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碩士學位論文

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保險公司因應死亡率風險之避險策略

Hedging Strategy against Mortality Risk for Insurance Company

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中華民國一百零一年七月

中文摘要

本篇論文主要討論在死亡率改善不確定性之下的避險策略。當保險公司負債面的人壽保單是比年金商品來得多的時候，公司會處於死亡率的風險之下。我們假設死亡率和利率都是隨機的情況，部分的死亡率風險可以經由自然避險而消除，而剩下的死亡率風險和利率風險則由零息債券和保單貼現商品來達到最適避險效果。我們考慮 mean variance、VaR 和 CTE 當成目標函數時的避險策略，其中在 mean variance 的最適避險策略可以導出公式解。由數值結果我們可以得知保單貼現的確是死亡率風險的有效避險工具。

關鍵字: 死亡率風險、Lee Carter model, CIR model, Maximum Entropy principle, Value at risk, Conditional tail expectation, Karush-Kuhn-Tucker.

Hedging Strategy against Mortality Risk for Insurance Company

Abstract

This paper proposes hedging strategies to deal with the uncertainty of mortality improvement. When insurance company has more life insurance contracts than annuities in the liability, it will be under the exposure of mortality risk. We assume both mortality and interest rate risk are stochastic. Part of mortality risk is eliminated by natural hedging and the remaining mortality risk and interest rate risk will be optimally hedged by zero coupon bond and life settlement contract. We consider the hedging strategies with objective functions of mean variance, value at risk and conditional tail expectation. The closed-form optimal hedging formula for mean variance assumption is derived, and the numerical result show the life settlement is indeed a effective hedging instrument against mortality risk.

Key words: Mortality risk, Lee Carter model, CIR model, Maximum Entropy principal. Value at risk, Conditional tail expectation, Karush-Kuhn-Tucker.

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1.Introduction

Life insurance companies are under the exposures of both longevity and mortality risk due to uncertainty of the mortality improvement. Recent researches and observations prove the significant improvement on the mortality rate of populations around the world. On the other hand, some pandemic diseases and catastrophic natural disaster also frequently cause mortality rate to rise unexpectedly. In order to transfer mortality risk, the insurance companies are seeking alternative hedging instruments. Other hedging instruments such as longevity bonds, longevity swap, q-forward are also discussed the feasibility of providing solution for transferring the mortality or longevity risk through capital market. For example, Blake, D et al.(2001) discussed how the survivor bond can hedge the mortality risk, and Dowd, K et al.(2006) introduced survivor swap as a hedging instrument for hedging longevity/mortality risk.

Another hedging strategy can be implemented by adjusting the mix of life insurance and annuity in the liability called natural hedging. Cox, S et al.(2007) proposed using natural hedging to stabilize the cash flow of aggregate liability. Wang, J.L et al.(2010) and Tsai, J.T. et al.(2010) investigated the optimal product mix of life insurance and annuity to naturally hedge the longevity and mortality risk. However, to adjust the product mix of life insurance and annuity in the liability to optimal condition is too difficult to implement in practice, controlling the distribution channel of insurance product is too costly to hedge the mortality or longevity risk. But we still cannot ignore the effect of natural hedging even it may not be able to achieve the optimal condition. In this paper, the hedging strategy is to reduce the natural hedged risks by incorporating the hedging instrument. Life settlement(senior life settlement) is a transaction that individuals aged 65 or above can sell their insurance policy to the investors in the secondary market, the investor will be responsible for paying the premium of this policy and have the right to get the insurance benefit when the insured of this policy is dead. The market for this kind of transactions is fast growing. Life settlement transaction can be a win-win situation for both investors and policyholders. The policyholders can sell their insurance policy with higher price than surrender value, the investors can obtain a relatively low volatility asset which is uncorrelated to the financial asset in the capital market. Because the payoff of life settlement is positive related to the mortality rate, it can be regarded as a hedging vehicle against the mortality risk for insurance company.

2.Models setting

2.1 Interest rate and mortality rate model

We focus on two type of risks: interest rate risk and mortality risk. These are two main risks affecting the value of insurance products. We assume the interest rate dynamic following CIR interest rate model(Cox, J. C. et al.(1985)). Under risk neutral measure Q, the stochastic differential equation of CIR model can be written as

$$dr(t) = a\big(b - r(t)\big)dt + \sigma\sqrt{r(t)}dW^Q(t)$$

provided $2ab \geq \sigma^2$, where the coefficient a represents the speed of mean reverting, b is the long-term average interest rate level and σ describes the volatility of interest rate.

Assume the market price of risk is of the following form

$$\lambda(t) = \lambda\sqrt{r(t)}$$

then the Radon-Nikodym derivatives is

$$\left.\frac{dQ}{dP}\right|_{F_t} = e^{\int_0^t \lambda\sqrt{r(u)}dW^P(u) - \frac{1}{2}\int_0^t \lambda^2 r(u)du}$$

Therefore

$$dW^Q(t) = dW^P(t) - \lambda\sqrt{r(t)}dt$$

Under the real-world probability measure P

$$dr(t) = [ab - (a + \lambda\sigma)r(t)]dt + \sigma\sqrt{r(t)}dW^{P}(t)$$
$$= (a + \lambda\sigma)\left[\frac{ab}{a + \lambda\sigma} - r(t)\right]dt + \sigma\sqrt{r(t)}dW^{P}(t)$$

The bond price formula under CIR interest rate model is

$$P(t, T) = A(T - t)e^{-B(T-t)r(t)}$$

where

$$B(x) = \frac{2(e^{\gamma x} - 1)}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}$$

and

$$A(x) = \left[\frac{2\gamma e^{(a+\gamma)\frac{x}{2}}}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}\right]^{\frac{2ab}{\sigma^2}}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

We use Lee Carter model (Lee, R.D et al.(1992)) to model the future mortality improvement. Although there are many newly developed models providing better prediction performance than Lee Cater model, Lee Carter model still has attractive properties including easy model structure and acceptable prediction errors. Moreover, we can extend the univariate mortality model to mutlivariate model by giving the correlated structures of $k_t{}'s$. For i-th population we can represent the mortality model as

$$\ln\left(m_{x,t}^{(i)}\right) = \alpha_x^{(i)} + \beta_x^{(i)}k_t^{(i)}$$

Furthermore, we adopt multivariate random walk model to describe the correlated dynamics of all $k_t^{(i)}$'s, which means we can use VAR with lags 0 to model the first difference of $k_t^{(i)}$'s.

$$\begin{pmatrix}\Delta k_t^{(1)}\\ \Delta k_t^{(2)}\\ \Delta k_t^{(3)}\\ \Delta k_t^{(4)}\end{pmatrix} = \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix} + \Sigma^{\frac{1}{2}}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\end{pmatrix}$$

where $\Sigma^{\frac{1}{2}}$ is the Cholesky decomposition of covariance matrix Σ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are four identical and independent standard normal random variables.

2.2.The profit function

Our goal is to construct the asset portfolio to hedge the interest rate risk and mortality risk in the liability. On the liability side, we consider life contracts and annuities with insured and annuitants of different ages and genders. On the asset side, to hedge the interest rate risk and mortality risk, we choose zero coupon bonds with different maturities and life settlement with insured of different ages and genders as the hedging instruments.

When insurance company calculate price of their insurance product including life contracts and annuities, they always use static reference mortality table instead of dynamic stochastic mortality rate. Since the static reference mortality table can not reflect the impact of uncertain mortality improvement on the price of insurance product. We define the profit function of life contracts or annuities as the difference between actuarial present value calculated by the cohort dynamic mortality rates and actuarial present value calculated by the reference static mortality rate. Since the dynamic mortality rates are stochastic, the profit function is a random variable.

We define the notation of profit function of female annuity product by

$$\pi^{fa}(x, r_t, m_{x,t}^{fa}) = V_{actual}^{fa}(x, r_t, m_{x,t}^{fa}) - V_{reference}^{fa}(x, r_t, m_{x,t}^{fa})$$

where V_{cohort}^{fa} is the stochastic actuarial present value calculated by using dynamic cohort mortality rates, V_{period}^{fa} is the actuarial present value calculated by using static reference mortality rates. x denotes age ,r_t represents the interest rate and $m^{fa}_{x,t}$ is the force of mortality for population in female annuity.

Similarly, the profit function for male annuity, female life and male life can be written accordingly.

$$\pi^{ma}(x, r_t, m^{ma}_{x,t}) = V^{ma}_{actual}(x, r_t, m^{ma}_{x,t}) - V^{ma}_{reference}(x, r_t, m^{ma}_{x,t})$$

$$\pi^{fl}(x, r_t, m^{fl}_{x,t}) = V^{fl}_{actual}(x, r_t, m^{fl}_{x,t}) - V^{fl}_{reference}(x, r_t, m^{fl}_{x,t})$$

$$\pi^{ml}(x, r_t, m^{ml}_{x,t}) = V^{ml}_{actual}(x, r_t, m^{ml}_{x,t}) - V^{ml}_{reference}(x, r_t, m^{ml}_{x,t})$$

The profit function of life settlement is the stochastic present value of cash flow generated by life settlement minus the cost of buying life settlement, here we assume the price of life settlement is determined by the suggested life expectancy of the insured who sells the life settlement of his/her life insurance contract. Therefore the cost of buying life settlement with benefit 1 is

$$\overline{V}^S(x, r_t, m_{x,t}) = \prod_{i=1}^{ET} \frac{1}{1 + r_i}$$

where r_i is the interest rate in year i, ET is the life expectancy suggested by the medical profession. Then the profit function of life settlement can be defined by the same concept as

$$\pi^S(x, r_t, m_{x,t}) = V^S(x, r_t, m^S_{x,t}) - \overline{V}^S(x, r_t)$$

where $V^S(x, r_t, m^L_{m,t})$ is the stochastic present value of cash flow generated by life settlement.

The definition of the profit function of zero coupon bond with face value 1 and maturity T is straightforward.

$$\pi^{P}(r_t, T) = \frac{1}{\prod_{i=1}^{T}(1 + r_i)} - P(r_t, T)$$

where $P(r_t, T)$ is the bond price calculated by using the closed form bond price formula of CIR model.

2.3.Adjusting mortality table

Without the mortality rate for the insured selling life settlement, we will not be able to analyze the distribution of profit function for life settlement. The available information about the insured sold life settlement is the age and life expectancy. Maximum entropy principle provide a reasonable and feasible methodology to adjust the "standard" mortality rates into a adjusted mortality rates by incorporating newly obtained information such as life expectancy, variance, median...etc . For example, Kogure., A. et al.(2010), Johnny Siu-Hang Li et al.(2010) and Johnny Siu-Hang Li et al.(2011) applied maximum entropy principle to change the physical probability measure to the objective probability measure for pricing mortality linked derivatives. We will applied the method in Brockett, P. L. (1991) to construct the life time distribution of life settlement seller.

Let $K(x)$ be the curtate life time of (x)

According to standard life table the probability mass function of K(x) is

$$(g_0, g_1, \dots, g_{\omega-x})$$

where $g_i = \Pr(K(x) = i)$

We want to find adjusted mortality table with curtate life time of (x) as

$$(f_0, f_1, \dots, f_{\omega-x})$$

$$\text{s.t} \sum_k f_k = 1 \text{ and } \sum_k k f_k = ET$$

where $f_i = \Pr(K(x) = i)$ under adjusted mortality table and ET is the expectation of lifetime based on newly obtained information.

To find the adjusted distribution of life time we have to solve the following optimization problem that minimizes the Kullback–Leibler information(Kullback, S et al.(1951))

$$\min_{f_k} \sum_k f_k \ln\left(\frac{f_k}{g_k}\right)$$

subject to

$$\sum_k f_k = 1$$

and

$$\sum_k k f_k = ET$$

The solution can be obtained by Lagrange multiplier method. First we consider the Lagrangian function

$$L(f,\beta) = \sum_k f_k \ln\left(\frac{f_k}{g_k}\right) - \beta_0\left(1 - \sum_k f_k\right) - \beta_1\left(ET - \sum_k f_k\right)$$

we need to solve $\nabla L(f,\beta) = 0$, which is equivalently to solving the following system of equations.

$$\begin{cases} \ln\left(\frac{f_k}{g_k}\right) + 1 + \beta_0 + k\beta_1 = 0 \\ -1 + \sum_k f_k = 0 \\ -ET + \sum_k k f_k = 0 \end{cases}$$

$$\Rightarrow f_k = g_k e^{-1-\beta_0-k\beta_1} \quad k = 0,1,\dots,n(=\omega - x)$$

where β_0 and β_1 are solution of

$$\begin{cases} \sum_k g_k e^{-1-\beta_0-k\beta_1} = 1 \\ \sum_k k g_k e^{-1-\beta_0-k\beta_1} = ET \end{cases}$$

or equivalently

$$\min_{\beta_0,\beta_1}\left\{\sum_k g_k e^{-\beta_0-1-\beta_1 k}\right\}+\beta_0+(ET)\beta_1$$

3.Hedging Approaches

As defining the profit function of assets and liabilities, we then define the profit function of the surplus to be the profit function of assets minus profit function of liabilities.

$$\begin{aligned}\pi(t)=\sum_{i=1}^{n_B} N_i\pi^B(r_t,T_i)+\sum_{i}^{n_S} M_i\pi^S\left(x_i,r_t,m_{m,t}^S\right)-\sum_i c_i^{fl}\pi^{fl}\left(x_i,r_t,m_{x,t}^{fl}\right)\\ -\sum_i c_i^{ml}\pi^{ml}\left(x_i,r_t,m_{x,t}^{ml}\right)-\sum_i c_i^{fa}\pi^{fa}\left(x_i,r_t,m_{x,t}^{fa}\right)\\ -\sum_i c_i^{ma}\pi^{ma}\left(x_i,r_t,m_{x,t}^{ma}\right)\end{aligned}$$

where c_i^{fl}, c_i^{ml} are female life and male life insurance benefit for the i-th insured. c_i^{fa}, c_i^{ma} are female annuity and male annuity annual payment amount for the i-th annuitant.

The insurance company need to manage the profit function of surplus. The mean variance optimization problem will be

$$\max_{N_1,\dots,N_{n_B},M_1,\dots,M_{n_S}} E[\pi(t)]-\theta Var[\pi(t)]$$

subject to

$$N_i, M_j \geq 0\ \forall i,j$$

and

$$\begin{aligned}\sum_{i=1}^{n_B} N_i P(r_t,T_i)+\sum_{i}^{n_S} M_i\overline{V}^S\left(x_i,r_t,m_{x,t}^S\right)\\ =\sum_i c_i^{fa}V_{period}^{fa}\left(x_i,r_t,m_{x,t}^{fa}\right)+\sum_i c_i^{ma}V_{period}^{ma}\left(x_i,r_t,m_{x,t}^{ma}\right)\\ +\sum_i c_i^{fl}V_{period}^{fl}\left(x_i,r_t,m_{x,t}^{fl}\right)+\sum_i c_i^{ml}V_{period}^{ml}\left(x_i,r_t,m_{x,t}^{ml}\right)\end{aligned}$$

The first constraint is to avoid short position of assets and the second constraint indicates the budget constraint.

Let

$$L = \sum_{i} c_i^{fa} V_{period}^{fa}\left(x_i, r_t, m_{x,t}^{fa}\right) + \sum_{i} c_i^{ma} V_{period}^{ma}\left(x_i, r_t, m_{x,t}^{ma}\right) + \sum_{i} c_i^{fl} V_{period}^{fl}\left(x_i, r_t, m_{x,t}^{fl}\right) + \sum_{i} c_i^{ml} V_{period}^{ml}\left(x_i, r_t, m_{x,t}^{ml}\right)$$

Then the budget constraint can be rewritten as

$$\sum_{i=1}^{n_B} N_i \frac{B(r_t, T_i)}{L} + \sum_{i}^{n_S} M_i \frac{\overline{V}^S\left(x_i, r_t, m_{x,t}^S\right)}{L} = 1$$

This optimization problem includes equality constraints and inequality constraints, the Karush-Kuhn-Tucker (KKT) optimality conditions(Kuhn et al.(1951)) in appendix 1 provide a method to solve this problem analytically.

Let $u = \left(M_1, \dots, M_{n_S}, N_{n_S+1}, \dots, N_{n_B+n_S}\right)' = (u_1, \dots, u_n)'$ be the units column vectors, the first n_S components are the units we need to buy life settlements with different ages, gender and life expectancies and the last n_B components are the units we need to buy bonds with different maturities.

Our target is to solve the problem:

$$\max_{u}\left\{[m', \overline{m}]\begin{bmatrix} u \\ -1 \end{bmatrix} - \theta[u', -1]\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} u \\ -1 \end{bmatrix}\right\}$$

$$\text{s.t.}\, u'a = 1 \text{ and } u' \geq 0$$

where m is the n by 1 mean column vectors of profit function of all the assets, $\overline{m}$ is the sum of expected value of profit function of all liabilities. Σ is the (n+1)*(n+1) covariance matrix of all assets and liabilities, we can decompose Σ into 4 sub-matrices

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Σ_{11} is n*n matrix represent covariance matrix of assets. Σ_{22} is 1*1 matrix equaling to the variance of sum of all liabilities. And

$$a=\begin{bmatrix}\dfrac{B(r_t,T_1)}{L}\\ \vdots \\ \dfrac{B\left(r_t,T_{N_B}\right)}{L}\\ \dfrac{\overline{V}^S\left(x_1,r_t,m_{x,t}^S\right)}{L}\\ \vdots \\ \dfrac{\overline{V}^S\left(x_{N_S},r_t,m_{x,t}^S\right)}{L}\end{bmatrix}$$

Denote

$$f(u)=-[m',\overline{m}]\begin{bmatrix}u\\-1\end{bmatrix}+\theta[u',-1]\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\begin{bmatrix}u\\-1\end{bmatrix}$$

$$=-m'u+\overline{m}+\theta[u'\Sigma_{11}u-\Sigma_{21}u-u'\Sigma_{12}+\Sigma_{22}]$$

$$=-m'u+\overline{m}+\theta[u'\Sigma_{11}u-2u'\Sigma_{12}+\Sigma_{22}]$$

Our optimization problem becomes

$$\min_u f(x)$$

$$\text{s.t.}\, u'a-1=0\ \text{ and } -u'\leq 0$$

The optimal solution is

$$u=\frac{1}{2\theta}\Sigma_{11}^{-1}(\mu+m-\lambda a)+\Sigma_{11}^{-1}\Sigma_{12}$$

where

$$\lambda=\frac{a'\Sigma_{11}^{-1}m-2\theta\left(1-a'\Sigma_{11}^{-1}\Sigma_{12}\right)}{a'\Sigma_{11}^{-1}a}$$

if $u_i>0\ \forall i$

The detailed derivation of the solution is in appendix 2.

Mean variance approach is easy to implement and has good properties such as closed-form optimal allocation formula, however using first two moments to determine hedging strategies may be too simple to capture the characteristics of profit function. We consider further objective functions such as value at risk(VaR) and conditional tail expectation(CTE) to offer a comparative hedging performance to the mean variance approach.

Set loss function as negative of profit function, that is

$$L = -\pi$$

The definition of VaR could be written as

$$\mathrm{VaR}_\beta(L) = \inf\{\xi | P(L \le \xi) = \beta\}$$

and we apply the result of Trindade et al.(2007) and Pflug, G. (2000) to obtain the value of CTE by solving the following optimization problem

$$\mathrm{CTE}_\beta(L) = \frac{1}{1-\beta}\int_{L \ge \mathrm{VaR}_\beta(L)} L f_L(l)dl$$

$$= \frac{1}{1-\beta} E\left[L I_{\{L \ge \mathrm{VaR}_\beta(L)\}}\right]$$

$$= \min_{\xi}\left\{\xi + \frac{1}{1-\beta}\int_{l\in R}[l-\xi]^+ f_L(l)dl\right\}$$

Therefore

$$\min_{u} CTE_\beta(L) = \min_{u,\xi}\left\{\xi + \frac{1}{1-\beta}\int_{l\in R}[l-\xi]^+ f_L(l)dl\right\}$$

$$= \min_{u,\xi}\left\{\xi + \frac{1}{1-\beta}\frac{1}{n}\sum_{i=1}^{n}[l_i-\xi]^+\right\}$$

4.Numerical examples

We first consider the mortality is stochastic and the interest rate is non-stochastic. Therefore the interest rate is assumed to be a constant rate 0.03 in this example, there will be 100,000 generated mortality sample paths for calculating profit function of the liabilities. On the asset side, we choose life settlement of insured aged 65 and with suggested life expectancy 10 for both male and female. On the liability side, we include life contracts of female aged 50 and male aged 65 with benefit payment 100, there are also annuities of female aged 55 and male aged 65 with annual payment 1 in the liability. Table 1 summarizes the assets and liabilities:

Asset	Liability
Life settlement:	**Life: (benefit=100)**
Male 65	Female 50
(suggested life expectancy=10)	Male 65
Female 65	**Annuity:**
(suggested life expectancy=10)	Female 55
	Male 65

Table 1: Assets and liabilities

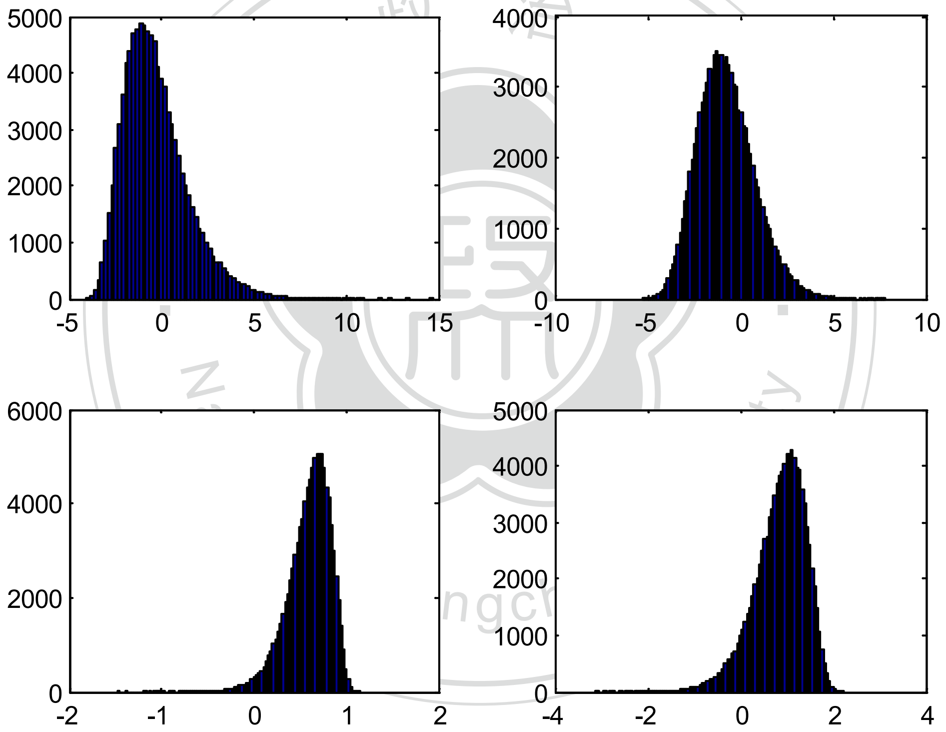

Figure 1: The profit functions on the liability side

(top left): Life Female 50, (top right): Life Male 65, (bottom left): Annuity Female 55, (bottom right): Annuity Male 65.

The distribution of profit functions are displayed on Figure 1 and Figure 2. We can discover due to mortality improvement, the expected value of profit function of life contracts are negative whereas they are positive for annuities. The averaged value of profit function of life settlements are also positive.

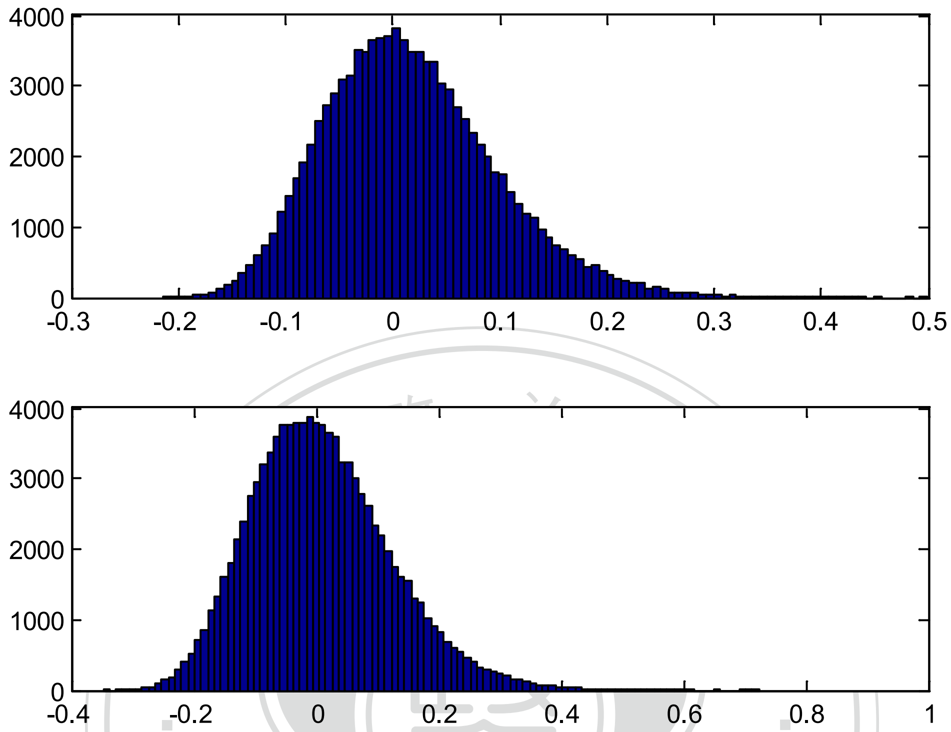

Figure 2: The profit functions on the asset side

(top):life settlement male aged 65 with life expectancy 10.

(bottom): life settlement female aged 65 with life expectancy 10.

The expected value and covariance matrix of profit functions are shown in Table 2 and Table 3. The life settlements have similar properties to the life insurance contracts, therefore it provide excellent hedging effectiveness against mortality risk from life insurance contracts.

Life settlement Male 65 ET1=0	Life settlement Female 65 ET1=0	Life Female 50	Life Male 65	Annuity Female 55	Annuity Male 65
0.016953	0.008119	-0.29314	-0.75086	0.598889	0.852789

Table 2: Mean of profit functions

Life settlement Male 65 ET1=0	Life settlement Female 65 ET1=0	Life Female 50	Life Male 65	Annuity Female 55	Annuity Male 65
0.006401	0.004383	0.055503	0.117803	-0.00458	-0.01429
0.004383	0.01301	0.136444	0.075508	0.001919	0.006579
0.055503	0.136444	3.080249	1.234243	0.035585	0.11696
0.117803	0.075508	1.234243	2.376275	-0.099	-0.3025
-0.00458	0.001919	0.035585	-0.099	0.054546	0.098543
-0.01429	0.006579	0.11696	-0.3025	0.098543	0.308113
0.006401	0.004383	0.055503	0.117803	-0.00458	-0.01429

Table 3: Covariance matrix of profit functions

The optimal hedging strategies according to different objective functions are in Table 4. As the parameter θ increases, the optimal weight for life settlement male will decrease but the optimal weight for life settlement female will increase. The result for VaR objective functions and CTE objective functions are similar, it put more weights on both life settlement of male and female comparing to the result with mean variance objective function.

	Life settlement male 65 ET=10	Life settlement female 65 ET=10
MV $\theta = 1$		
Units	17.7247	11.2857
Weight	0.1010	0.0746
MV $\theta = 2$		
Units	17.0029	11.3729
Weight	0.0969	0.0752
VaR(0.05)		
Units	22.2499	11.8484
Weight	0.1268	0.0783
CTE(0.05)		
Units	21.9820	13.8237
Weight	0.1253	0.0914

Table 4 optimal hedging strategies

Figure3~6 display the hedging effectiveness of different objective function. In Figure 3 and Figure 4, we can see under the mean variance hedging strategies, the distributions are less volatile after hedging, because the goal is to reduce the variance of portfolio and maximize the mean of profit function simultaneously. Figure 5 and Figure 6 have different hedging outcomes, the hedged distribution will retain the weight on the right tail and reduce the weight on the left tail. This is the most desirable result, it means our hedging strategies may reduce the down side risk of our portfolio but at the same time it will not harm the opportunity of making profit.

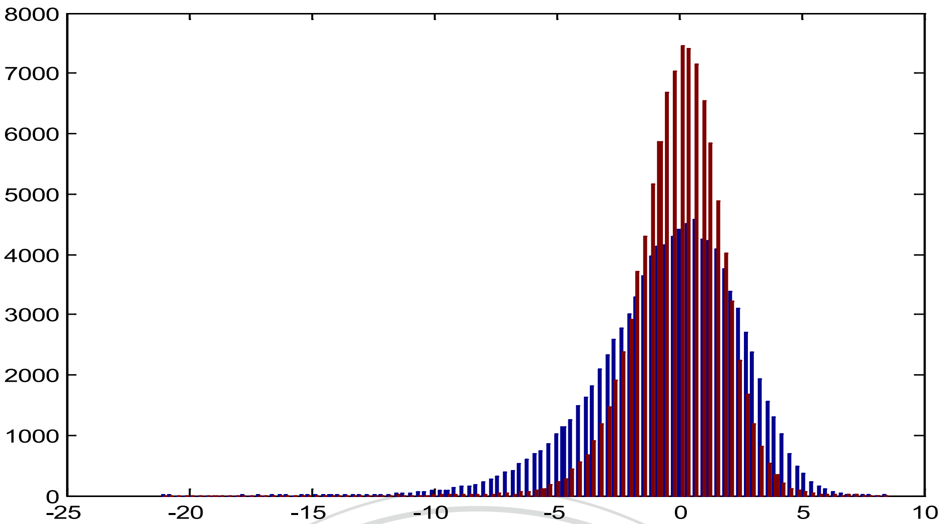

Figure 3: hedging effectiveness with objective function mv $\theta = 1$

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

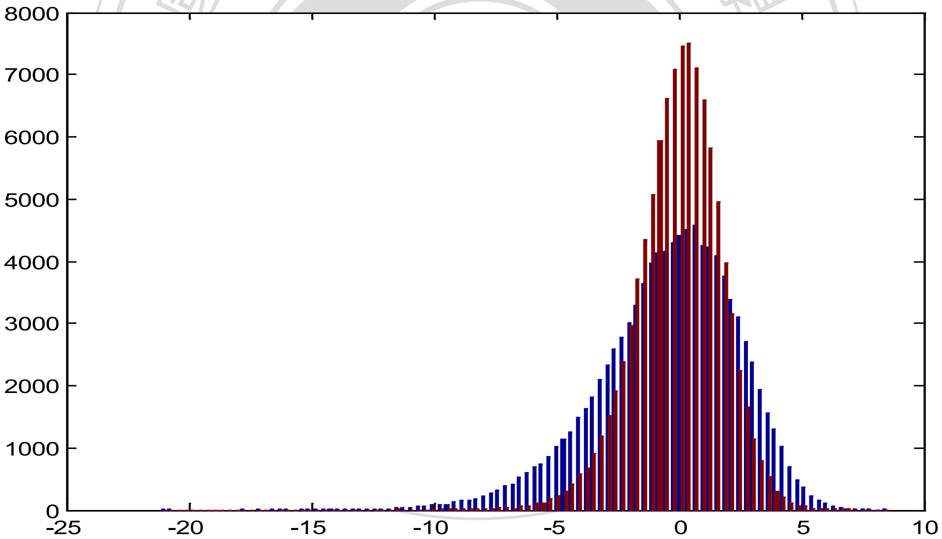

Figure 4: hedging effectiveness with objective function mv $\theta = 2$

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

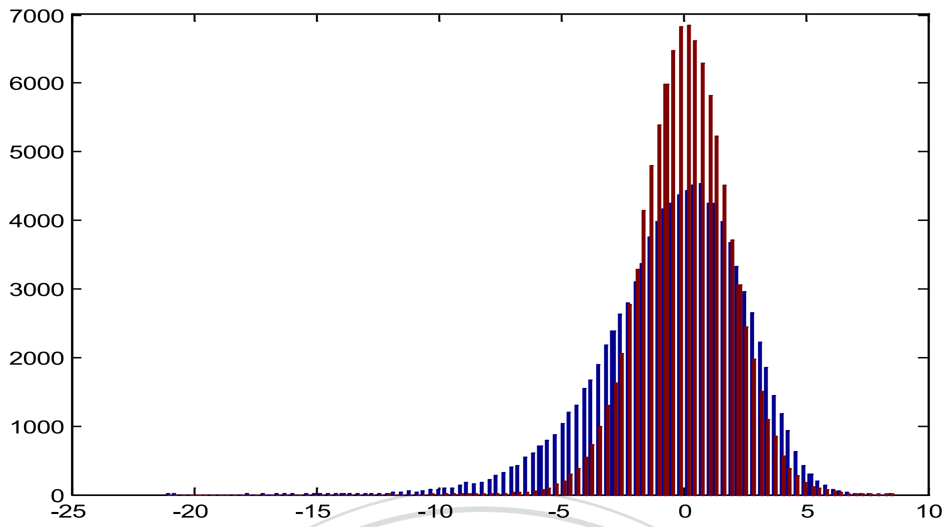

Figure 5: hedging effectiveness with objective function VaR

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

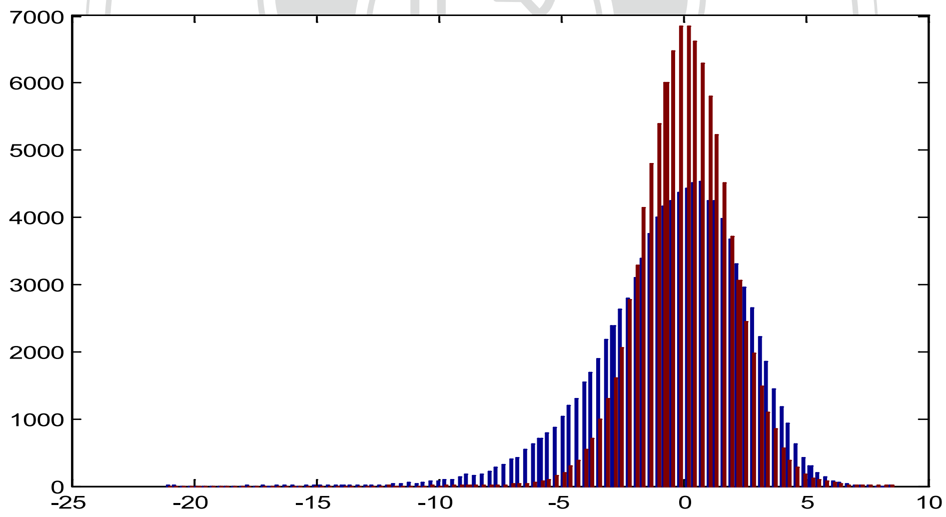

Figure 6: hedging effectiveness with objective function CTE

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

Next step, we will discuss the hedging strategies by incorporating both interest risk and mortality risk. The constant interest rate is replaced by100,000 sample paths of interest rate generated according to CIR model with parameters $a = 0.2$, $b = 0.03$,

$\sigma = 0.04$ and $\lambda = 0.3$. Here we include additional asset, zero coupon bond with maturity 20 years to hedge the interest rate risk.

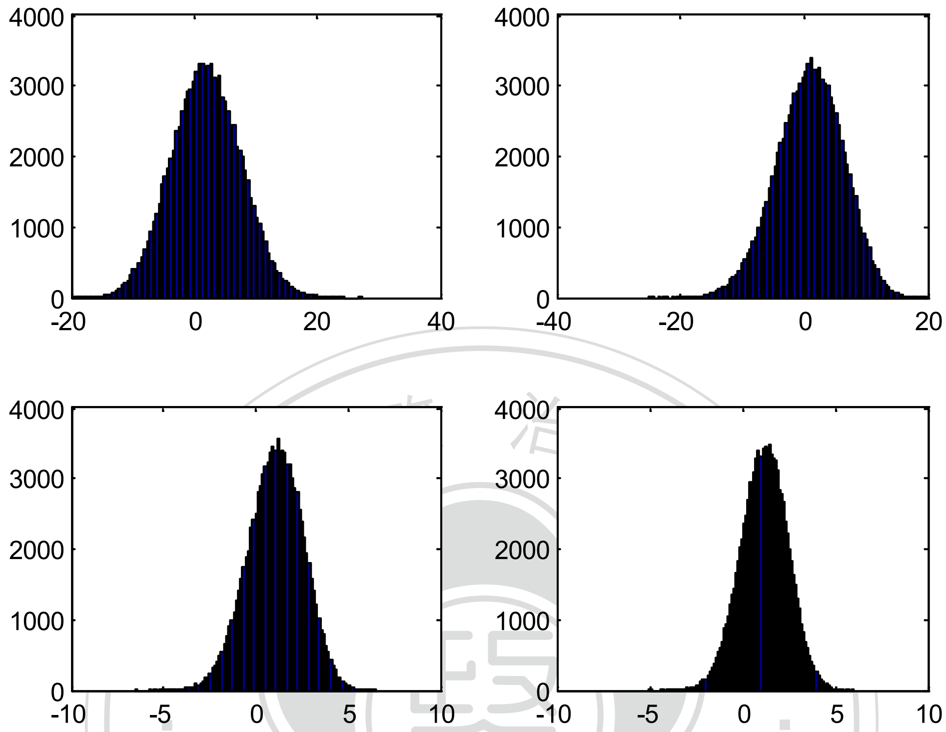

Figure 7: The profit functions on the liability side

(top left): Life Female 50, (top right): Life Male 65, (bottom left): Annuity Female 55, (bottom right): Annuity Male 65.

Fig 7 and Fig 8 have similar distribution shape for each asset and liability comparing to the case without interest rate risk but the dispersion is larger due to stochastic interest rate contribute more randomness to the distributions of profit functions.

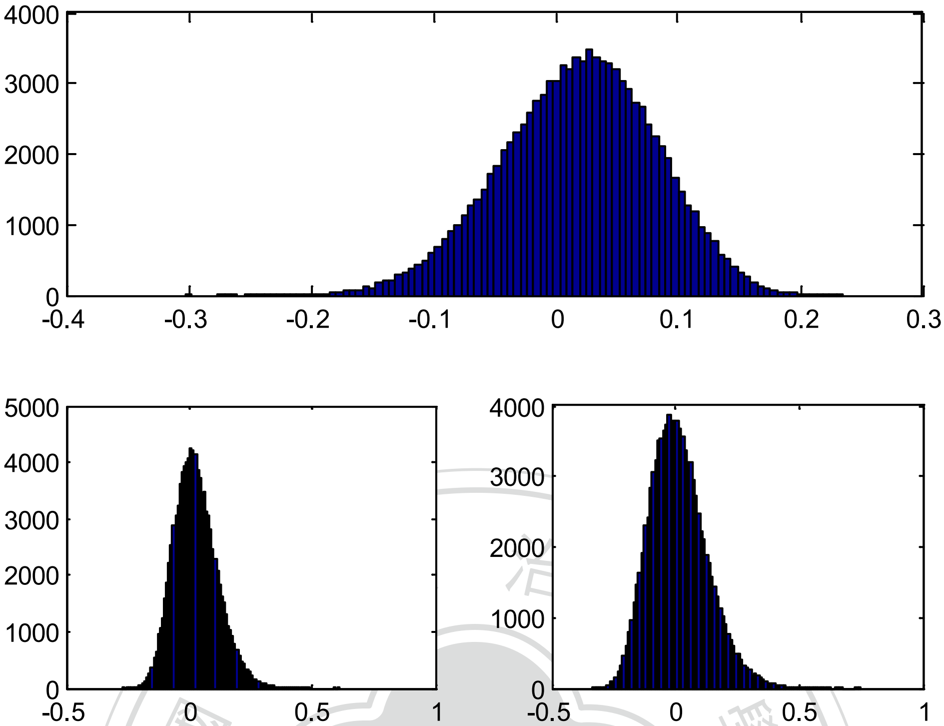

Figure 8: The profit functions on the asset side

(top): zero coupon bond with maturity 20 years.

(bottom left):life settlement male aged 65 with life expectancy 10.

(bottom right): life settlement female aged 65 with life expectancy 10.

Table 5 is the optimal hedging allocation incorporating additional interest rate risk, we can observe the large portion of weight is put on the zero coupon bond, hence under our assumption, the interest rate risk dominates the mortality risk. Similarly, as the parameter θ increases, the weight on zero coupon bond and life settlement male decrease but the weight on life settlement male increases. This result indicates that life settlement female seems has better effect on reducing portfolio variance. While considering the VaR and CTE criterion, we find it put more weight on zero coupon bond and life settlement male. This is quite different form the result of mean variance hedging strategies.

	Zero coupon bond T=20	Life settlement male 65 ET=10	Life settlement female 65 ET=10
MV θ = 1			
Units	183.5250	20.1810	12.7361
Weight	0.8008	0.1151	0.0842
MV θ = 2			
Units	183.4617	19.7923	13.1131
Weight	0.8005	0.1128	0.0867
VaR(0.05)			
Units	188.1349	24.4197	6.0362
Weight	0.8209	0.1392	0.0399
CTE(0.05)			
Units	189.3252	20.0600	9.0111
Weight	0.8261	0.1144	0.0596

Table 5 optimal hedging strategies

Fig 9~12 show the hedging effectiveness of our example. As mentioned earlier, the interest rate risk dominates the mortality risk. We cannot easily recognize the differences between theses 4 figures.

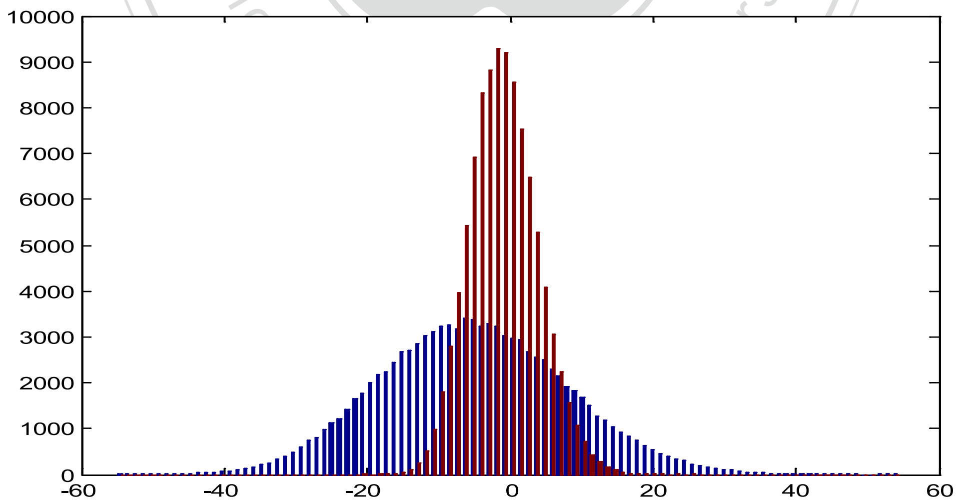

Figure 9: hedging effectiveness with objective function mv θ = 1

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

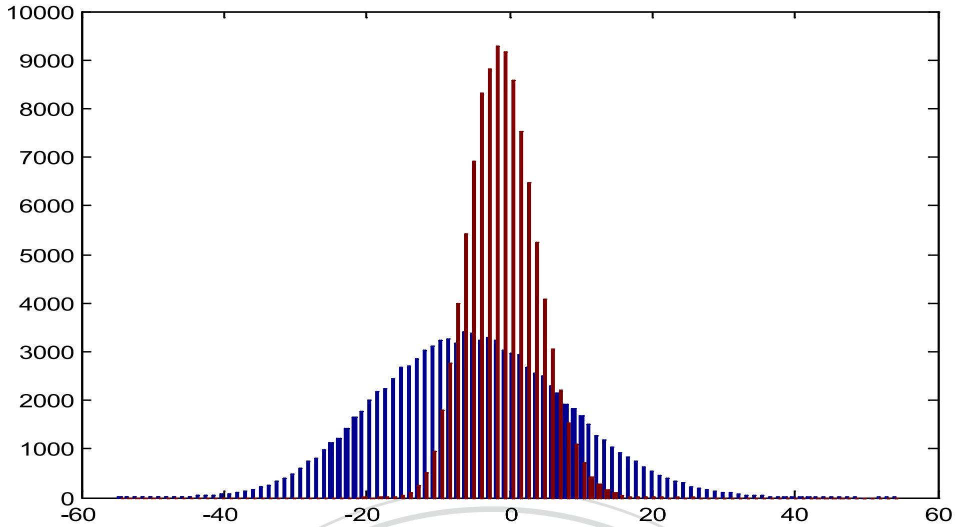

Figure 10: hedging effectiveness with objective function mv $\theta = 2$

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

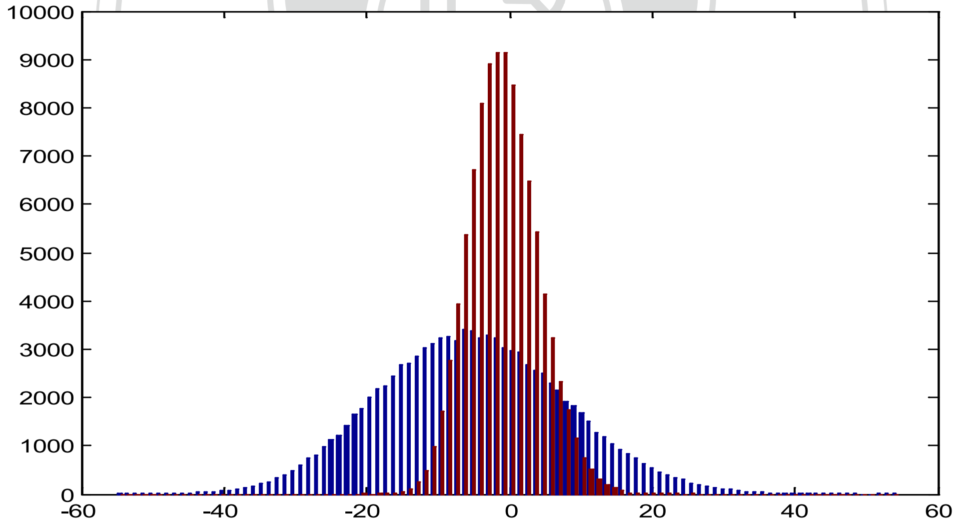

Figure 11: hedging effectiveness with objective function VaR

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

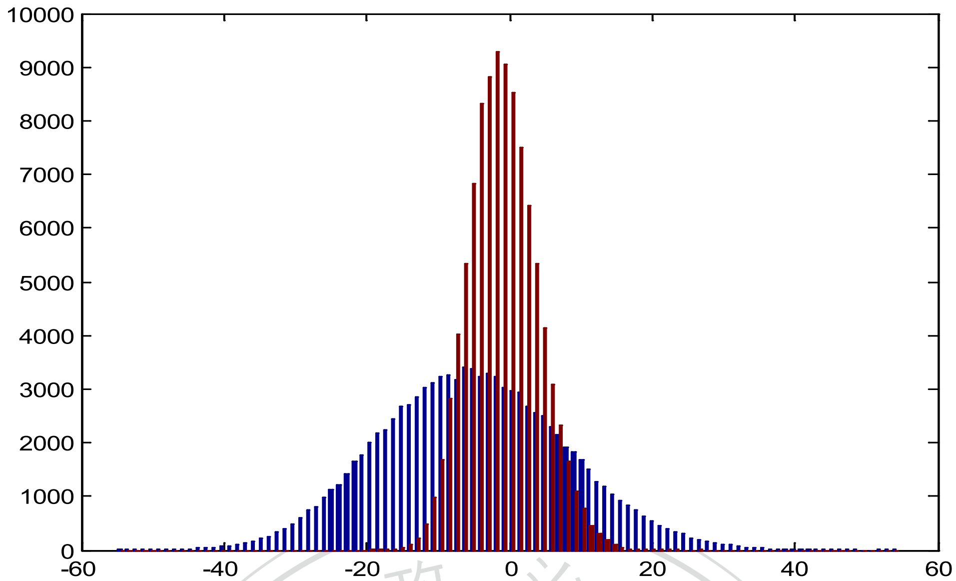

Figure 12: hedging effectiveness with objective function CTE

(Red):hedged profit function of portfolio. (blue): unhedged profit function of portfolio

5.Conclusions

This paper proposes the methodology to hedge mortality risk by life settlement. Using zero coupon bonds and life settlement to hedging the interest rate and mortality risk, we find the risk on the liability side is effectively reduced. Furthermore we have derived the closed-form optimal solution under mean variance assumption. Hedging strategies with mean variance objective function can adjust the parameter θ to reflect their risk aversion. We also investigate alternative objective function such as VaR and CTE, the result is more attractive for insurance companies, it reduces the downside risk without sacrificing upside profit.

Our hedging approaches is flexible. Even we change the interest rate or mortality rate model, the methodology in this paper is still adoptable. This hedging strategy is also applicable in practice for insurance companies which have complicated liabilities structures. Under the mean variance objective function assumption, the larger value θ

is, the more emphasis on reducing variance of portfolio. The mean variance hedging strategy is similar to the strategy of VaR and CTE objective functions, the main target is to control the downside risk. In order to control the downside risk, we not only need to care about the variance but also need to take the mean of portfolio into account. Therefore life settlement can be regard as effective hedging instrument to controlling the mortality risk for insurance companies.

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Appendix:

1.Karush-Kuhn-Tucker (KKT) optimality conditions:

Consider the constrained optimization problem:

$$\min_{x} f(x)$$

$$\text{s.t.}\begin{cases} g_j(x) \le 0, & j = 1, \dots, m \\ h_l(x) = 0, & l = 1, \dots, r \end{cases}$$

The Lagrangian Function is given by

$$L(x, \mu, \lambda) = f(x) + \sum_{j=1}^{m} \mu_j g_j(x) + \sum_{l=1}^{r} \lambda_l h_l(x)$$

IF x* is an optimal solution of the problem, then there exist Lagrange multipliers μ^* and λ^* such that

$$\nabla f(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) + \sum_{l=1}^{r} \lambda_l^* \nabla h_l(x^*) = 0$$

$$g_j(x^*) \le 0 \ \ \forall\, j = 1, \dots, m$$

$$h_l(x^*) = 0 \ \ \forall\, l = 1, \dots, r$$

$$\mu_j^* \ge 0 \ \ \forall\, j = 1, \dots, m$$

$$\mu_j^* g_j(x^*) = 0 \ \ \forall\, j = 1, \dots, m$$

This is the KKT condition

2.Solution of the optimal hedging problem

The Lagrangian function can be written as

$$L(u, \mu, \lambda) = f(u) - \mu' u + \lambda(u' a - 1)$$

The KKT conditions imply the following system of equations:

$$\nabla f(u) - \mu + \lambda a = 0 \quad (1)$$

where

$$\nabla f(u) = \frac{\partial f(u)}{\partial u} = -m + \theta[2\Sigma_{11} u - 2\Sigma_{12}]$$

$$u'a - 1 = 0 \quad (2)$$

$$-u \leq 0 \quad (3)$$

$$\mu \geq 0 \quad (4)$$

$$\mu_i u_i = 0 \; i = 1, \dots, N \quad (5)$$

Case 1: $u_i > 0 \; \forall i$

(1)=>

$$-m + \theta[2\Sigma_{11}u - 2\Sigma_{12}] - \mu + \lambda a = 0$$

$$\Rightarrow \Sigma_{11}u = \frac{1}{2\theta}(\mu + m - \lambda a) + \Sigma_{12}$$

$$u = \frac{1}{2\theta}\Sigma_{11}^{-1}(\mu + m - \lambda a) + \Sigma_{11}^{-1}\Sigma_{12}$$

From (5), IF $u_i \neq 0 \; \forall$ i then $\mu_i = 0 \; \forall$ i so we have

$$u = \frac{1}{2\theta}\Sigma_{11}^{-1}(m - \lambda a) + \Sigma_{11}^{-1}\Sigma_{12} \quad (*)$$

substitute u into (2) we can solve λ easily

$$a'u = 1$$

$$\Rightarrow a'\left\{\frac{1}{2\theta}\Sigma_{11}^{-1}(m - \lambda a) + \Sigma_{11}^{-1}\Sigma_{12}\right\} = 1$$

$$\Rightarrow \frac{1}{2\theta}a'\Sigma_{11}^{-1}(m - \lambda a) + a'\Sigma_{11}^{-1}\Sigma_{12} = 1$$

$$\Rightarrow a'\Sigma_{11}^{-1}m - \lambda a'\Sigma_{11}^{-1}a = 2\theta\left(1 - a'\Sigma_{11}^{-1}\Sigma_{12}\right)$$

$$\lambda = \frac{a'\Sigma_{11}^{-1}m - 2\theta\left(1 - a'\Sigma_{11}^{-1}\Sigma_{12}\right)}{a'\Sigma_{11}^{-1}a} \quad (**)$$

substitute (**) into (*), we get the desired optimal asset allocation.

$$u = \frac{1}{\theta}\Sigma_{11}^{-1}\left(m - \frac{a'\Sigma_{11}^{-1}m - 2\theta\left(1 - a'\Sigma_{11}^{-1}\Sigma_{12}\right)}{a'\Sigma_{11}^{-1}a}a\right) + 2\Sigma_{11}^{-1}\Sigma_{12}$$

Case 2: $u_i = 0$ for some i's

suppose there are k $u's$ being zero say

$$u_{(1)} = u_{(2)} = \cdots = u_{(k)} = 0 \quad \{(1), \dots, (k)\} \in \{1, \dots, N\}$$

and $(1) \leq (2) \leq \cdots \leq (k)$

The other $u's$ are nonzero called

$$u_{(1)'}, u_{(2)'}, \dots, u_{(N-k)'}$$

By (5) of KKT condition, we can say $\mu_{(1)}, \dots, \mu_{(k)}$ are nonzero, The others are all zero called $\mu_{(1)'}, \dots, \mu_{(N-k)'}$

From the expression of u

$$u = \frac{1}{2\theta}\Sigma_{11}^{-1}(\mu + m - \lambda a) + \Sigma_{11}^{-1}\Sigma_{12}$$

Define $u_A = (u_{(1)}, u_{(2)}, \dots, u_{(k)})'$,we have

$$u_A = 0 = \frac{1}{2\theta}\Sigma_{11}^{-1}((1){:}\,(k), :)(\mu + m - \lambda a) + \Sigma_{11}^{-1}((1){:}\,(k), :)\Sigma_{12}$$

here $\Sigma_{11}^{-1}((1){:}\,(k), :)$ denotes the matrix obtained by picking rows (1), (2),...,(k) from Σ_{11}^{-1}.

we also define $\mu_A = (\mu_{(1)}, \dots, \mu_{(k)})'$, then

$$\frac{1}{2\theta}\Sigma_{11}^{-1}\big((1){:}\,(k), (1){:}\,(k)\big)\mu_A + \frac{1}{2\theta}\Sigma_{11}^{-1}\big((1){:}\,(k), :\big)m - \frac{\lambda}{2\theta}\Sigma_{11}^{-1}\big((1){:}\,(k), :\big)a$$
$$+ \Sigma_{11}^{-1}((1){:}\,(k), :)\Sigma_{12} = 0$$

$$\Rightarrow \Sigma_{11}^{-1}\big((1){:}\,(k), (1){:}\,(k)\big)\mu_A$$
$$= \lambda\Sigma_{11}^{-1}\big((1){:}\,(k), :\big)a - \Sigma_{11}^{-1}\big((1){:}\,(k), :\big)m - 2\theta\Sigma_{11}^{-1}((1){:}\,(k), :)\Sigma_{12}$$

$$\Rightarrow \mu_A$$
$$= \Sigma_{11}\big((1){:}\,(k), (1){:}\,(k)\big)\big[\lambda\Sigma_{11}^{-1}\big((1){:}\,(k), :\big)a - \Sigma_{11}^{-1}\big((1){:}\,(k), :\big)m$$
$$- 2\theta\Sigma_{11}^{-1}((1){:}\,(k), :)\Sigma_{12}\big]$$

where $\Sigma_{11}^{-1}\big((1){:}\,(k), (1){:}\,(k)\big)$ means picking rows (1), (2),...,(k) and columns (1), (2),...,(k) from Σ_{11}^{-1} to form the new submatrix.

The remaining part is to solve λ

By the (2) of KKT condition, and define $a_B = (a_{(1)'}, a_{(2)'}, \dots, a_{(N-k)'})'$ and

$u_B = (u_{(1)'}, u_{(2)'}, \dots, u_{(N-k)'})'$

$$a_B' u_B = 1$$

we have

$$a_B' \left\{ \frac{1}{2\theta} \Sigma_{11}^{-1}((1)' : (N-k)', :)(m - \lambda a) + \Sigma_{11}^{-1}((1)' : (N-k)', :)\Sigma_{12} \right\} = 1$$

$$\Rightarrow a_B' \Sigma_{11}^{-1}((1)' : (N-k)', :)(m - \lambda a) = 2\theta\left[1 - a_B' \Sigma_{11}^{-1}((1)' : (N-k)', :)\Sigma_{12}\right]$$

$$\lambda = \frac{a_B' \Sigma_{11}^{-1}((1)' : (N-k)', :)m - 2\theta\left[1 - a_B' \Sigma_{11}^{-1}((1)' : (N-k)', :)\Sigma_{12}\right]}{a_B' \Sigma_{11}^{-1}((1)' : (N-k)', :)a}$$