram:

$$
\begin{array}{ccc}
U_{\tau \cap \sigma} & \overset{h_2}{\hookrightarrow} & U_\sigma \\
{\scriptstyle h_1} \downarrow & \swarrow {\scriptstyle h_1 \circ h_2^{-1}} & \\
U_\tau & &
\end{array}
$$

Example 4.2.6. Let $F = \{\{0\} = \sigma_0, \mathbb{R}_{\geq 0} = pos(1) = \sigma_1\}$ be in $N_\mathbb{R} \simeq \mathbb{R}$. The dual cone $\sigma_0^\vee = \{v \in \mathbb{R} \mid v \cdot 0 \geq 0\} = \mathbb{R}$. So the semigroup $S_{\sigma_0} = \sigma_0^\vee \cap M = \mathbb{Z}_{\geq 0}(1) \oplus \mathbb{Z}_{\geq 0}(-1)$, and thus the affine toric variety $U_{\sigma_0} = hom(S_{\sigma_0}, \mathbb{T}) = \mathbb{R}$.

The dual cone $\sigma_1^\vee = \{v \in \mathbb{R} \mid v \cdot u \geq 0, \forall u \in \sigma_1\} = pos\{1\} \simeq \mathbb{R}_{\geq 0}$. So the semigroup $S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}$. Since we take $f(0) = 0$ and $f(1) = x_1$ where f is in $hom(S_{\sigma_1}, \mathbb{T})$, the affine toric variety $U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) \simeq \mathbb{T}$.

Hence $\mathbb{X}_F(\mathbb{T}) = (U_{\sigma_0} \coprod U_{\sigma_1})/\sim = (\mathbb{R} \coprod \mathbb{T})/\sim = \mathbb{T}$

Definition 4.2.7. Let $\mathbb{X}_F(\mathbb{T})$ be a tropical toric variety. A point p is said to be regular or smooth in $\mathbb{X}_F(\mathbb{T})$ if there is a unimodular cone σ in F such that p is in U_σ. A tropical toric variety $\mathbb{X}_F(\mathbb{T})$ is called regular or smooth if all points of $\mathbb{X}_F(\mathbb{T})$ are regular or smooth. A tropical toric variety $\mathbb{X}_F(\mathbb{T})$ is called singular if $\mathbb{X}_F(\mathbb{T})$ is not regular.

Definition 4.2.8. Let F be fan in $N_\mathbb{R}$. A set Δ is called a subfan of F if Δ is a subset of F and is also a fan.

Definition 4.2.9. Let the lattice $N = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, and let another lattice $N' = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m$. Let F be a fan in $N_\mathbb{R}$, and let Δ be another fan in N'. Let $\phi : N_\mathbb{R} \to N'_\mathbb{R}$ be a linear map such that $\phi(N) \subseteq N'$ and there is a cone σ' in Δ containing $\phi(\sigma)$ for all cone σ in F. Then the map $\phi : F \to \Delta$ is said a map of fans.

Theorem 4.2.10. Let $N \simeq \mathbb{Z}^n$ and $N' \simeq \mathbb{Z}^m$ be two different lattice, and let $M = hom(N, \mathbb{Z})$ and $M' = hom(N', \mathbb{Z})$ be the dual lattice. Let F be a fan in $N_\mathbb{R}$ and let Δ be another fan in $N'_\mathbb{R}$. Let $\phi : F \to \Delta$ be a map of fans. Then ϕ extends to a continuous map $\phi : \mathbb{X}_F(\mathbb{T}) \to \mathbb{X}_\Delta(\mathbb{T})$.

Proof. Since $\phi : F \to \Delta$ is a map of fans, there is a cone σ' in Δ such that $\phi(\sigma) \subset \sigma'$. Let u be in $S_{\sigma'}$. Then u is in $hom(N', \mathbb{Z})$ and is in $\sigma'^\vee$, and thus we have the map $u : N' \to \mathbb{Z}$ via $u(n') = < u, n' >$. Since $\phi : F \to \Delta$ is a

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map of fans, $\phi(N) \subset N' \xrightarrow{u} \mathbb{Z}$. Let $\mu : N \to \mathbb{Z}$ via $\mu(n) =< u, \phi(n) >$,

then μ is in $hom(N, \mathbb{Z})$. Since $\phi(\sigma)$ is a subset of σ', $\phi(n)$ is in σ' for all n in

σ. Since u is in $\sigma'^{\vee}$, $< u, \phi(n) >\geq 0$, and this implies μ is in $\sigma^{\vee}$. So μ is in

$\sigma^{\vee} \cap hom(N, \mathbb{Z}) = S_{\sigma}$. Therefore, we obtain a map $S_{\sigma'} \to S_{\sigma}$, and thus have

a map $U_{\sigma} \to U_{\sigma'}$. And we have the following commutative diagram:

$$U_{\sigma_1} \hookleftarrow U_{\sigma_1 \cap \sigma_2} \hookrightarrow U_{\sigma_2}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$U_{\sigma_1'} \hookleftarrow U_{\sigma_1' \cap \sigma_2'} \hookrightarrow U_{\sigma_2'}$$

Hence $\mathbb{X}_F(\mathbb{T}) \to \mathbb{X}_\Delta(\mathbb{T})$.

Definition 4.2.11. For all r in $\mathbb{Z}$, we define the tropical Hirzebruch surface

H_r to be

$$\mathbb{T}H_r := \{([x_0 : x_1], [y_0 : y_1 : y_2]) \in \mathbb{T}\mathbb{P}^1 \times \mathbb{T}\mathbb{P}^2 \mid rx_0 + y_0 = rx_1 + y_1\}.$$

For more information on tropical Hirzebruch surfaces, see [3] § 2.
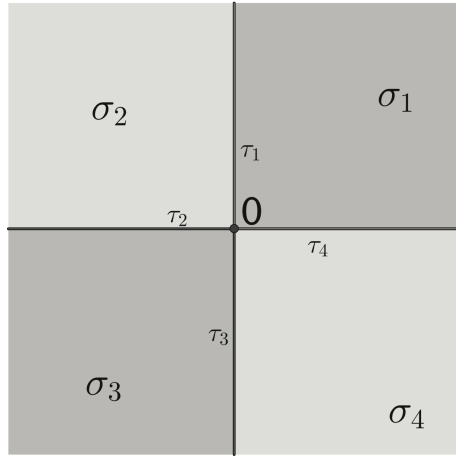
Figure 4.1: the fan Δ
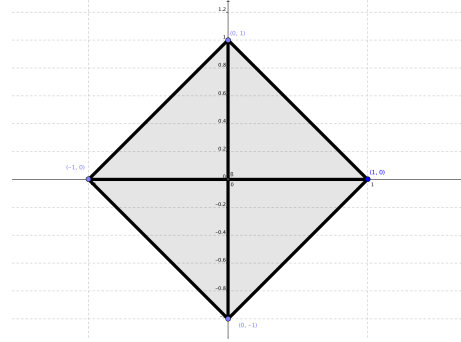
Figure 4.2: the polytope P

4.3 Smooth two-dimensional tropical toric Fano varieties

In this section, we know that there are only five smooth Fano polytopes in $\mathbb{R}^2$ up to the action of $GL(2, \mathbb{Z})$, so I will calculate these cases of smooth two-dimensional tropical toric Fano varieties.

Example 4.3.1. Given the lattice $N = \mathbb{Z}(1, 0) \oplus \mathbb{Z}(0, 1) \simeq \mathbb{Z}^2$, then $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the dual lattice $M \simeq \mathbb{Z}^2$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Let the fan Δ in $N_{\mathbb{R}}$. Suppose that the fan Δ (the figure 4.1) has

$$\sigma_1 = pos\{(1, 0), (0, 1)\}, \ \sigma_2 = pos\{(-1, 0), (0, 1)\},$$

$$\sigma_3 = pos\{(-1, 0), (0, -1)\}, \ \sigma_4 = pos\{(1, 0), (0, -1)\},$$

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(0,1)\}, \ \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(-1,0)\},$$

$$\tau_3 = \sigma_3 \cap \sigma_4 = pos\{(0,-1)\}, \ \tau_4 = \sigma_4 \cap \sigma_1 = pos\{(1,0)\},$$

and the origin. Then the dual cones

$$\sigma_1^\vee = pos\{(1,0),(0,1)\}, \ \sigma_2^\vee = pos\{(-1,0),(0,1)\},$$

$$\sigma_3^\vee = pos\{(-1,0),(0,-1)\}, \ \sigma_4^\vee = pos\{(1,0),(0,-1)\}.$$

Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\sigma_2} = \sigma_2^\vee \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\sigma_3} = \sigma_3^\vee \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\sigma_4} = \sigma_4^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1) \oplus \mathbb{Z}_{\geq 0}(-1,0),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_4} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\tau_4} = S_{\sigma_4} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1) \oplus \mathbb{Z}_{\geq 0}(1,0),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1).$$

Let f_i be in $U_{\sigma_i} = hom(S_{\sigma_i}, \mathbb{T})$ for all $i = 1, 2, 3, 4$, then we have some

maps

$$f_1 : S_{\sigma_1} \to \mathbb{T} \text{ via } f_1(1,0) = x \text{ and } f_1(0,1) = y,$$

$$f_2 : S_{\sigma_2} \to \mathbb{T} \text{ via } f_2(-1,0) = -x \text{ and } f_2(0,1) = y,$$

$$f_3 : S_{\sigma_3} \to \mathbb{T} \text{ via } f_3(-1,0) = -x \text{ and } f_3(0,-1) = -y,$$

$$f_4 : S_{\sigma_4} \to \mathbb{T} \text{ via } f_4(1,0) = x \text{ and } f_4(0,-1) = -y.$$

Therefore, the affine toric varieties

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{T}) = \mathbb{T}^2,$$

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$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_4} = hom(S_{\sigma_4}, \mathbb{T}) = \mathbb{T}^2,$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_4} = hom(S_{\tau_4}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) = \mathbb{R}^2.$$

The gluing of the affine toric varieties U_{σ_1} and U_{σ_2} along their common subset U_{τ_1} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $((x_0 : x_1), y)$ where $x = x_1 - x_0$. The gluing of the affine toric varieties U_{σ_2} and U_{σ_3} along their common subset U_{τ_2} gives $\mathbb{T} \times \mathbb{TP}^1$ with coordinates $(-x, (y_0 : y_1))$ where $y = y_1 - y_0$. The gluing of the affine toric varieties U_{σ_3} and U_{σ_4} along their common subset U_{τ_3} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $((x_0 : x_1), -y)$ where $x = x_1 - x_0$. The gluing of the affine toric varieties U_{σ_4} and U_{σ_1} along their common subset U_{τ_4} gives $\mathbb{T} \times \mathbb{TP}^1$ with coordinates $(x, (y_0 : y_1))$ where $y = y_1 - y_0$. The

following commutative diagram:

$$U_{\sigma_1} \longleftarrow U_{\tau_1} \longhookrightarrow U_{\sigma_2}$$

$$U_{\tau_4} \qquad\qquad U_{\tau_2}$$

$$U_{\sigma_4} \longleftarrow U_{\tau_3} \longhookrightarrow U_{\sigma_3}$$

Hence the gluing of these two gives the tropical toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = (\coprod_{\sigma \in \Delta} U_\sigma)/\sim = \mathbb{TP}^1 \times \mathbb{TP}^1.$$

The polytopes $P = conv\{0, (1,0), (0,1), (-1,0), (0,-1)\}$ (the figure 4.2)
in $N_\mathbb{R}$. Since $conv\{(1,0), (0,1)\}$, $conv\{(0,1), (-1,0)\}$, $conv\{(-1,0), (0,-1)\}$,
$conv\{(0,-1), (1,0)\}$ are the facets of P, and they are the convex hull of a
basis of N, X_P is a smooth Fano polytope (by Theorem 3.4.2).

Example 4.3.2. Given the lattice $N \simeq \mathbb{Z}^2$, then $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the
dual lattice $M \simeq \mathbb{Z}^2$ and $M_\mathbb{R} = M \otimes \mathbb{R}$.

Let the fan Δ in $N_\mathbb{R}$. Suppose that the fan Δ (the figure 4.3) has

$\sigma_1 = pos\{(1,0), (0,1)\}$, $\sigma_2 = pos\{(-1,-1), (0,1)\}$, $\sigma_3 = pos\{(1,0), (-1,-1)\}$,
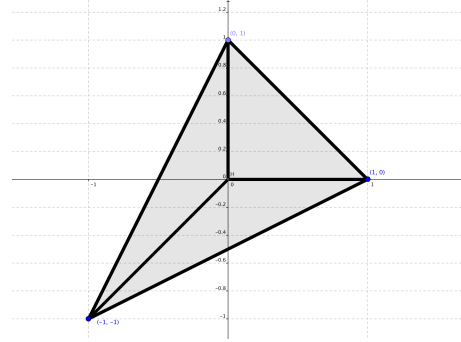
Figure 4.4: the polytope P

Figure 4.3: the fan Δ

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(0,1)\}, \ \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(-1,-1)\},$$

$$\tau_3 = \sigma_3 \cap \sigma_1 = pos\{(1,0)\}, \text{ and the origin.}$$

Then the dual cones

$$\sigma_1^\vee = pos\{(1,0),(0,1)\}, \ \sigma_2^\vee = pos\{(-1,0),(-1,1)\}, \ \sigma_3^\vee = pos\{(1,-1),(0,-1)\}.$$

Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\sigma_2} = \sigma_2^\vee \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\sigma_3} = \sigma_3^\vee \cap M = \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1) \oplus \mathbb{Z}_{\geq 0}(-1,-1),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1).$$

Let f_i be in $U_{\sigma_i} = hom(S_{\sigma_i}, \mathbb{T})$ for all $i = 1,2,3$, then we have some maps

$$f_1 : S_{\sigma_1} \to \mathbb{T} \text{ via } f_1(1,0) = x \text{ and } f_1(0,1) = y,$$

$$f_2 : S_{\sigma_2} \to \mathbb{T} \text{ via } f_2(-1,0) = -x \text{ and } f_2(-1,1) = -x + y,$$

$$f_3 : S_{\sigma_3} \to \mathbb{T} \text{ via } f_3(1,-1) = x - y \text{ and } f_3(0,-1) = -y,$$

Therefore, the affine toric variety

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{T}) = \mathbb{T}^2,$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) = \mathbb{R}^2.$$

The gluing of the affine toric varieties U_{σ_1} and U_{σ_2} along their common subset U_{τ_1} gives $\mathbb{TP}^2$ with coordinates $(x_0 : x_1 : x_2)$ where $x = x_1 - x_0$ and $y = x_2 - x_0$. The gluing of the affine toric varieties U_{σ_2} and U_{σ_3} along their common subset U_{τ_2} gives $\mathbb{TP}^2$ with coordinates $(x_0 : x_1 : x_2)$ where $-x + y = x_1 - x_0$ and $-x = x_2 - x_0$. The gluing of the affine toric varieties U_{σ_3} and U_{σ_4} along their common subset U_{τ_3} gives $\mathbb{TP}^2$ with coordinates $(x_0 : x_1 : x_2)$ where $y = x_1 - x_0$ and $x = x_2 - x_0$.

The following commutative diagram:

$$
\begin{array}{ccc}
U_{\sigma_1} \longleftarrow U_{\tau_1} \longrightarrow U_{\sigma_2} \\
\nwarrow \qquad \qquad \uparrow \\
U_{\tau_3} \qquad U_{\tau_2} \\
\searrow \qquad \downarrow \\
U_{\sigma_3}
\end{array}
$$

Hence the gluing of these two gives the tropical toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = (\coprod_{\sigma \in \Delta} U_\sigma)/\sim = \mathbb{T}\mathbb{P}^2.$$

The polytopes $P = conv\{0, (1,0), (0,1), (-1,-1)\}$ (the figure 4.4) in $N_\mathbb{R}$.Since $conv\{(1,0),(0,1)\}$, $conv\{(0,1),(-1,-1)\}$, $conv\{(-1,-1),(1,0)\}$ are the facets of P, and they are the convex hull of a basis of N, X_P is a smooth Fano polytope (by Theorem 3.4.2).

Example 4.3.3. Given the lattice $N \simeq \mathbb{Z}^2$, then $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the dual lattice $M \simeq \mathbb{Z}^2$ and $M_\mathbb{R} = M \otimes \mathbb{R}$.

Let the fan Δ in $N_\mathbb{R}$. Suppose that the fan Δ (the figure 4.5) has

$$\sigma_1 = pos\{(1,0),(1,1)\}, \ \sigma_2 = pos\{(1,1),(0,1)\},$$

$$\sigma_3 = pos\{(0,1),(-1,-1)\}, \ \sigma_4 = pos\{(-1,-1),(1,0)\},$$

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(1,1)\}, \ \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(0,1)\},$$

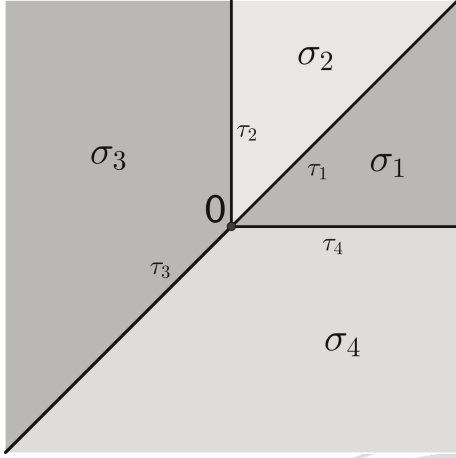$$\tau_3 = \sigma_3 \cap \sigma_4 = pos\{(-1,-1)\}, \ \tau_4 = \sigma_4 \cap \sigma_1 = pos\{(1,0)\},$$

Figure 4.6: the polytope P

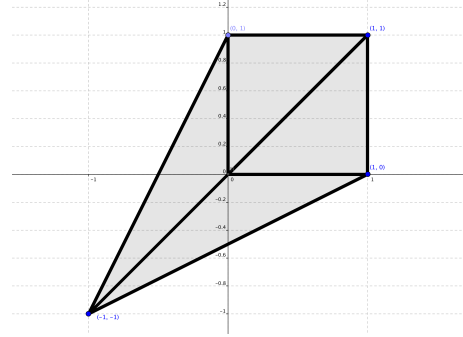Figure 4.5: the fan Δ

and the origin. Then the dual cones

$$\sigma_1^\vee = pos\{(-1,-1),(0,1)\},\ \sigma_2^\vee = pos\{(1,0),(-1,1)\},$$

$$\sigma_3^\vee = pos\{(-1,0),(-1,1)\},\ \sigma_4^\vee = pos\{(0,-1),(1,-1)\}.$$

Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(1,-1),$$

$$S_{\sigma_2} = \sigma_2^\vee \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\sigma_3} = \sigma_3^\vee \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\sigma_4} = \sigma_4^\vee \cap M = \mathbb{Z}_{\geq 0}(0,-1) \oplus \mathbb{Z}_{\geq 0}(1,-1),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1,1) \oplus \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_4} = \mathbb{Z}_{\geq 0}(-1,-1) \oplus \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\tau_4} = S_{\sigma_4} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1).$$

Let f_i be in $U_{\sigma_i} = hom(S_{\sigma_i}, \mathbb{T})$ for all $i = 1, 2, 3, 4$, then we have some maps

$$f_1 : S_{\sigma_1} \to \mathbb{T} \text{ via } f_1(0,1) = y \text{ and } f_1(1,-1) = x - y,$$

$$f_2 : S_{\sigma_2} \to \mathbb{T} \text{ via } f_2(1,0) = x \text{ and } f_2(-1,1) = -x + y,$$

$$f_3 : S_{\sigma_3} \to \mathbb{T} \text{ via } f_3(-1,0) = -x \text{ and } f_3(-1,1) = -x + y,$$

$$f_4 : S_{\sigma_4} \to \mathbb{T} \text{ via } f_4(0,-1) = -y \text{ and } f_4(1,-1) = x - y.$$

Therefore, the affine toric variety

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_4} = hom(S_{\sigma_4}, \mathbb{T}) = \mathbb{T}^2,$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_4} = hom(S_{\tau_4}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) = \mathbb{R}^2.$$

The gluing of the affine toric varieties U_{σ_2} and U_{σ_3} along their common subset U_{τ_2} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $((x_0 : x_1), -x + y)$ where $x = x_0 - x_1$. The gluing of the affine toric varieties U_{σ_4} and U_{σ_1} along their common subset U_{τ_4} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $((y_0 : y_1), x - y)$ where $y = y_0 - y_1$.
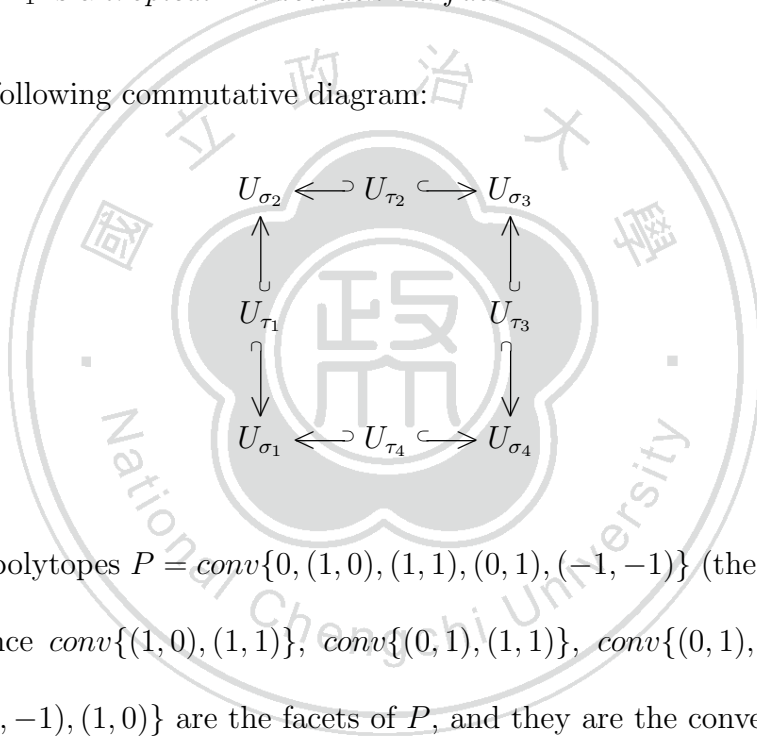
The two copies of $\mathbb{TP}^1 \times \mathbb{T}$ are glued along their second coordinates gives $\mathbb{TP}^1 \times \mathbb{TP}^2$ with coordinates $([z_0 : z_1], [x_0 - x_1 : y_0 - y_1 : 0]) = ([z_0 : z_1], [x_0 - x_1 : y_0 - y_1 : 0])$ where $x - y = z_0 - z_1$. Since $z_0 - z_1 = (x_0 - x_1) - (y_0 - y_1)$, $(x_0 - x_1) - z_0 = (y_0 - y_1) - z_1$. Hence the toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = \left(\coprod_{\sigma \in \Delta} U_\sigma \right) \Big/ {\sim}$$

$$= \{ ([z_0 : z_1], [x_0 - x_1 : y_0 - y_1 : 0]) \in \mathbb{TP}^1 \times \mathbb{TP}^2 \mid (x_0 - x_1) - z_0 =$$

$$(y_0 - y_1) - z_1\},$$

$$= \{([z_0 : z_1], [x_0 + y_1 : y_0 + x_1 : x_1 + y_1]) \in \mathbb{TP}^1 \times \mathbb{TP}^2 \mid (x_0 - x_1) -$$

$$z_0 = (y_0 - y_1) - z_1\},$$

$$= \mathbb{T}H_{-1}$$

where $\mathbb{T}H_{-1}$ is a *tropical Hirzebruch surface*.

The following commutative diagram:

$$\begin{array}{ccc} U_{\sigma_2} \longleftarrow U_{\tau_2} \longhookrightarrow U_{\sigma_3} \\ \uparrow \qquad\qquad\qquad \uparrow \\ U_{\tau_1} \qquad\qquad\qquad U_{\tau_3} \\ \downarrow \qquad\qquad\qquad \downarrow \\ U_{\sigma_1} \longleftarrow U_{\tau_4} \longhookrightarrow U_{\sigma_4} \end{array}$$

The polytopes $P = conv\{0, (1, 0), (1, 1), (0, 1), (-1, -1)\}$ (the figure 4.6) in $N_{\mathbb{R}}$. Since $conv\{(1, 0), (1, 1)\}$, $conv\{(0, 1), (1, 1)\}$, $conv\{(0, 1), (-1, -1)\}$, $conv\{(-1, -1), (1, 0)\}$ are the facets of P, and they are the convex hull of a basis of N, X_P is a smooth Fano polytope (by Theorem 3.4.2).

Example 4.3.4. Given the lattice $N \simeq \mathbb{Z}^2$, then $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the dual lattice $M \simeq \mathbb{Z}^2$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Let the fan Δ in $N_{\mathbb{R}}$. Suppose that the fan Δ (the figure 4.7) has

$$\sigma_1 = pos\{(1,0),(1,1)\}, \;\; \sigma_2 = pos\{(1,1),(0,1)\},$$

$$\sigma_3 = pos\{(0,1),(-1,-1)\}, \;\; \sigma_4 = pos\{(-1,-1),(0,-1)\},$$

$$\sigma_5 = pos\{(0,-1),(1,0)\},$$

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(1,1)\}, \;\; \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(0,1)\},$$

$$\tau_3 = \sigma_3 \cap \sigma_4 = pos\{(-1,-1)\}, \;\; \tau_4 = \sigma_4 \cap \sigma_5 = pos\{(0,-1)\},$$

$$\tau_5 = \sigma_5 \cap \sigma_1 = pos\{(1,0)\},$$

and the origin. Then the dual cones

$$\sigma_1^\vee = pos\{(1,-1),(0,1)\}, \;\; \sigma_2^\vee = pos\{(1,0),(-1,1)\},$$

$$\sigma_3^\vee = pos\{(-1,0),(-1,1)\}, \;\; \sigma_4^\vee = pos\{(-1,0),(1,-1)\},$$

$$\sigma_5^\vee = pos\{(0,-1),(1,0)\}.$$

Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^\vee \cap M = \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(1, -1),$$

$$S_{\sigma_2} = \sigma_2^\vee \cap M = \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 1),$$

$$S_{\sigma_3} = \sigma_3^\vee \cap M = \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 1),$$

$$S_{\sigma_4} = \sigma_4^\vee \cap M = \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(1, -1),$$

$$S_{\sigma_5} = \sigma_5^\vee \cap M = \mathbb{Z}_{\geq 0}(0, -1) \oplus \mathbb{Z}_{\geq 0}(1, 0),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1, 1) \oplus \mathbb{Z}_{\geq 0}(1, -1) \oplus \mathbb{Z}_{\geq 0}(-1, 1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 0),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_4} = \mathbb{Z}_{\geq 0}(-1, -1) \oplus \mathbb{Z}_{\geq 0}(1, -1) \oplus \mathbb{Z}_{\geq 0}(-1, 1),$$

$$S_{\tau_4} = S_{\sigma_4} + S_{\sigma_5} = \mathbb{Z}_{\geq 0}(0, -1) \oplus \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(-1, 0),$$

$$S_{\tau_5} = S_{\sigma_5} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(0, -1),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}_{\geq 0}(0, 1) \oplus \mathbb{Z}_{\geq 0}(-1, 0) \oplus \mathbb{Z}_{\geq 0}(0, -1).$$

Therefore, the affine toric variety

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_4} = hom(S_{\sigma_4}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_5} = hom(S_{\sigma_5}, \mathbb{T}) = \mathbb{T}^2,$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_4} = hom(S_{\tau_4}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_5} = hom(S_{\tau_5}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) = \mathbb{R}^2.$$

The gluing of the affine toric varieties U_{σ_2} and U_{σ_3} along their common subset U_{τ_2} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $([x_0 : x_1], -x+y)$ where $x = x_0 - x_1$. The gluing of the affine toric varieties U_{σ_3} and U_{σ_4} along their common subset U_{τ_3} gives $\mathbb{TP}^1 \times \mathbb{T}$ with coordinates $(-x, [z_0 : z_1])$ where $-x + y = z_1 - z_0$. The two copies of $\mathbb{TP}^1 \times \mathbb{T}$ are glued along their coordinates gives $\mathbb{TP}^1 \times \mathbb{TP}^1$ with coordinates $([x_0 :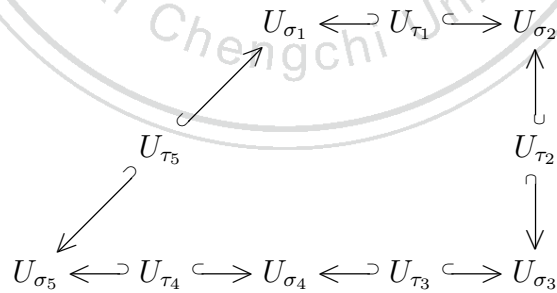 x_1], [z_0 : z_1])$. Since we have some embedding $h_{44} : U_{\tau_4} \hookrightarrow U_{\sigma_4}$ via $h_{44}(x_0 - x_1, x_1 - x_0, x_1 - x_0 + z_0 - z_1) = (0, x_1 - x_0, z_0 - z_1)$

and $h_{45} : U_{\tau_4} \hookrightarrow U_{\sigma_5}$ via $h_{45}(x_0 - x_1, x_1 - x_0, x_1 - x_0 + z_0 - z_1) = (x_0 - x_1, 0, x_1 - x_0 + z_0 - z_1)$, we have tropical isomorphism $h_{45} \circ h_{44}^{-1} : U_{\sigma_4} \to U_{\sigma_1}$ via $h_{45} \circ h_{44}^{-1}(0, x_1 - x_0, z_0 - z_1) = (x_0 - x_1, 0, x_1 - x_0 + z_0 - z_1)$. Similarly, $h_{51} \circ h_{55}^{-1} :$ $U_{\sigma_5} \to U_{\sigma_1}$ via $h_{51} \circ h_{55}^{-1}(x_0 - x_1, 0, x_1 - x_0 + z_0 - z_1) = (z_0 - z_1, x_0 - x_1 + z_1 - z_0, 0)$, and $h_{12} \circ h_{11}^{-1} : U_{\sigma_1} \to U_{\sigma_2}$ via $h_{12} \circ h_{11}^{-1}(z_0 - z_1, x_0 - x_1 + z_1 - z_0, 0) = (0, z_1 - z_0, x_0 - x_1)$. Hence the toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = \left(\coprod_{\sigma \in \Delta} U_\sigma\right)\Big/_\sim$$

$$= \{([x_0 : x_1], [z_0 : z_1], [a : b : c]) \in \mathbb{TP}^1 \times \mathbb{TP}^1 \times \mathbb{TP}^2 \mid a + x_1 = b + x_0, \ a + z_1 = c + z_0\},$$

that is, $\mathbb{X}_\Delta(\mathbb{T})$ is the blow up of $\mathbb{TP}^2$ at the two points $[0 : -\infty : -\infty]$ and $[-\infty : 0 : -\infty]$.

The following commutative diagram:

$$U_{\sigma_1} \longleftarrow\hspace{-1em}\hookrightarrow U_{\tau_1} \hookrightarrow U_{\sigma_2}$$

$$U_{\tau_5} \qquad\qquad U_{\tau_2}$$

$$U_{\sigma_5} \longleftarrow\hspace{-1em}\hookrightarrow U_{\tau_4} \hookrightarrow U_{\sigma_4} \longleftarrow\hspace{-1em}\hookrightarrow U_{\tau_3} \hookrightarrow U_{\sigma_3}$$

The polytopes $P = conv\{0, (1, 0), (1, 1), (0, 1), (-1, -1), (0, -1)\}$ (the figure 4.8) in $N_\mathbb{R}$. Since $conv\{(1, 0), (1, 1)\}$, $conv\{(0, 1), (1, 1)\}$, $conv\{(0, 1), (-1, -1)\}$,
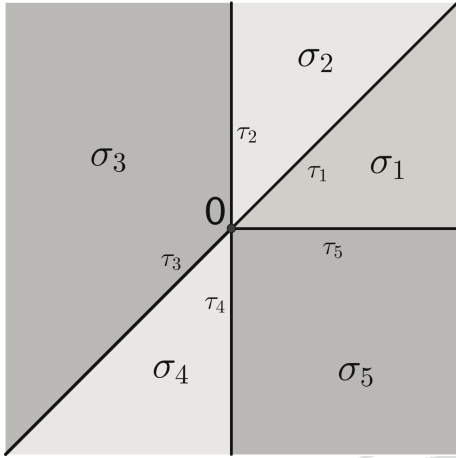
Figure 4.7: the fan Δ

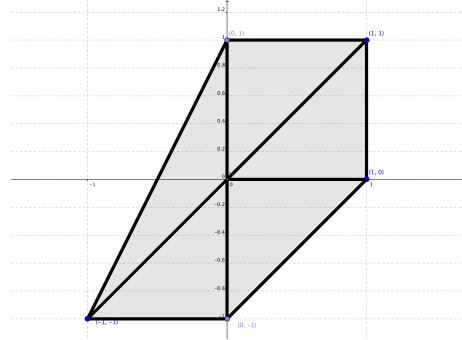Figure 4.8: the polytope P

$conv\{(-1,-1),(0,-1)\}$, $conv\{(0,-1),(1,0)\}$ are the facets of P, and they are the convex hull of a basis of N, X_P is a smooth Fano polytope (by Theorem 3.4.2).

Example 4.3.5. Given the lattice $N \simeq \mathbb{Z}^2$, then $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^2$, the dual lattice $M \simeq \mathbb{Z}^2$ and $M_\mathbb{R} = M \otimes \mathbb{R}$.

Let the fan Δ in $N_\mathbb{R}$. Suppose that the fan Δ (the figure 4.9) has

$$\sigma_1 = pos\{(1,0),(1,1)\}, \ \sigma_2 = pos\{(1,1),(0,1)\},$$

$$\sigma_3 = pos\{(0,1),(-1,0)\}, \ \sigma_4 = pos\{(-1,0),(-1,-1)\},$$

$$\sigma_5 = pos\{(-1,-1),(0,-1)\}, \ \sigma_6 = pos\{(0,-1),(1,0)\},$$

Figure 4.10: the polytope P
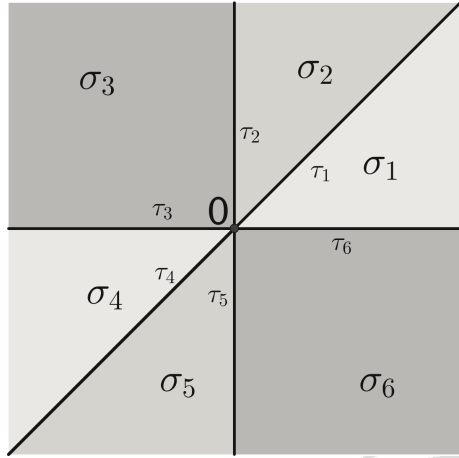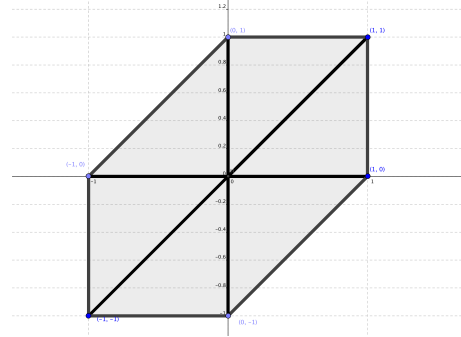
Figure 4.9: the fan Δ

together with

$$\tau_1 = \sigma_1 \cap \sigma_2 = pos\{(1,1)\}, \ \tau_2 = \sigma_2 \cap \sigma_3 = pos\{(0,1)\},$$

$$\tau_3 = \sigma_3 \cap \sigma_4 = pos\{(-1,0)\}, \ \tau_4 = \sigma_4 \cap \sigma_5 = pos\{(-1,-1)\},$$

$$\tau_5 = \sigma_5 \cap \sigma_6 = pos\{(0,-1)\}, \ \tau_6 = \sigma_6 \cap \sigma_1 = pos\{(1,0)\},$$

and the origin. Then the dual cones

$$\sigma_1^\vee = pos\{(-1,-1),(0,1)\}, \ \sigma_2^\vee = pos\{(1,0),(-1,1)\},$$

$$\sigma_3^\vee = pos\{(-1,0),(0,1)\}, \ \sigma_4^\vee = pos\{(-1,1),(0,-1)\},$$

$$\sigma_5^\vee = pos\{(-1,0),(1,-1)\}, \ \sigma_6^\vee = pos\{(0,-1),(1,0)\}.$$

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Moreover, the corresponding semigroups

$$S_{\sigma_1} = \sigma_1^{\vee} \cap M = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(1,-1),$$

$$S_{\sigma_2} = \sigma_2^{\vee} \cap M = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\sigma_3} = \sigma_3^{\vee} \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1),$$

$$S_{\sigma_4} = \sigma_4^{\vee} \cap M = \mathbb{Z}_{\geq 0}(-1,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\sigma_5} = \sigma_5^{\vee} \cap M = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(1,-1),$$

$$S_{\sigma_6} = \sigma_6^{\vee} \cap M = \mathbb{Z}_{\geq 0}(0,-1) \oplus \mathbb{Z}_{\geq 0}(1,0),$$

together with

$$S_{\tau_1} = S_{\sigma_1} + S_{\sigma_2} = \mathbb{Z}_{\geq 0}(1,1) \oplus \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\tau_2} = S_{\sigma_2} + S_{\sigma_3} = \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0),$$

$$S_{\tau_3} = S_{\sigma_3} + S_{\sigma_4} = \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\tau_4} = S_{\sigma_4} + S_{\sigma_5} = \mathbb{Z}_{\geq 0}(-1,-1) \oplus \mathbb{Z}_{\geq 0}(1,-1) \oplus \mathbb{Z}_{\geq 0}(-1,1),$$

$$S_{\tau_5} = S_{\sigma_5} + S_{\sigma_6} = \mathbb{Z}_{\geq 0}(0,-1) \oplus \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(-1,0),$$

$$S_{\tau_6} = S_{\sigma_6} + S_{\sigma_1} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(0,-1),$$

$$S_{\{0\}} = \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}_{\geq 0}(0,1) \oplus \mathbb{Z}_{\geq 0}(-1,0) \oplus \mathbb{Z}_{\geq 0}(0,-1).$$

Therefore, the affine toric variety

$$U_{\sigma_1} = hom(S_{\sigma_1}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_2} = hom(S_{\sigma_2}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_3} = hom(S_{\sigma_3}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_4} = hom(S_{\sigma_4}, \mathbb{T}) = \mathbb{T}^2,$$

$$U_{\sigma_5} = hom(S_{\sigma_5}, \mathbb{T}) = \mathbb{T}^2, \ U_{\sigma_6} = hom(S_{\sigma_6}, \mathbb{T}) = \mathbb{T}^2,$$

together with

$$U_{\tau_1} = hom(S_{\tau_1}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_2} = hom(S_{\tau_2}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_3} = hom(S_{\tau_3}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_4} = hom(S_{\tau_4}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\tau_5} = hom(S_{\tau_5}, \mathbb{T}) = \mathbb{R} \times \mathbb{T}, \ U_{\tau_6} = hom(S_{\tau_6}, \mathbb{T}) = \mathbb{R} \times \mathbb{T},$$

$$U_{\{0\}} = hom(S_{\{0\}}, \mathbb{T}) = \mathbb{R}^2.$$

Let f_i be in $U_{\sigma_i} = hom(S_{\sigma_i}, \mathbb{T})$ for all $i = 1, 2, 3, 4, 5, 6$, then we have

some maps

$$f_1 : S_{\sigma_1} \to \mathbb{T} \text{ via } f_1(0,1) = y \text{ and } f_1(1,-1) = x - y,$$

$$f_2 : S_{\sigma_2} \to \mathbb{T} \text{ via } f_2(1,0) = x \text{ and } f_2(-1,1) = -x + y,$$

$$f_3 : S_{\sigma_3} \to \mathbb{T} \text{ via } f_3(-1,0) = -x \text{ and } f_3(0,1) = y,$$

$$f_4 : S_{\sigma_4} \to \mathbb{T} \text{ via } f_4(0,-1) = -y \text{ and } f_4(1,-1) = -x + y,$$

$$f_5 : S_{\sigma_4} \to \mathbb{T} \text{ via } f_4(-1,0) = -x \text{ and } f_4(1,-1) = x - y,$$

$$f_5 : S_{\sigma_4} \to \mathbb{T} \text{ via } f_4(0,-1) = -y \text{ and } f_4(1,0) = x.$$

Since we have some embedding $h_{11} : U_{\tau_1} \hookrightarrow U_{\sigma_1}$ via $h_{11}(x + y, x - y, -x + y) = (0, x - y, y)$ and $h_{12} : U_{\tau_1} \hookrightarrow U_{\sigma_2}$ via $h_{12}(x + y, x - y, -x + y) = (x, -x + y, 0)$, we have tropical isomorphism $h_{12} \circ h_{11}^{-1} : U_{\sigma_1} \to U_{\sigma_2}$ via $h_{12} \circ h_{11}^{-1}(0, x - y, y) = (x, -x + y, 0)$. Similarly, $h_{23} \circ h_{22}^{-1} : U_{\sigma_2} \to U_{\sigma_3}$ via $h_{23} \circ h_{22}^{-1}(x, -x + y, 0) = (-x, 0, y)$, and $h_{34} \circ h_{33}^{-1} : U_{\sigma_3} \to U_{\sigma_4}$ via $h_{34} \circ h_{33}^{-1}(-x, 0, y) = (0, -x + y, -y)$, $h_{45} \circ h_{44}^{-1} : U_{\sigma_4} \to U_{\sigma_5}$ via $h_{45} \circ h_{44}^{-1}(0, -x + y, -y) = (-x, x - y, 0)$, $h_{56} \circ h_{55}^{-1} : U_{\sigma_5} \to U_{\sigma_6}$ via $h_{56} \circ h_{55}^{-1}(-x, x - y, 0) = (x, 0, -y)$, $h_{61} \circ h_{66}^{-1} : U_{\sigma_6} \to U_{\sigma_1}$ via $h_{61} \circ h_{66}^{-1}(x, 0, -y) = (0, x - y, y)$. Consider the product $\mathbb{TP}^1 \times \mathbb{TP}^1 \times \mathbb{TP}^1 \times \mathbb{TP}^2$ with homogeneous coordinates $[x_0 : x_1]$,
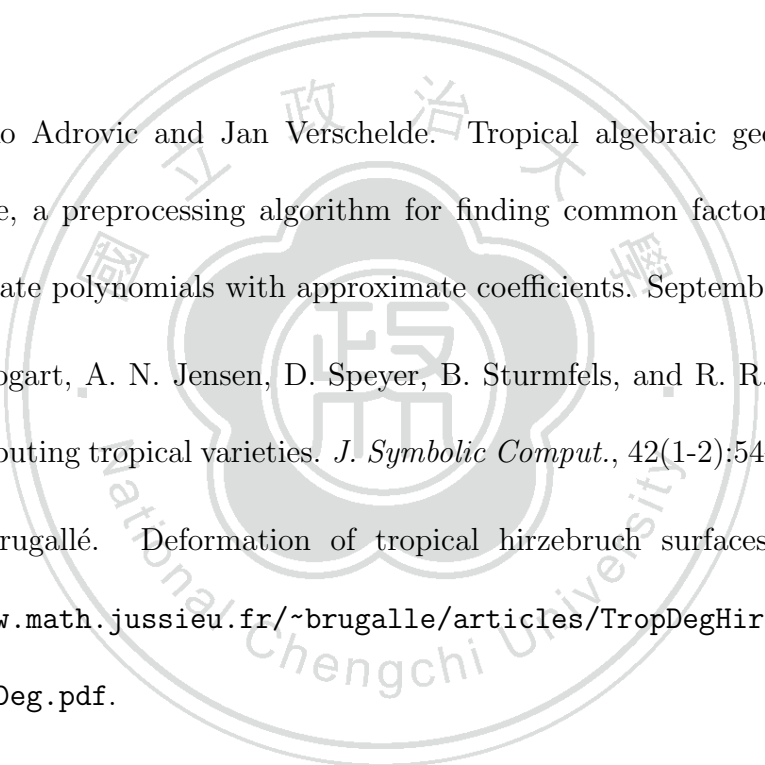
$[z_0 : z_1], [y_0 : y_1]$ on respective $\mathbb{TP}^1$ and homogeneous coordinate $[a : b : c]$ on $\mathbb{TP}^2$. We set $x = x_0 - x_1$, $x - y = z_0 - z_1$, $y = y_0 - y_1$. Hence the toric variety

$$\mathbb{X}_\Delta(\mathbb{T}) = \left(\coprod_{\sigma \in \Delta} U_\sigma\right)\Big/_{\sim}$$

$$= \{([x_0 : x_1], [z_0 : z_1], [y_0, y_1], [a : b : c]) \in \mathbb{TP}^1 \times \mathbb{TP}^1 \times \mathbb{TP}^1$$

$$\times \mathbb{TP}^2 \mid b + x_1 = c + x_0, \ a + z_1 = b + z_0, \ c + y_1 = a + y_0\},$$

that is, $\mathbb{X}_\Delta(\mathbb{T})$ is the blow up of $\mathbb{TP}^2$ at the three points $[0 : -\infty : -\infty]$, $[-\infty : 0 : -\infty]$, and $[-\infty : -\infty : 0]$.

The polytopes $P = conv\{0, (1, 0), (1, 1), (0, 1), (0, -1), (-1, -1), (0, -1)\}$ (the figure 4.10) in $N_\mathbb{R}$. Since $conv\{(1, 0), (1, 1)\}$, $conv\{(0, 1), (1, 1)\}$, $conv\{(0, 1), (-1, 0)\}$, $conv\{(-1, 0), (-1, -1)\}$, $conv\{(-1, -1), (0, -1)\}$, $conv\{(0, -1), (1, 0)\}$ are the facets of P, and they are the convex hull of a basis of N, X_P is a smooth Fano polytope (by Theorem 3.4.2).

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