1. Introduction

In 1927, L. H. Thomas and E. Fermi [15] independently gave a method of studying the electron distribution in an atom, using the statistics for a degenerate gas. This led to a nonlinear second-order differential equation

$$x^{\frac{1}{2}} \cdot y^{''}(x) = [y(x)]^{\frac{3}{2}}.$$

The physicists were interested in three boundary value problem for this equation,

$$y(0) = 1, \quad ry^{'}(r) = y(r),$$

$$y(0) = 1, \quad \lim_{x \to \infty} y(x) = 0,$$

$$y(0) = 1, \quad y(a) = 0.$$

In the first problem, r is the Bohr atom radius. The second problem corresponds to the neutral atom, whereas the last is the ion case. The physicists presumably have got out of the Thomas-Fermi theory all that is of interest to them. They have enriched the mathematical literature by the equation above and it seems to be time for mathematicians to react to the challenge by discovering the astounding properties of the solutions. The present note is a preliminary account of results obtained for which proofs will be published elsewhere [15].

In R. Bellman [14], the important nonlinear second-order equation

$$\frac{d}{dt}\left(t^{p}\frac{du}{dt}\right) \pm t^{\sigma}u^{n} = 0,$$

this equation has several interesting physical applications, occurring in astrophysics in the form of the Emden equation and in atomic physics in the form of the Fermi-Thomas equation. There seems little doubt that nonlinear equations of this type would enter with greater frequency into mathematical physics, were it more widely known with what ease the properties of the physical solutions can be determined.

Mathematically, the equation possesses great interest: it is a nontrivial, nonlinear differential equation with a large class of solutions whose behavior can be ascertained with astonishing accuracy, despite the fact that the solutions, in general, cannot be obtained explicitly.

In order to isolate this large class of tractable solutions, we employ the concept of proper solution, which is a real and continuous solution for $t \geq t_0$. Henceforth we shall confine ourselves to the consideration of proper solutions alone. In order to remind the reader of this fact, we shall constantly insert this assumption into our hypotheses. This assumption is a natural one as far as physical applications are concerned.

In papers Li $[1-8]$ the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, some interesting results on blow-up, blow-up rate and estimates for the life-span of solutions are obtained. We want to use the methods in $[9-13]$ to study the case of Emden-Fowler type wave equation in $0-$dimensional form, that is, to consider the equation $t^2 u'' - \triangle u = u^p$, with space dimension $n = 0$ for $t \geq 1$ under the given initial conditions: $u(1) = u_0$, $u'(1) = u_1$. Using the transformation $t = e^s$, $u(t) = v(s)$, we have

$$\begin{cases} v_{ss}(s) - v_s(s) = v(s)^p, & p \in \mathbb{N} - \{1\}, \\ v(0) = u_0, \quad v_s(0) = u_1. \end{cases}$$

We discuss this problem into three different cases under some given conditions: (a) $u_1 = 0$, $u_0 > 0$, (b) $u_1 > 0$, $u_0 > 0$, (c) $u_1 < 0$, $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$ and we have the following main result for $u_1 < 0$, $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$:

$$u(t) \leq (u_0 - (u_1 + u_0^p)) + (u_1 + u_0^p)\, t - u_0^p \ln t.$$

Furthermore, for $E(0) \geq 0$,

$$u(t) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} \ln t \right)^{\frac{2}{1-p}}.$$

2. Local Existence of Solutions

In this chapter we establish the local existence and uniqueness result for the problem of particular type of Emden-Fowler equation.

Theorem 1. *For $p \in \mathbb{N}$, these functions $t^{-2}u^p$ for $t \geq 1$ are locally Lipschitz, the local existence and uniqueness of solutions of the equation*

$$\begin{cases} t^2 u^{''} = u^p, \qquad p \in \mathbb{N} - \{1\}, \\ u(1) = u_0, \quad u^{'}(1) = u_1, \end{cases} \tag{*}$$

can be obtained.

Proof. Let us consider the transformation $t = e^s$, $u(t) = v(s)$, then

$$u^{'}(t) = v_s(s)\frac{ds}{dt} = t^{-1}v_s(s),$$

$$u^{''}(t) = \frac{d}{dt}\left(t^{-1}v_s(s)\right) = -t^{-2}v_s(s) + t^{-2}v_{ss}(s),$$

$$t^2 u^{''}(t) = -v_s(s) + v_{ss}(s),$$

$$u(t)^p = v(s)^p = -v_s(s) + v_{ss}(s)$$

and

$$v(0) = u(1) = u_0; \quad v_s(0) = u^{'}(1) = u_1.$$

Therefore, we obtain

$$\begin{cases} v_{ss}(s) - v_s(s) = v(s)^p, \qquad p \in \mathbb{N} - \{1\}, \\ v(0) = u_0, \quad v_s(0) = u_1. \end{cases} \tag{2.1}$$

Thus, the local existence of solution u for $(*)$ in $[1, T)$ is equivalent to the local existence of solution v for (2.1) in $[0, \ln T)$.

Since $v_{ss}(s) - v_s(s) = v(s)^p$, by integrating this equation with respect to s, we obtain

$$v_s(s) = u_1 + \int_0^s v(r)^p\,dr + v(s) - u_0$$

$$= (u_1 - u_0) + v(s) + \int_0^s v(r)^p\,dr.$$

Again, by integrating the above equation with respect to s, we have

$$v(s) = u_0 + (u_1 - u_0)s + \int_0^s v(r)\,dr + \int_0^s \int_0^\eta v(r)^p\,dr\,d\eta$$

$$= u_0 + (u_1 - u_0)s + \int_0^s v(r)\,dr + \int_0^s (s-r)\,v(r)^p\,dr$$

$$= u_0 + (u_1 - u_0)s + \int_0^s \Big(v(r) \ + \ (s-r)\,v(r)^p\Big)\,dr.$$

Let us denote

$$F(v(s)) := u_0 + (u_1 - u_0)s + \int_0^s \Big(v(r) \ + \ (s-r)\,v(r)^p\Big)\,dr \tag{2.2}$$

and for $k \in (0,1)$,

$$X = \Big\{ v \in C^0\Big[0\ ,\ \ln T\Big)\ ,\ \|v\|_\infty \le M \Big\},$$

$$M = \frac{k\Big|u_1 \ - \ u_0\Big| + |u_0|}{\Big(1 \ - \ k\Big)\Big(1 - \dfrac{1}{p}\Big)}, \quad k = \ln T,$$

with

$$pM^{p-1}k^2 \ + \ 2k < 2.$$

For $s < \ln T$, $v \in X$, by (2.2), we have

$$|F(v(s))| \le |u_0| + \Big|u_1 \ - \ u_0\Big|s + Ms + M^p \cdot \frac{1}{2}s^2$$

$$= |u_0| + \Big(\Big|u_1 \ - \ u_0\Big| + M\Big)s + \frac{1}{2}M^p s^2$$

$$= |u_0| + \Big(\Big|u_1 \ - \ u_0\Big| + M\Big)s + \Big(\frac{p}{2}M^{p-1}s^2\Big)\frac{M}{p}$$

and

$$\begin{aligned}|F(v(s))| &\le |u_0| + \left(\left|u_1 - u_0\right| + M\right)k + \left(\frac{p}{2}M^{p-1}k^2\right)\frac{M}{p} \\ &\le |u_0| + \left(\left|u_1 - u_0\right| + M\right)k + \frac{M}{p}(1-k) \\ &\le |u_0| + \frac{M}{p} + \left(\left|u_1 - u_0\right| + M - \frac{M}{p}\right)k \\ &= |u_0| + \frac{M}{p} + \left(1 - \frac{1}{p}\right)M - |u_0| = M.\end{aligned}$$

Therefore we obtain that $F : X \to X$, this means that F maps X into X itself.

Next, we claim that F is a contractive map. By (2.2) again, we have

$$F(v(s)) - F(w(s)) = \int_0^s \left(v(r) - w(r)\right) dr + \int_0^s (s-r)\left(v(r)^p - w(r)^p\right) dr$$

and

$$\begin{aligned}&\left|F(v(s)) - F(w(s))\right| \\ &\le \int_0^s \left|v(r) - w(r)\right| dr + \int_0^s (s-r)\left|v(r)^p - w(r)^p\right| dr \\ &\le \int_0^s \left|v(r) - w(r)\right| dr + \int_0^s (s-r)\,pM^{p-1}\left|v(r) - w(r)\right| dr \\ &= \int_0^s \left(p(s-r)M^{p-1} + 1\right)\left|v(r) - w(r)\right| dr \\ &\le \left(pM^{p-1}\cdot\frac{1}{2}s^2 + s\right)\left\|v - w\right\|_\infty \\ &\le \left(\frac{1}{2}pM^{p-1}\left(\ln T\right)^2 + \ln T\right)\left\|v - w\right\|_\infty .\end{aligned}$$

From the definition of M and T we can find that $\frac{1}{2}pM^{p-1}\left(\ln T\right)^2 + \ln T < 1$, thus F is contractive in X and the local existence of solution of v of (2.1) can be obtained.■

3. Notation and Fundamental Lemmas

For a given function v in this work we use the following abbreviations

$$a(s) = v(s)^2,\ E(0) = u_1^2 - \frac{2}{p+1} u_0^{p+1},\ J(s) = a(s)^{-\frac{p-1}{4}}.$$

After some calculations we can obtain the follwing lemmas 2 and 3.

Lemma 2. *Suppose that $v \in C^2[0, \ln T)$ is the positive solution of* (2.1), *then*

$$E(s) = v_s(s)^2 - 2\int_0^s v_s(r)^2\, dr - \frac{2}{p+1} v(s)^{p+1} = E(0), \tag{3.1}$$

$$(p+3)\, v_s(s)^2 = (p+1)\, E(0) + a''(s) - a'(s) + 2(p+1)\int_0^s v_s(r)^2\, dr, \tag{3.2}$$

$$J''(s) = \frac{p^2-1}{4} J(s)^{\frac{p+3}{p-1}} \left(E(0) - \frac{a'(s)}{p+1} + 2\int_0^s v_s(r)^2\, dr \right) \tag{3.3}$$

and

$$J'(s)^2 = J'(0)^2 + \frac{(p-1)^2}{4} E(0) \left(J(s)^{\frac{2(p+1)}{p-1}} - J(0)^{\frac{2(p+1)}{p-1}} \right) + \frac{(p-1)^2}{2} J(s)^{\frac{2(p+1)}{p-1}} \int_0^s v_s(r)^2\, dr. \tag{3.4}$$

Proof. i) We claim the conservation (3.1). By (2.1) and the definition of $E(s)$,

$$\begin{aligned}
\frac{dE(s)}{ds} &= \frac{d}{ds}\left(v_s(s)^2 - 2\int_0^s v_s(r)^2\, dr - \frac{2}{p+1} v(s)^{p+1} \right) \\
&= 2v_s(s)\, v_{ss}(s) - 2v_s(s)^2 - 2v(s)^p\, v_s(s) \\
&= 2v_s(s)\left(v_{ss}(s) - v_s(s) - v(s)^p \right) \\
&= 0.
\end{aligned}$$

ii) To (3.2), using the definition of a,

$$a^{'}(s) = 2v(s)\, v_s(s),$$

$$\begin{aligned} a^{''}(s) &= 2v(s)\, v_{ss}(s) + 2v_s(s)^2 \\ &= 2v(s)\left(v_s(s) + v(s)^p\right) + 2v_s(s)^2 \\ &= a^{'}(s) + 2\left(v_s(s)^2 + v(s)^{p+1}\right). \end{aligned} \tag{3.5}$$

By (3.1),

$$E(0) = v_s(s)^2 - 2\int_0^s v_s(r)^2\, dr - \frac{2}{p+1} v(s)^{p+1},$$

$$(p+1)\, E(0) = (p+1)\, v_s(s)^2 - 2(p+1)\int_0^s v_s(r)^2\, dr - 2v(s)^{p+1}.$$

By (3.5),

$$2v(s)^{p+1} = a^{''}(s) - a^{'}(s) - 2v_s(s)^2,$$

$$(p+3)\, v_s(s)^2 = (p+1)\, E(0) + a^{''}(s) - a^{'}(s) + 2(p+1)\int_0^s v_s(r)^2\, dr.$$

iii) For (3.3), use the definition of $J(s)$, we have

$$J^{'}(s) = -\frac{p-1}{4} a(s)^{-\frac{p+3}{4}} a^{'}(s),$$

$$J^{''}(s) = \frac{p-1}{4} J(s)^{\frac{p+3}{p-1}} \left((p+3)\frac{a(s)^{-1} a^{'}(s)^2}{4} - a^{''}(s)\right). \tag{3.6}$$

By (3.6) and the definition of a, we obtain

$$\begin{aligned} J^{''}(s) &= \frac{p^2-1}{4} J(s)^{\frac{p+3}{p-1}} \left((p+3)\frac{a(s)^{-1} a^{'}(s)^2}{4(p+1)} - \frac{a^{''}(s)}{p+1}\right) \\ &= \frac{p^2-1}{4} J(s)^{\frac{p+3}{p-1}} \left(\frac{(p+3)\, v_s(s)^2 - a^{''}(s)}{p+1}\right). \end{aligned} \tag{3.7}$$

By (3.2) and (3.7), then we have

$$J^{''}(s) = \frac{p^2-1}{4} J(s)^{\frac{p+3}{p-1}} \left(E(0) - \frac{a^{'}(s)}{p+1} + 2\int_0^s v_s(r)^2\, dr\right).$$

iv) From the definition of J, we have

$$J^{'}(s)^2 - J^{'}(0)^2 = \frac{(p-1)^2}{4^2}\left(a(s)^{-\frac{p+3}{2}} a^{'}(s)^2 - a(0)^{-\frac{p+3}{2}} a^{'}(0)^2\right)$$

and

$$a^{'}(s)^2 = 4v(s)^2 v_s(s)^2 = 4a(s) v_s(s)^2,$$

we get

$$\begin{aligned} J^{'}(s)^2 - J^{'}(0)^2 &= \frac{(p-1)^2}{4}\left(a(s)^{-\frac{p+1}{2}} v_s(s)^2 - a(0)^{-\frac{p+1}{2}} v_s(0)^2\right) \\ &= \frac{(p-1)^2}{4}\left(J(s)^{\frac{2(p+1)}{p-1}} v_s(s)^2 - J(0)^{\frac{2(p+1)}{p-1}} v_s(0)^2\right). \end{aligned}$$

By (3.1), then

$$\begin{aligned} & J^{'}(s)^2 - J^{'}(0)^2 \\ &= \frac{(p-1)^2}{4} J(s)^{\frac{2(p+1)}{p-1}}\left(E(0) + 2\int_0^s v_s(r)^2\,dr + \frac{2}{p+1} v(s)^{p+1}\right) \\ &- \frac{(p-1)^2}{4} J(0)^{\frac{2(p+1)}{p-1}}\left(E(0) + \frac{2}{p+1} u_0^{p+1}\right) \\ &= \frac{(p-1)^2}{4} E(0) J(s)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{2(p+1)} J(s)^{\frac{2(p+1)}{p-1}} v(s)^{p+1} \\ &+ \frac{(p-1)^2}{2} J(s)^{\frac{2(p+1)}{p-1}} \int_0^s v_s(r)^2\,dr - \frac{(p-1)^2}{4} J(0)^{\frac{2(p+1)}{p-1}} E(0) \\ &- \frac{(p-1)^2}{2(p+1)} J(0)^{\frac{2(p+1)}{p-1}} u_0^{p+1}, \end{aligned}$$

$$\begin{aligned} & J^{'}(s)^2 - J^{'}(0)^2 \\ &= \frac{(p-1)^2}{4} E(0) J(s)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{2(p+1)} v(s)^{-\frac{p-1}{2}\frac{2(p+1)}{p-1}} v(s)^{p+1} \\ &+ \frac{(p-1)^2}{2} J(s)^{\frac{2(p+1)}{p-1}} \int_0^s v_s(r)^2\,dr - \frac{(p-1)^2}{4} J(0)^{\frac{2(p+1)}{p-1}} E(0) \\ &- \frac{(p-1)^2}{2(p+1)} v(0)^{-\frac{p-1}{2}\frac{2(p+1)}{p-1}} u_0^{p+1} \end{aligned}$$

and

$$J^{'}(s)^2 - J^{'}(0)^2 = \frac{(p-1)^2}{4} E(0)\left(J(s)^{\frac{2(p+1)}{p-1}} - J(0)^{\frac{2(p+1)}{p-1}}\right) + \frac{(p-1)^2}{2} J(s)^{\frac{2(p+1)}{p-1}} \int_0^s v_s(r)^2\, dr.$$

Lemma 3. *For $u_0 > 0$, the positive solution v of the equation (2.1), we have:*

$$\text{i) } u_1 \geq 0, \text{ then } v_s(s) > 0 \quad \textit{for all} \quad s > 0. \tag{3.8}$$

$$\text{ii) } u_1 < 0,\ u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right), \textit{ then } v_s(s) < 0 \quad \textit{for all} \quad s > 0. \tag{3.9}$$

Proof. i) Since $v_{ss}(s) = v_s(s) + v(s)^p$ and $v_{ss}(0) = v_s(0) + v(0)^p = u_1 + u_0^p > 0$, we know that $v_{ss}(s) > 0$ in $[0, s_1)$ for some $s_1 > 0$ and $v_s(s)$ is increasing in $[0, s_1)$ for some $s_1 > 0$. Then, $v_s(s) > v_s(0) = u_1 \geq 0$ for all $s \in [0, s_1)$ and $v(s)$ is increasing in $[0, s_1)$ for some $s_1 > 0$.

Moreover, since v and v_s are increasing in $[0, s_1)$,

$$v_{ss}(s_1) = v_s(s_1) + v(s_1)^p > v_s(0) + v(0)^p > 0 \quad \text{for all} \quad s \in [0, s_1)$$

and

$$v_s(s_1) > v_s(s) > 0 \quad \text{for all} \quad s \in [0, s_1),$$

we know that there exists a positive number $s_2 > 0$, such that $v_s(s) > 0$ for all $s \in [0, s_1 + s_2)$.

Continuing such process, we obtain $v_s(s) > 0$ for all $s > 0$.

ii) Since $v_{ss}(s) = v_s(s) + v(s)^p$ and $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$, $v_{ss}(0) = v_s(0) + v(0)^p = u_1 + u_0^p < 0$, there exists a positive number $s_1 > 0$ such that

$$v_{ss}(s) < 0 \quad \text{in} \quad [0, s_1),$$

$v_s(s)$ is decreasing in $[0, s_1)$; therefore, $v_s(s) < v_s(0) = u_1 < 0$ for all $s \in [0, s_1)$ and $v(s)$ is decreasing in $[0, s_1)$ for some $s_1 > 0$.

Moreover, since v and v_s are decreasing in $[0, s_1)$,

$$v_{ss}(s_1) = v_s(s_1) + v(s_1)^p < v_s(0) + v(0)^p < 0 \quad \text{for all} \quad s \in [0, s_1)$$

and

$$v_s(s_1) < v_s(s) < 0 \quad \text{for all} \quad s \in [0, s_1),$$

we know that there exists a positive number $s_2 > 0$, such that $v_s(s) < 0$ for all $s \in [0, s_1 + s_2)$.

Continuing such process, we obtain $v_s(s) < 0$ for all $s > 0$.■

4. Estimates for the life-span of positive solution u of $(*)$ under $u_1 = 0$, $u_0 > 0$

In this chapter we want to estimate the life-span of positive solution u of $(*)$ under $u_1 = 0$, $u_0 > 0$. Here the life-span T^* of u means that u is the positive solution of equation $(*)$ and u exists only in $[1, T^*)$ so that the problem $(*)$ possesses the positive solution $u \in C^2[1, T^*)$ for $T < T^*$.

Theorem 4. *For $u_1 = 0$, $u_0 > 0$, the positive solution u of $(*)$ blows up in finite time; that is, there exists a bound number T^* so that*

$$u(t)^{-1} \to 0 \quad \textit{for} \quad t \to T^*.$$

Proof. By lemma 3, (3.8), we know that $v_s(s) > 0$ for all $s > 0$ under $u_1 = 0$, $u_0 > 0$.

From the definition of $a(s)$, we have

$$a'(s) = 2v(s)\, v_s(s) > 0 \quad \text{for all} \quad s > 0.$$

By lemma 2, (3.5), we have $a''(s) - a'(s) = 2\left(v_s(s)^2 + v(s)^{p+1}\right)$.

By multiplying e^{-s} to the above equation, we obtain

$$\left(a'(s)\, e^{-s}\right)' = e^{-s}\left(a''(s) - a'(s)\right) = 2e^{-s}\left(v_s(s)^2 + v(s)^{p+1}\right).$$

By integrating the above equation with respect to s, then

$$\begin{aligned} a'(s)\, e^{-s} - a'(0) &= 2\int_0^s e^{-r}\left(v_s(r)^2 + v(r)^{p+1}\right) dr \\ &\geq 4\int_0^s e^{-r}\left(v_s(r) \cdot v(r)^{\frac{p+1}{2}}\right) dr \\ &= 4\int_0^s e^{-r} \cdot \frac{2}{p+3}\left(v(r)^{\frac{p+3}{2}}\right)_s dr \end{aligned}$$

and in addition, $a^{'}(0) = 0$, we have

$$a^{'}(s)\, e^{-s} \geq \frac{8}{p+3}\left(v(r)^{\frac{p+3}{2}} e^{-r} \mid_{r=0}^{s} + \int_0^s v(r)^{\frac{p+3}{2}} e^{-r} dr\right)$$

$$= \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} e^{-s} - v(0)^{\frac{p+3}{2}}\right) + \frac{8}{p+3}\int_0^s v(r)^{\frac{p+3}{2}} e^{-r} dr.$$

Since $a^{'}(s) > 0$ for all $s > 0$, we know that a is increasing in $(0, \infty)$. From the definition of $a(s)$, v is increasing in $(0, \infty)$ and

$$a^{'}(s)\, e^{-s} \geq \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} e^{-s} - v(0)^{\frac{p+3}{2}}\right) + \frac{8}{p+3}\int_0^s v(0)^{\frac{p+3}{2}} e^{-r} dr$$

$$= \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} e^{-s} - v(0)^{\frac{p+3}{2}}\right) + \frac{8}{p+3} v(0)^{\frac{p+3}{2}}\left(1 - e^{-s}\right)$$

and

$$a^{'}(s) \geq \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} - v(0)^{\frac{p+3}{2}} e^{s} + v(0)^{\frac{p+3}{2}}\left(e^{s} - 1\right)\right)$$

$$= \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} - v(0)^{\frac{p+3}{2}}\right)$$

$$= \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} - u_0^{\frac{p+3}{2}}\right). \tag{4.1}$$

Since $v_{ss}(s) = v_s(s) + v(s)^p$, $u_1 = 0$ and by integrating this equation with respect to s, we have

$$v_s(s) = v_s(0) + v(s) - v(0) + \int_0^s v(r)^p\, dr$$

$$= u_1 + v(s) - u_0 + \int_0^s v(r)^p\, dr$$

$$= v(s) - u_0 + \int_0^s v(r)^p\, dr$$

and v is increasing in $(0, \infty)$, then

$$v_s(s) \geq v(s) - u_0 + \int_0^s v(0)^p\, dr$$

$$= v(s) - u_0 + u_0^p \cdot s$$

and

$$v_s\left(s\right)-v\left(s\right)\geq u_0^p\cdot s-u_0.$$

Multiplying the above inequality by e^{-s}, we have

$$\left(e^{-s}v\left(s\right)\right)_s=e^{-s}\left(v_s\left(s\right)\ -\ v\left(s\right)\right)\geq e^{-s}\left(u_0^p\cdot s\ -\ u_0\right).$$

By integrating the above inequality with respect to s, we obtain

$$\begin{aligned}e^{-s}v\left(s\right)-u_0&\geq u_0^p\int_0^s re^{-r}\,dr-u_0\int_0^s e^{-r}\,dr\\&=u_0^p\left(-se^{-s}\ -\ e^{-s}+1\right)-u_0\left(1\ -\ e^{-s}\right)\end{aligned}$$

and

$$v\left(s\right)\geq u_0+u_0^p\left(e^s\ -\ 1-s\right). \tag{4.2}$$

According to (4.2), we get

$$v\left(s\right)^{\frac{p+3}{2}}\geq\left(u_0+u_0^p\left(e^s\ -\ 1-s\right)\right)^{\frac{p+3}{2}}$$

and for all $\epsilon\in(0,1)$, we have

$$\epsilon v\left(s\right)^{\frac{p+3}{2}}\geq\epsilon\left(u_0+u_0^p\left(e^s\ -\ 1-s\right)\right)^{\frac{p+3}{2}}$$

and

$$\begin{aligned}\epsilon v\left(s\right)^{\frac{p+3}{2}}-8u_0^{\frac{p+3}{2}}&\geq\epsilon\left(u_0+u_0^p\left(e^s\ -\ 1-s\right)\right)^{\frac{p+3}{2}}-8u_0^{\frac{p+3}{2}}\\&\geq\epsilon\left(u_0^{\frac{p+3}{2}}+u_0^{\frac{p(p+3)}{2}}\left(e^s\ -\ 1-s\right)^{\frac{p+3}{2}}\right)-8u_0^{\frac{p+3}{2}}\\&=\left(\epsilon-8\right)u_0^{\frac{p+3}{2}}+\epsilon u_0^{\frac{p(p+3)}{2}}\left(e^s\ -\ 1-s\right)^{\frac{p+3}{2}}.\end{aligned}$$

Now, we want to find a number $s_0>0$ such that

$$\left(\epsilon-8\right)u_0^{\frac{p+3}{2}}+\epsilon u_0^{\frac{p(p+3)}{2}}\left(e^{s_0}\ -\ 1-s_0\right)^{\frac{p+3}{2}}=0,$$

that is,

$$\left(e^{s_0} - 1 - s_0\right)^{\frac{p+3}{2}} = \frac{8-\epsilon}{\epsilon} u_0^{\frac{p+3}{2}(1-p)}$$

and

$$e^{s_0} - s_0 = 1 + \left(\frac{8-\epsilon}{\epsilon} u_0^{\frac{p+3}{2}(1-p)}\right)^{\frac{2}{p+3}}. \tag{4.3}$$

This means that there exists a number $s_0 > 0$ satisfying (4.3) with $\epsilon \in (0,1)$ such that

$$\epsilon v(s)^{\frac{p+3}{2}} - 8u_0^{\frac{p+3}{2}} \geq 0 \quad \text{for all} \quad s \geq s_0.$$

By (4.1), we have

$$\begin{aligned} a'(s) &\geq \frac{8}{p+3} v(s)^{\frac{p+3}{2}} - \frac{8}{p+3} u_0^{\frac{p+3}{2}} \\ &= \frac{8-\epsilon}{p+3} v(s)^{\frac{p+3}{2}} + \frac{\epsilon \cdot v(s)^{\frac{p+3}{2}} - 8u_0^{\frac{p+3}{2}}}{p+3} \\ &\geq \frac{8-\epsilon}{p+3} v(s)^{\frac{p+3}{2}} \quad \text{for all} \quad s \geq s_0. \end{aligned}$$

From the definition of a and for all $s \geq s_0$, $\epsilon \in (0,1)$, we obtain

$$2v(s) v_s(s) \geq \frac{8-\epsilon}{p+3} v(s)^{\frac{p+3}{2}},$$

$$v(s)^{-\frac{p+1}{2}} v_s(s) \geq \frac{8-\epsilon}{2(p+3)},$$

$$\frac{2}{1-p}\left(v(s)^{\frac{1-p}{2}}\right)_s \geq \frac{8-\epsilon}{2(p+3)}$$

and

$$\left(v(s)^{\frac{1-p}{2}}\right)_s \leq \frac{8-\epsilon}{2(p+3)} \cdot \frac{1-p}{2}.$$

Integrating the above inequality with respect to s, we have

$$v(s)^{\frac{1-p}{2}} \leq v(s_0)^{\frac{1-p}{2}} - \frac{8-\epsilon}{2(p+3)} \cdot \frac{p-1}{2}(s-s_0).$$

Thus, there exists a finite number

$$s_1^* \leq s_0 + \frac{2(p+3)}{8-\epsilon} \cdot \frac{2}{p-1} v(s_0)^{\frac{1-p}{2}} := k_1$$

such that $v(s)^{-1} \to 0$ for $s \to s_1^*$, that is,

$$u(t)^{-1} \to 0 \quad \text{for} \quad t \to \exp(k_1),$$

which implies that the life-span T^* of positive solution u is finite and $T^* \leq \exp(k_1)$.■

Graphs of positive solution u of $(*)$ under $u_1 = 0$, $u_0 > 0$:

According to $e^{s_0} - s_0 = 1 + u_0^{1-p}\left(\frac{8-\epsilon}{\epsilon}\right)^{\frac{2}{p+3}}$, $v(s_0)^{\frac{p+3}{2}} = \frac{8}{\epsilon}u_0^{\frac{p+3}{2}}$, $v(s_0) = \left(\frac{8}{\epsilon}\right)^{\frac{2}{p+3}} u_0$, we get that $v(s)^{\frac{1-p}{2}} \leq v(s_0)^{\frac{1-p}{2}} - \frac{8-\epsilon}{2(p+3)}\frac{p-1}{2}(s-s_0) = \left(\frac{8}{\epsilon}\right)^{\frac{1-p}{p+3}} u_0^{\frac{1-p}{2}} - \frac{8-\epsilon}{2(p+3)}\frac{p-1}{2}(s-s_0)$, this means that $u(t) \geq \left(\left(\frac{8}{\epsilon}\right)^{\frac{1-p}{p+3}} u_0^{\frac{1-p}{2}} - \frac{8-\epsilon}{2(p+3)}\frac{p-1}{2}(s-s_0)\right)^{\frac{2}{1-p}}$.

(a) Graphs for different ϵ:

1. Given $\epsilon = 0.1$, $p = 2$, $u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4 \cdot 79^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4 \cdot 79^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 3.3059$, $u \geq \frac{1}{(2.1384 - 0.395 \ln t)^2}$, $2.1384 - 0.395 \ln T = 0$, $T \sim 224.45$.

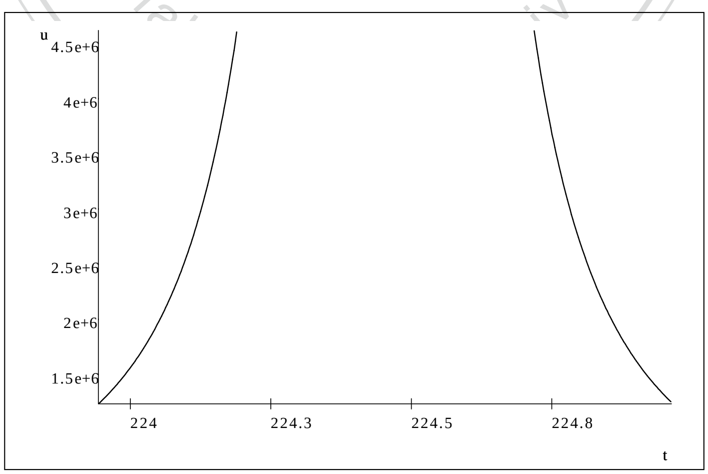

Figure 1: Graph of u

2. Given $\epsilon = 0.3$, $p = 2$, $u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4\left(\frac{77}{3}\right)^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4\left(\frac{77}{3}\right)^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 2.9216$, $u \geq \frac{1}{(2.162 - 0.385\ln t)^2}$, $2.162 - 0.385\ln T = 0$, $T \sim 274.67$.

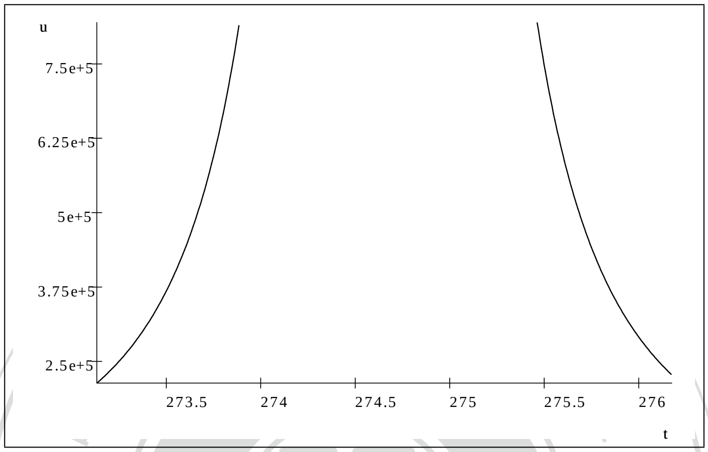

Figure 2: Graph of u

3. Given $\epsilon = 0.6$, $p = 2$, $u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4\left(\frac{74}{6}\right)^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4\left(\frac{74}{6}\right)^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 2.6816$, $u \geq \frac{1}{(2.1835 - 0.37\ln t)^2}$, $2.1835 - 0.37\ln T = 0$, $T \sim 365.53$.

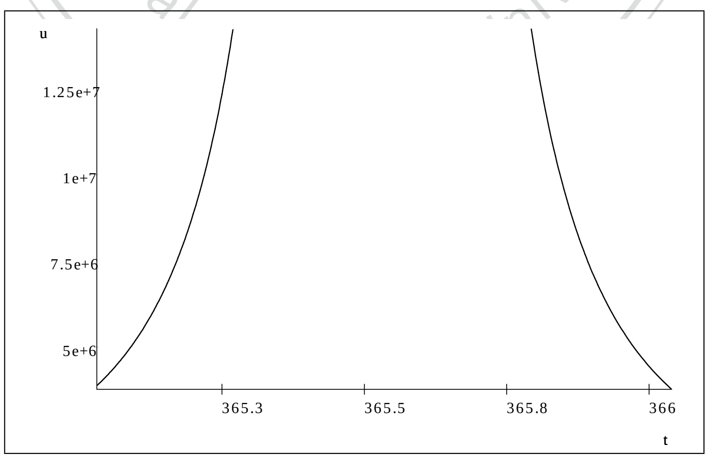

Figure 3: Graph of u

4. Given $\epsilon = 1, p = 2, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4 \cdot 7^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4 \cdot 7^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 2.5062$, $0.25 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln T + 0.87591 = 0$, $T \sim 529.87$, $u \geq \frac{1}{\left(0.25 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln t + 0.87591\right)^2}$.

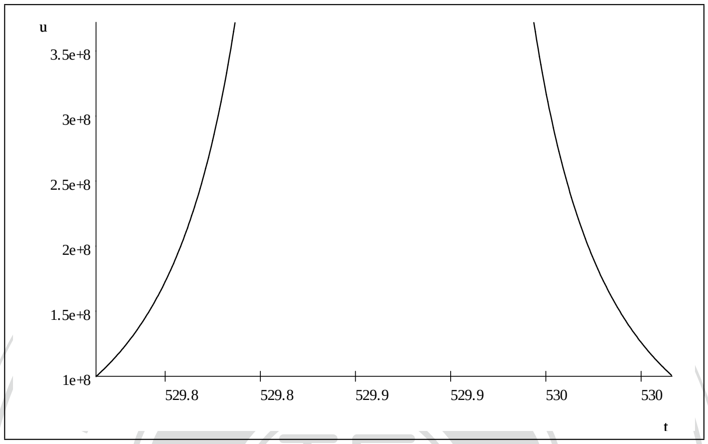

Figure 4: Graph of u

5. Given $\epsilon = 2, p = 2, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4 \cdot 3^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4 \cdot 3^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 2.2464$, $\frac{1}{2} \times 4^{\frac{4}{5}} - \frac{3}{10} \ln T + 0.67392 = 0$, $T \sim 1478.5$, $u \geq \frac{1}{\left(\frac{1}{2} \times 4^{\frac{4}{5}} - \frac{3}{10} \ln t + 0.673\,92\right)^2}$.

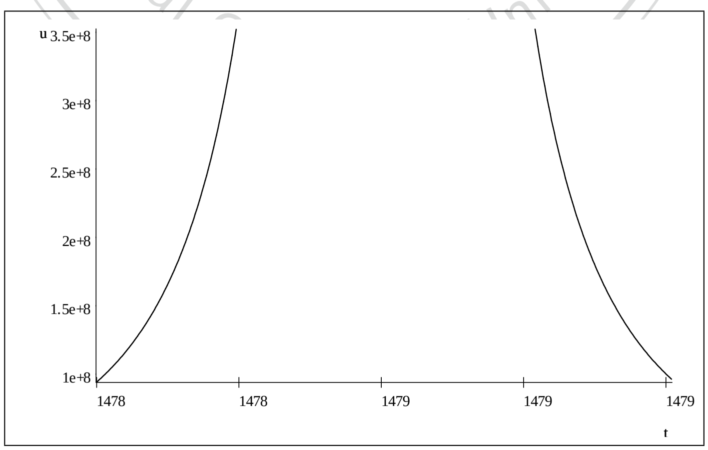

Figure 5: Graph of u

6. Given $\epsilon = 7, p = 2, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4\left(\frac{1}{7}\right)^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4\left(\frac{1}{7}\right)^{\frac{2}{5}}\right)$, using the soft for Solving Equation, the solution is: $s_0 \sim 2.2464$, $u \geq \frac{1}{\left(\frac{\sqrt[5]{7}}{4} 8^{\frac{4}{5}} - \frac{1}{20}\ln t + 0.07286\right)^2}$, $\frac{\sqrt[5]{7}}{4} 8^{\frac{4}{5}} - \frac{1}{20}\ln T + 0.07286 = 0$, $T \sim 3.5224 \times 10^{17}$, let $t = r \cdot 10^{17}$, $u \geq \frac{1}{\left(\frac{\sqrt[5]{7}}{4} 8^{\frac{4}{5}} - \frac{1}{20}\ln r + 0.07286 - \frac{17\ln 10}{20}\right)^2}$.

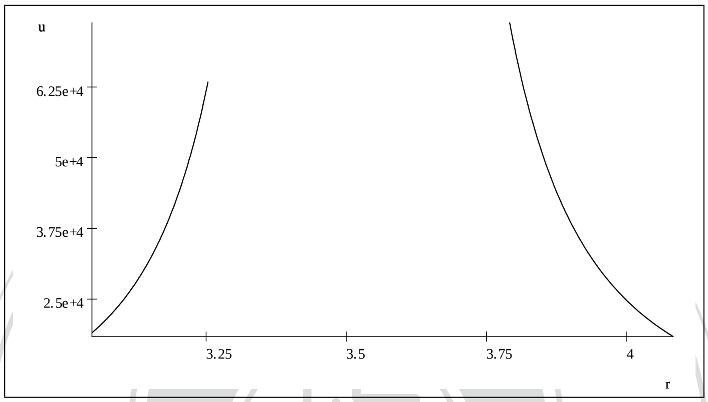

Figure 6: Graph of u

(b) Graphs for different p:

1. Given $\epsilon = 1, p = 2, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4 \cdot 7^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4 \cdot 7^{\frac{2}{5}}\right)$. Solution is: $s_0 \sim 2.5026$, $u \geq \frac{1}{\left(0.25 \times 8^{\frac{4}{5}} - \frac{7}{20}\ln t + 0.87591\right)^2}$, $0.25 \times 8^{\frac{4}{5}} - \frac{7}{20}\ln T + 0.87591 = 0$, $T \sim 529.87$.

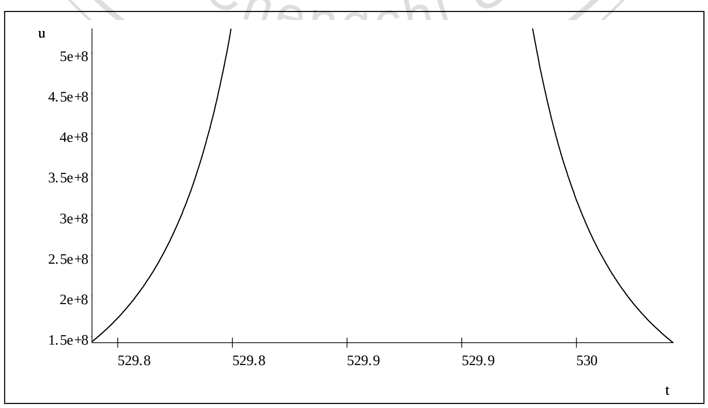

Figure 7: Graph of u

2. Given $\epsilon = 1, p = 3, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 16 \cdot 7^{\frac{1}{3}}$, $s_0 = \ln\left(s_0 + 1 + 16 \cdot 7^{\frac{1}{3}}\right)$. Solution is: $s_0 \sim 3.5601$, $u \geq \frac{1}{4.0767 - \frac{7}{12}\ln t}$, $4.0767 - \frac{7}{12}\ln T = 0$, $T \sim 1084.2$, let $t = 1000r$, $u \geq \frac{1}{-\frac{7}{12}\ln r + 4.0767 - \frac{7}{4}\ln 10}$.

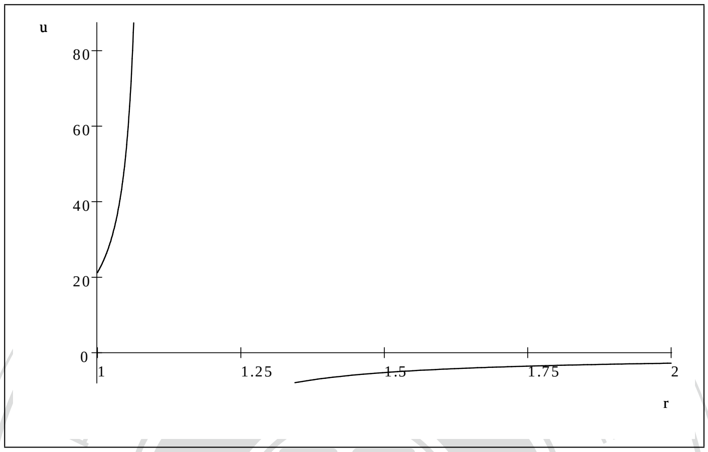

Figure 8: Graph of u

3. Given $\epsilon = 1$, $p = 5$, $u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 256 \cdot 7^{\frac{1}{4}}$, $s_0 = \ln\left(s_0 + 1 + 256 \cdot 7^{\frac{1}{4}}\right)$. Solution is: $s_0 \sim 6.0484$, $u \geq \frac{1}{\sqrt{\sqrt{32} - \frac{7}{8}\ln t + 5.2924}}$, $\sqrt{32} - \frac{7}{8}\ln T + 5.2924 = 0$, $T \sim 2.7197 \times 10^5$, let $t = 10^5 r$, $u \geq \frac{1}{\sqrt{\sqrt{32} - \frac{35}{8}\ln 10 - \frac{7}{8}\ln r + 5.2924}}$.

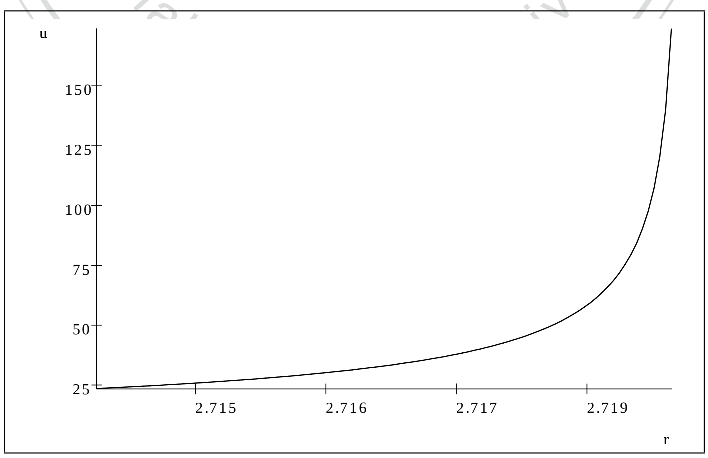

Figure 9: Graph of u

(c) Graphs for different u_0:

1. Given $\epsilon = 1, p = 2, u_0 = 0.25$: $e^{s_0} - s_0 = 1 + 4 \cdot 7^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 4 \cdot 7^{\frac{2}{5}}\right)$. Solution is: $s_0 \sim 2.5026$, $u \geq \frac{1}{\left(0.25 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln t + 0.87591\right)^2}$, $0.25 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln T + 0.87591 = 0$, $T \sim 529.87$.

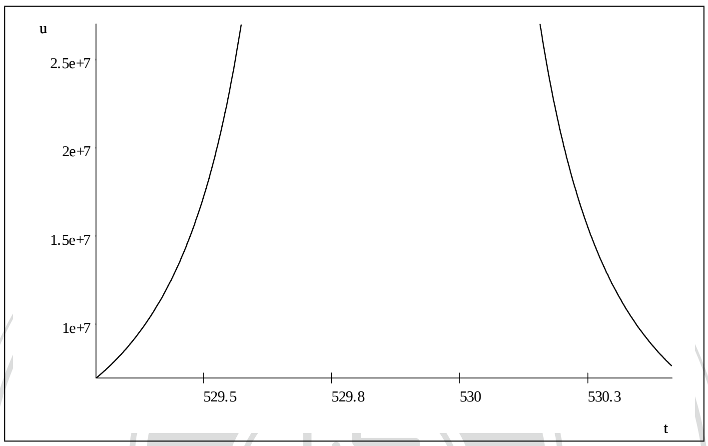

Figure 10: Graph of u

2. Given $\epsilon = 1, p = 2, u_0 = 0.5$: $e^{s_0} - s_0 = 1 + 2 \cdot 7^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + 2 \cdot 7^{\frac{2}{5}}\right)$. Solution is: $s_0 \sim 1.9948$, $u \geq \frac{1}{\left(0.17678 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln t + 0.69818\right)^2}$, $0.17678 \times 8^{\frac{4}{5}} - \frac{7}{20} \ln T + 0.69818 = 0$, $T \sim 105.71$.

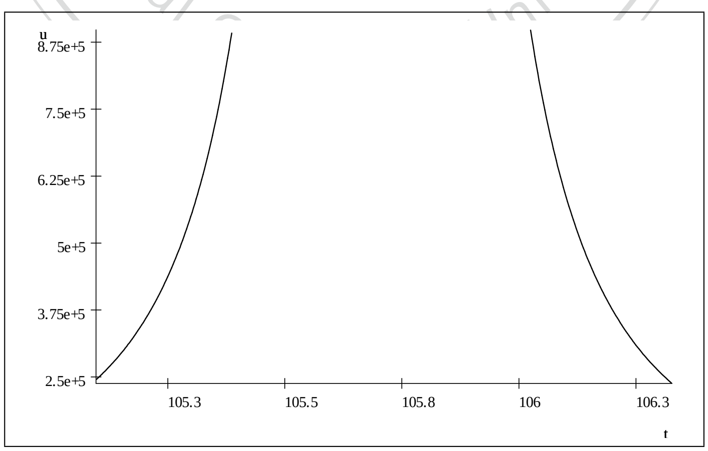

Figure 11: Graph of u

3. Given $\epsilon = 1$, $p = 2$, $u_0 = 2$: $e^{s_0} - s_0 = 1 + \frac{1}{2} \cdot 7^{\frac{2}{5}}$, $s_0 = \ln\left(s_0 + 1 + \frac{1}{2} \cdot 7^{\frac{2}{5}}\right)$. Solution is: $s_0 \sim 1.1864$, $u \geq \frac{1}{\left(\frac{\sqrt{2}}{16} \cdot 8^{\frac{4}{5}} - \frac{7}{20} \ln t + 0.41524\right)^2}$, $\frac{\sqrt{2}}{16} \cdot 8^{\frac{4}{5}} - \frac{7}{20} \ln T + 0.41524 = 0$, $T \sim 12.42$.

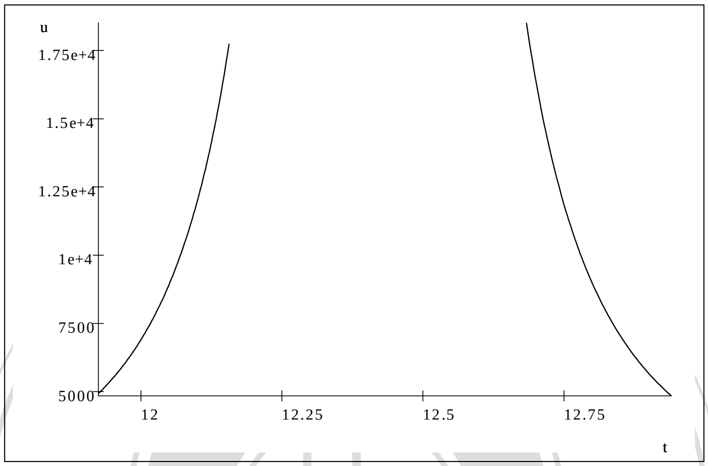

Figure 12: Graph of u

5. Estimates for the life-span of positive solution u of $(*)$ under $u_1 > 0$, $u_0 > 0$

In this chapter we start to estimate the life-span of positive solution u of $(*)$ under $u_1 > 0$, $u_0 > 0$.

Theorem 5. *For $u_1 > 0$, $u_0 > 0$, the positive solution u of $(*)$ blows up in finite time; that is, there exists a bound number T^* so that*

$$u(t)^{-1} \to 0 \quad for \quad t \to T^*.$$

Proof. We separate the proof into two parts, $E(0) \geq 0$ and $E(0) < 0$.

i) $E(0) \geq 0$. By lemma 2, (3.1), we have

$$v_s(s)^2 - \frac{2}{p+1} v(s)^{p+1} \geq E(0)$$

and

$$v_s(s)^2 \geq \frac{2}{p+1} v(s)^{p+1} + E(0).$$

By lemma 3, (3.8), we obtain

$$v_s(s) \geq \sqrt{\frac{2}{p+1} v(s)^{p+1} + E(0)}.$$

Under the condition $E(0) \geq 0$, we have

$$v_s(s) \geq \sqrt{\frac{2}{p+1}} v(s)^{\frac{p+1}{2}},$$

$$v(s)^{-\frac{p+1}{2}} v_s(s) \geq \sqrt{\frac{2}{p+1}},$$

$$\frac{2}{1-p} \left(v(s)^{\frac{1-p}{2}} \right)_s \geq \sqrt{\frac{2}{p+1}}$$

and

$$\left(v(s)^{\frac{1-p}{2}} \right)_s \leq \frac{1-p}{2} \sqrt{\frac{2}{p+1}}.$$

Integrating the above inequality with respect to s, we have

$$
\begin{aligned}
v(s)^{\frac{1-p}{2}} &\leq u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}s \\
&= u_0^{\frac{1-p}{2}} - \frac{p-1}{2}\sqrt{\frac{2}{p+1}}s.
\end{aligned}
$$

Thus, we obtain that there exists a finite time

$$
s_2^* \leq \frac{2}{p-1}\sqrt{\frac{p+1}{2}}u_0^{\frac{1-p}{2}} := k_2
$$

such that $v(s)^{-1} \to 0$ for $s \to s_2^*$, that is,

$$
u(t)^{-1} \to 0 \quad \text{for} \quad t \to \exp(k_2),
$$

which implies that the life-span T^* of positive solution u is finite and $T^* \leq \exp(k_2)$.

ii) $E(0) < 0$. From the definition of $J(s)$, we have

$$
J'(s) = -\frac{p-1}{4}a(s)^{-\frac{p+3}{4}}a'(s).
$$

By lemma 3, (3.8), we have $a'(s) > 0$, $v_s(s) > 0$ for all $s > 0$ and $J'(s) < 0$ for all $s > 0$, that is, J is decreasing in $(0, \infty)$.

Under the condition $E(0) < 0$ and by lemma 2, (3.4), we obtain

$$
\begin{aligned}
J'(s)^2 = J'(0)^2 &+ \frac{(p-1)^2}{4}E(0)\left(J(s)^{\frac{2(p+1)}{p-1}} - J(0)^{\frac{2(p+1)}{p-1}}\right) \\
&+ \frac{(p-1)^2}{2}J(s)^{\frac{2(p+1)}{p-1}}\int_0^s v_s(r)^2\,dr \geq J'(0)^2
\end{aligned}
$$

and

$$
J'(s) \leq J'(0).
$$

Integrating the above inequality with respect to s, we have

$$
\begin{aligned}
J(s) &\leq J(0) + J'(0)s \\
&= a(0)^{-\frac{p-1}{4}} - \frac{p-1}{4}a(0)^{-\frac{p+3}{4}}a'(0)s.
\end{aligned}
$$

Thus, there exists a finite number

$$s_3^* \leq \frac{4}{p-1} \frac{a(0)}{a'(0)} = \frac{2}{p-1} \frac{u_0}{u_1} := k_3$$

such that $J(s_3^*) = 0$ and $a(s)^{-1} \to 0$ for $s \to s_3^*$, that is,

$$u(t)^{-1} \to 0 \quad \text{for} \quad t \to \exp(k_3).$$

This means that the life-span T^* of u is finite and $T^* \leq \exp(k_3)$.■

Graphs of positive solution u of $(*)$ under $u_1 > 0$, $u_0 > 0$:

i) $E(0) \geq 0$: $u(t) \geq \left(u_0^{\frac{1-p}{2}} - \frac{p-1}{2}\sqrt{\frac{2}{p+1}} \ln t\right)^{\frac{2}{1-p}}$.

(a) Graphs for different p:

1. Given $p = 2$, $u_0 = 0.25$: $u \geq \left(2 - \frac{\sqrt{6}}{6} \ln t\right)^{-2}$, $2\sqrt{6} - \ln T = 0$, $T \sim 134.15$.

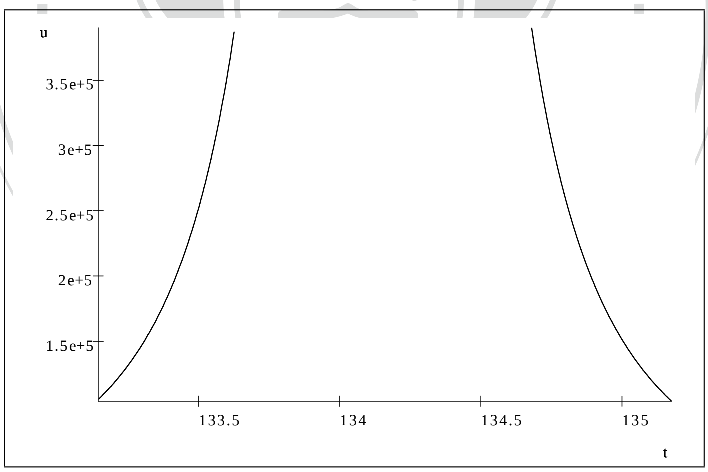

Figure 13: Graph of u

2. Given $p = 3, u_0 = 0.25$: $u \geq \frac{2}{8-\sqrt{2}\ln t}$, $4\sqrt{2} - \ln T = 0$, $T = e^{4\sqrt{2}} \sim 286.25$.

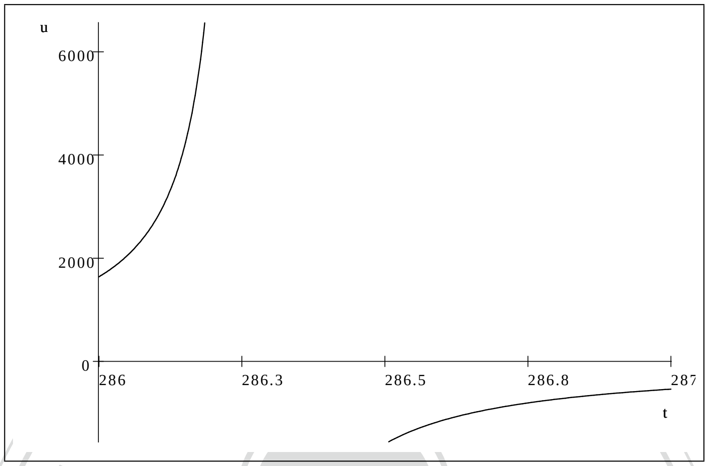

Figure 14: Graph of u

3. Given $p = 7$, $u_0 = 0.25$: $u \geq \frac{1}{\sqrt[3]{64-\frac{3}{2}\ln t}}$, $64 - \frac{3}{2}\ln T = 0$, $T \sim 3.3876 \times 10^{18}$, let $t = 10^{18} r$, $u \geq \frac{1}{\sqrt[3]{64-27\ln 10-\frac{3}{2}\ln r}}$.

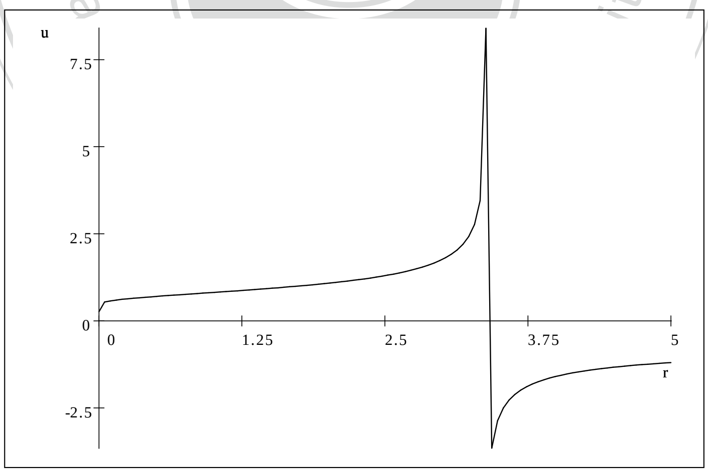

Figure 15: Graph of u

(b) Graphs for different u_0:

1. Given $p = 2$, $u_0 = 0.25$: $u \geq \left(2 - \frac{\sqrt{6}}{6}\ln t\right)^{-2}$, $2 - \frac{\sqrt{6}}{6}\ln T = 0$, $T \sim 134.15$.

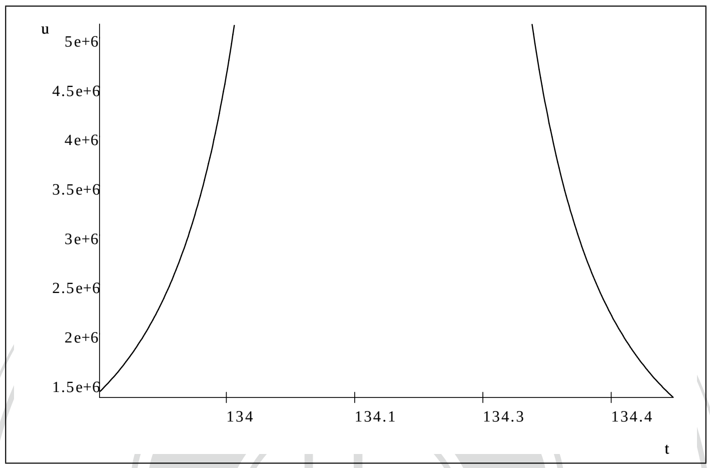

Figure 16: Graph of u

2. Given $p = 2$, $u_0 = 0.5$: $u \geq \left(1.4142 - \frac{\sqrt{6}}{6}\ln t\right)^{-2}$, $1.4142 - \frac{\sqrt{6}}{6}\ln T = 0$, $T \sim 31.947$.

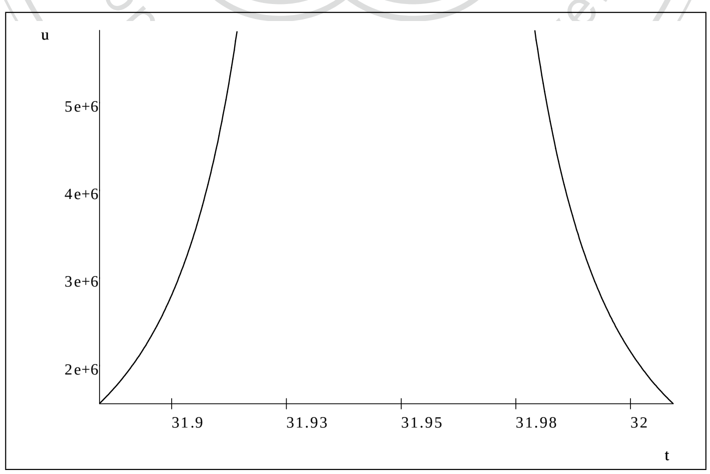

Figure 17: Graph of u

3. Given $p = 2,\ u_0 = 1.5$: $u \geq \left(0.8165 - \frac{\sqrt{6}}{6}\ln t\right)^{-2},\ 0.8165 - \frac{\sqrt{6}}{6}\ln T = 0,\ T \sim 7.3891$.

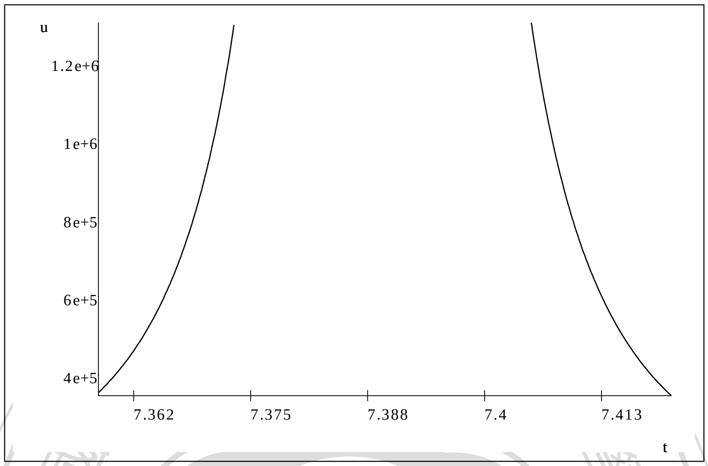

Figure 18: Graph of u

ii) $E(0) < 0$: $u(t) \geq \left(u_0^{\frac{1-p}{2}} - \frac{p-1}{2} u_0^{-\frac{p+1}{2}} u_1 \ln t\right)^{\frac{2}{1-p}}$.

(a) Graphs for different p:

1. Given $p = 2$, $u_0 = 0.25$, $u_1 = 0.25$: $u \geq \frac{1}{(2-\ln t)^2},\ 2 - \ln T = 0,\ T = e^2$.

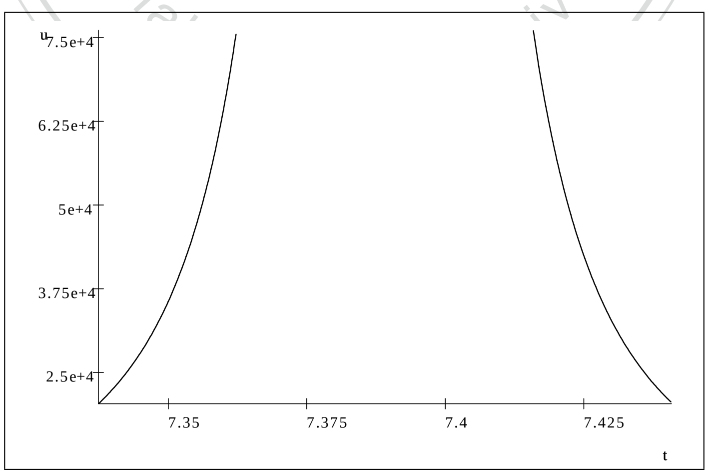

Figure 19: Graph of u

2. Given $p = 3$, $u_0 = 0.25$, $u_1 = 0.25$: $u \geq \frac{1}{4-4\ln t}$, $1 - \ln T = 0$, $T = e$.

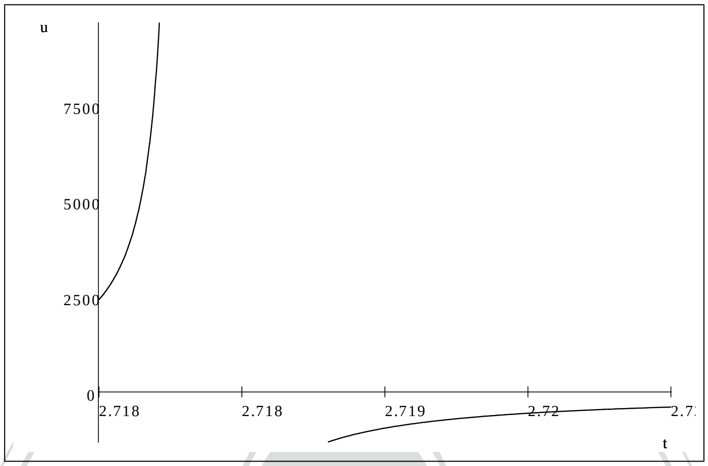

Figure 20: Graph of u

3. Given $p = 5$, $u_0 = 0.25$, $u_1 = 0.25$: $u \geq \frac{1}{4\sqrt{1-2\ln t}}$, $1 - 2\ln T = 0$, $T = e^{\frac{1}{2}} \sim 1.6487$.

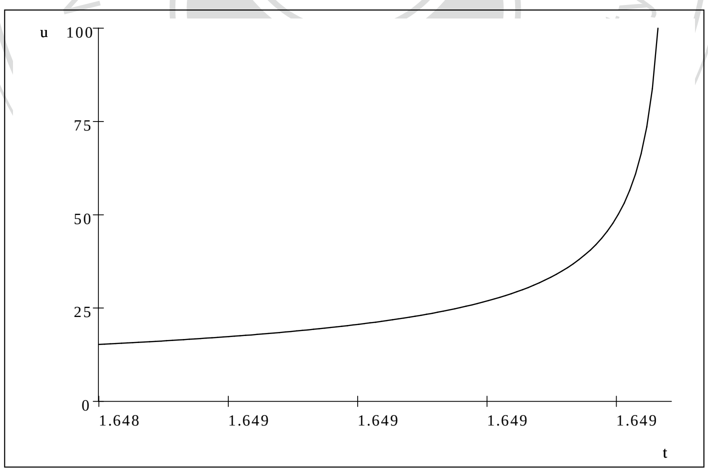

Figure 21: Graph of u

(b) Graphs for different u_0:

1. Given $p = 2$, $u_0 = 0.25$, $u_1 = 0.25$: $u \geq \frac{1}{(2-\ln t)^2}$, $2 - \ln T = 0$, $T = e^2$.

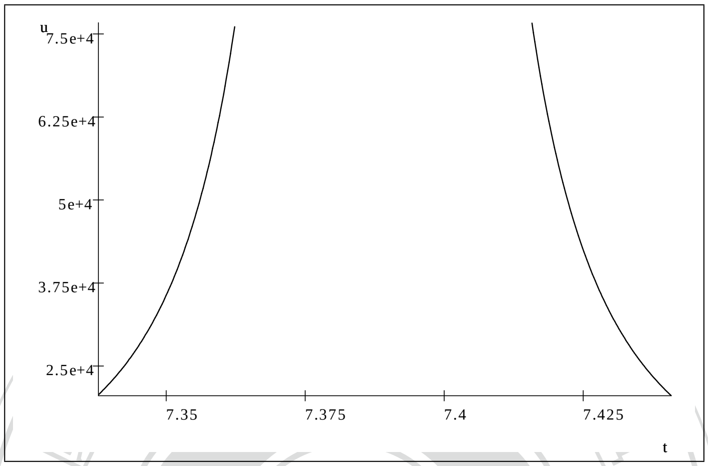

Figure 22: Graph of u

2. Given $p = 2$, $u_0 = 0.5$, $u_1 = 0.25$: $u \geq \frac{1}{(1.4142-0.35355\ln t)^2}$,
$1.4142 - 0.35355 \ln T = 0$, $T = e^4 \sim 54.598$.

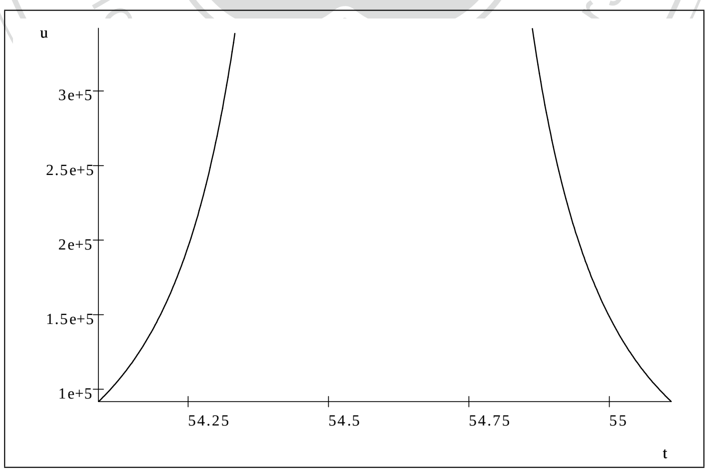

Figure 23: Graph of u

3. Given $p = 2$, $u_0 = 1$, $u_1 = 0.25$: $u \geq \frac{1}{\left(1-\frac{1}{8}\ln t\right)^2}$, $8 - \ln T = 0$, $T = e^8 \sim 2981$.

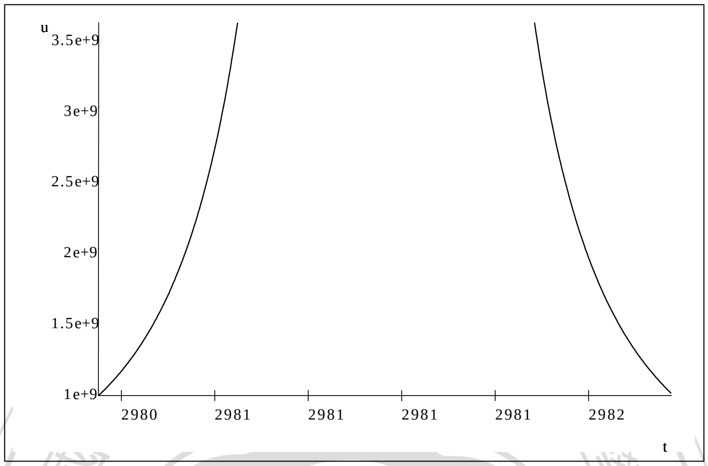

Figure 24: Graph of u

(c) Graphs for different u_1:

1. Given $p = 2$, $u_0 = 0.25$, $u_1 = 0.25$: $u \geq \frac{1}{(2-\ln t)^2}$, $2 - \ln T = 0$, $T = e^2$.

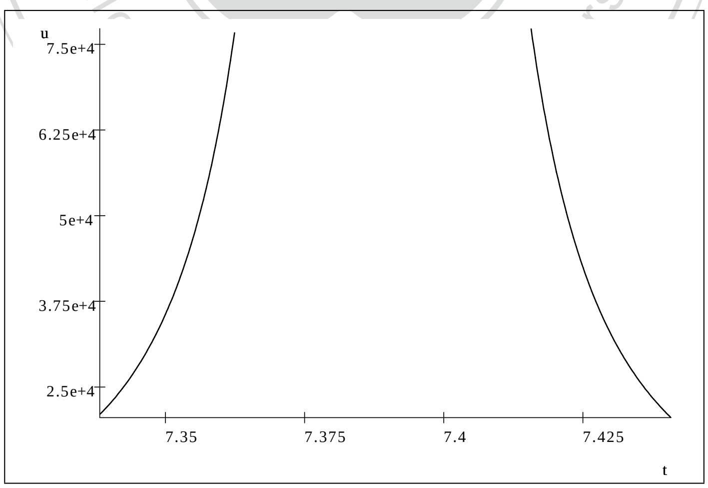

Figure 25: Graph of u

2. Given $p = 2$, $u_0 = 0.25$, $u_1 = 0.5$: $u \geq \frac{1}{(2-2\ln t)^2}$, $2 - 2\ln T = 0$, $T = e$.

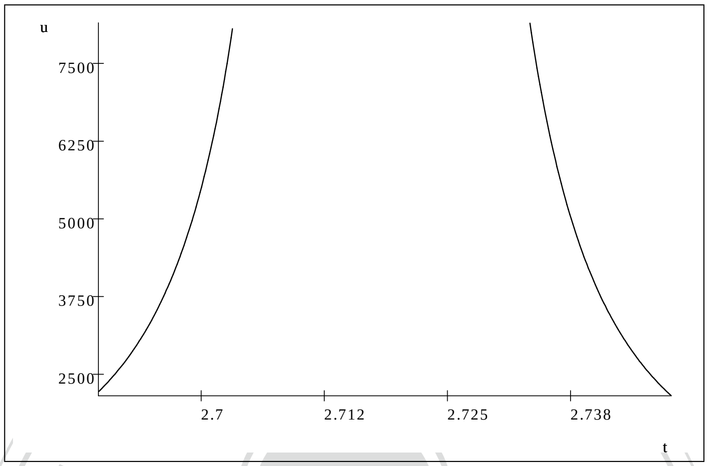

Figure 26: Graph of u

3. Given $p = 2$, $u_0 = 0.25$, $u_1 = 1$: $u \geq \frac{1}{(2-4\ln t)^2}$, $2 - 4\ln T = 0$, $T = e^{\frac{1}{2}} \sim 1.6487$.

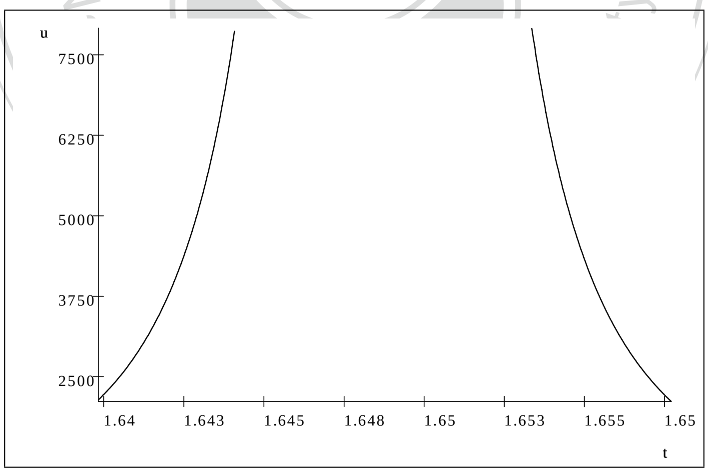

Figure 27: Graph of u

6. Magnitude of positive solution u of $(*)$ under $u_1 < 0,\ 0 < u_0 < (-u_1)^{\frac{1}{p}}$

Finally, we study the behavior of positive solution u of $(*)$ under $u_1 < 0$, $0 < u_0 < (-u_1)^{\frac{1}{p}}$ in this chapter.

Theorem 6. *For $u_1 < 0$, $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$, we have:*

$$u(t) \leq (u_0 - (u_1 + u_0^p)) + (u_1 + u_0^p)\,t - u_0^p \ln t.$$

Furthermore, for $E(0) \geq 0$,

$$u(t) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}}.$$

Proof. Since $v_{ss}(s) = v_s(s) + v(s)^p$ and by integrating this equation with respect to s, we have

$$\begin{aligned} v_s(s) &= v_s(0) + v(s) - v(0) + \int_0^s v(r)^p\,dr \\ &= (u_1 - u_0) + v(s) + \int_0^s v(r)^p\,dr. \end{aligned}$$

By lemma 3, (3.9), we have v is decreasing and

$$\begin{aligned} v_s(s) &\leq (u_1 - u_0) + v(s) + \int_0^s v(0)^p\,dr \\ &= (u_1 - u_0) + v(s) + u_0^p \cdot s \end{aligned}$$

and

$$v_s(s) - v(s) \leq (u_1 - u_0) + u_0^p \cdot s.$$

Multiplying the above inequality by e^{-s}, we have

$$\left(e^{-s}v(s)\right)_s = e^{-s}\left(v_s(s) - v(s)\right) \leq e^{-s}\left((u_1 - u_0) + u_0^p \cdot s\right).$$

By integrating the above inequality with respect to s, we obtain

$$\begin{aligned} e^{-s}v(s) - u_0 &\leq (u_1 - u_0)\int_0^s e^{-r}dr + u_0^p \int_0^s re^{-r}dr \\ &= (u_1 - u_0)\left(1 - e^{-s}\right) + u_0^p\left(-se^{-s} - e^{-s} + 1\right) \end{aligned}$$

and

$$v(s) \le (u_0 - u_1) + u_1 e^s + u_0^p \left(e^s \; - \; 1 - s\right),$$

that is,

$$u(t) \le (u_0 - u_1) + u_1 t + u_0^p \left(t \; - \; 1 - \ln t\right)$$

$$= (u_0 - (u_1 + u_0^p)) + (u_1 + u_0^p)\, t - u_0^p \ln t.$$

We plot the graphs of u for fixed p and u_1 under the condictions:

$u_1 < 0,\; 0 < u_0 < (-u_1)^{\frac{1}{p}}$, $u(t) \le (u_0 - (u_1 + u_0^p)) + (u_1 + u_0^p)\, t - u_0^p \ln t.$

1. $p = 2,\; u_1 = -1,\, u(t) \le 1 + u_0 - u_0^2 + \left(-1 + u_0^2\right) t - u_0^2 \ln t := f(t, u_0).$

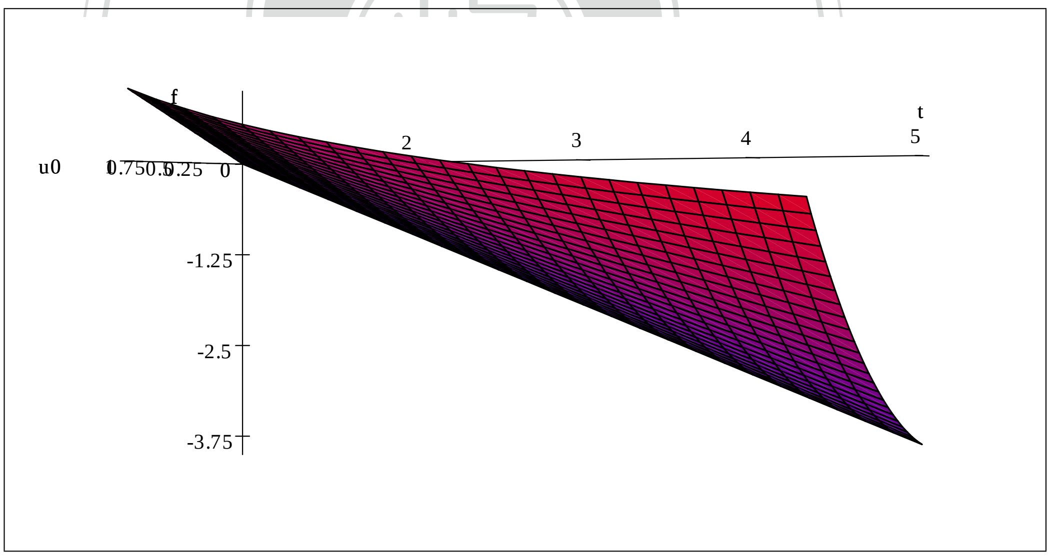

Figure 28: Graph of $f(t, u_0)$

2. $p = 2,\ u_1 = -2,\ u(t) \leq 2 + u_0 - u_0^2 + \left(-2 + u_0^2\right) t - u_0^2 \ln t := f(t, u_0)$.

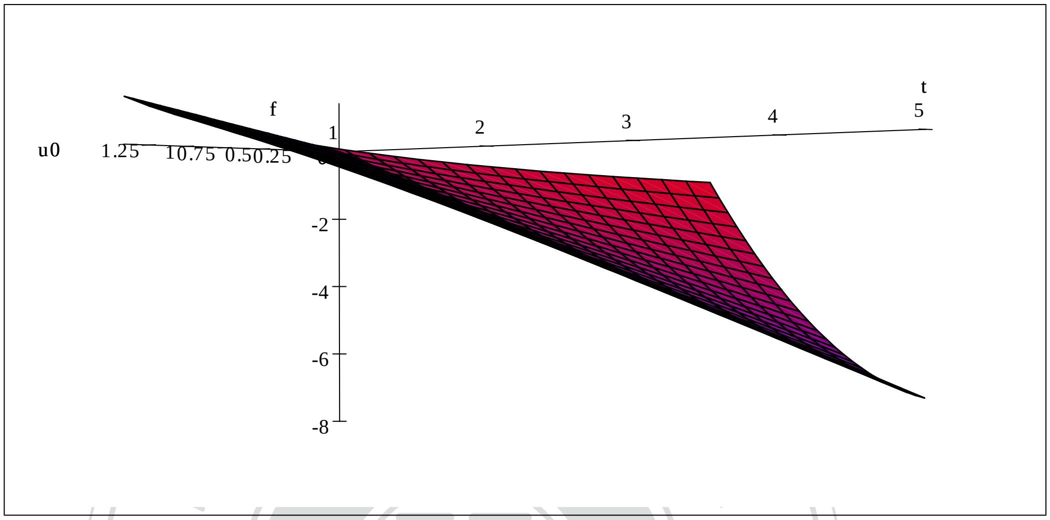

Figure 29: Graph of $f(t, u_0)$

3. $p = 2,\ u_1 = -5,\ u(t) = 5 + u_0 - u_0^2 + \left(-5 + u_0^2\right) t - u_0^2 \ln t := f(t, u_0)$.

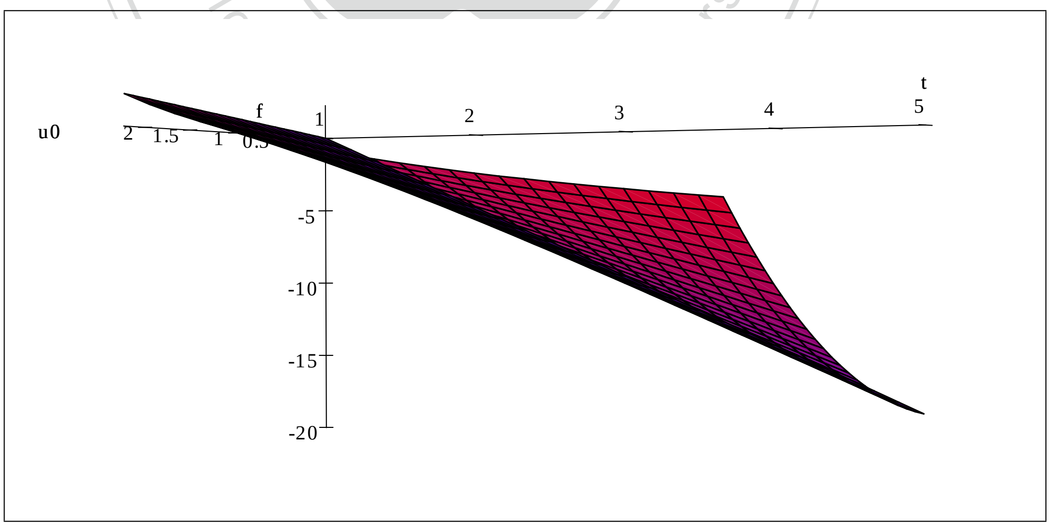

Figure 30: Graph of $f(t, u_0)$

4. $p = 4,\ u_1 = -1,\ u(t) \leq 1 + u_0 - u_0^4 + \left(-1 + u_0^4\right) t - u_0^4 \ln t := f(t, u_0)$.

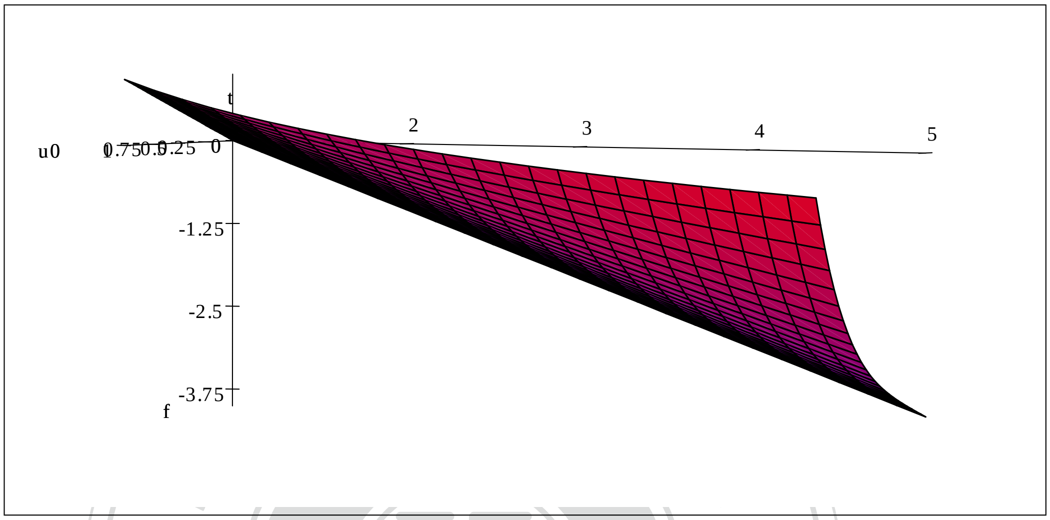

Figure 31: Graph of $f(t, u_0)$

5. $p = 4,\ u_1 = -2,\ u(t) \leq 2 + u_0 - u_0^4 + \left(-2 + u_0^4\right) t - u_0^4 \ln t := f(t, u_0)$.

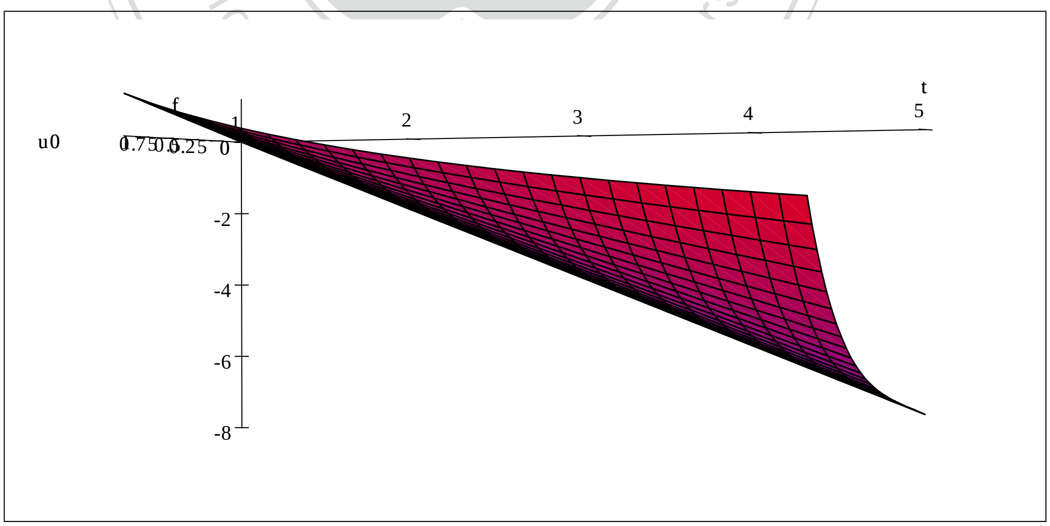

Figure 32: Graph of $f(t, u_0)$

6. $p=4,\ u_1=-5,\ u(t)\le 5+u_0-u_0^4+\left(-5+u_0^4\right)t-u_0^4\ln t:=f(t,u_0)$.

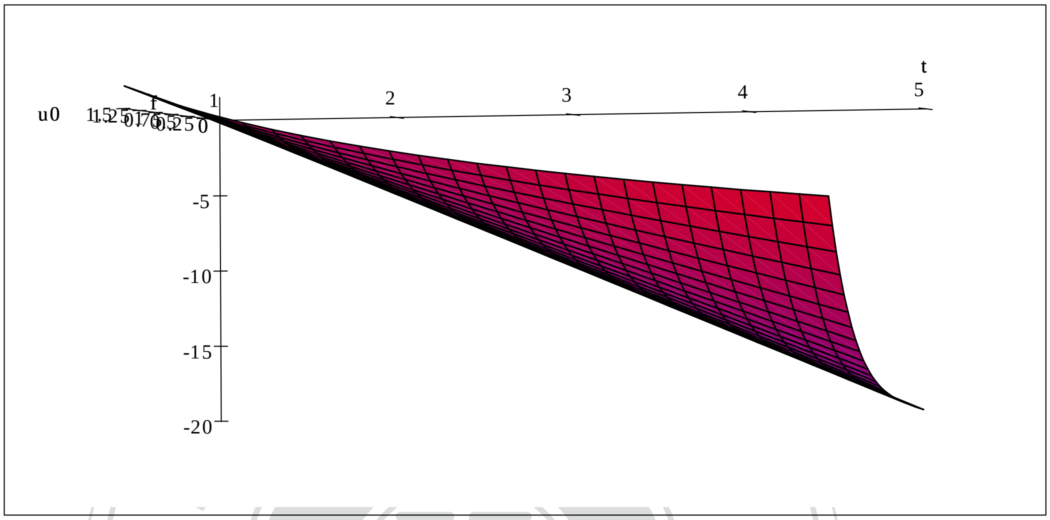

Figure 33: Graph of $f(t,u_0)$

7. $p=8,\ u_1=-1,\ u(t)\le 1+u_0-u_0^8+\left(-1+u_0^8\right)t-u_0^8\ln t:=f(t,u_0)$.

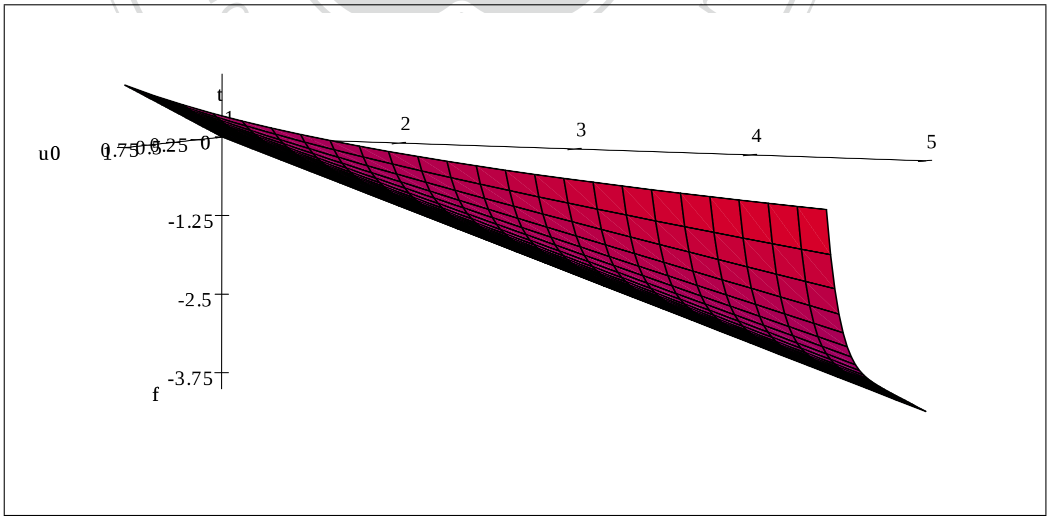

Figure 34: Graph of $f(t,u_0)$

8. $p = 8,\ u_1 = -2,\ u(t) \leq 2 + u_0 - u_0^8 + \left(-2 + u_0^8\right) t - u_0^8 \ln t := f(t, u_0)$.

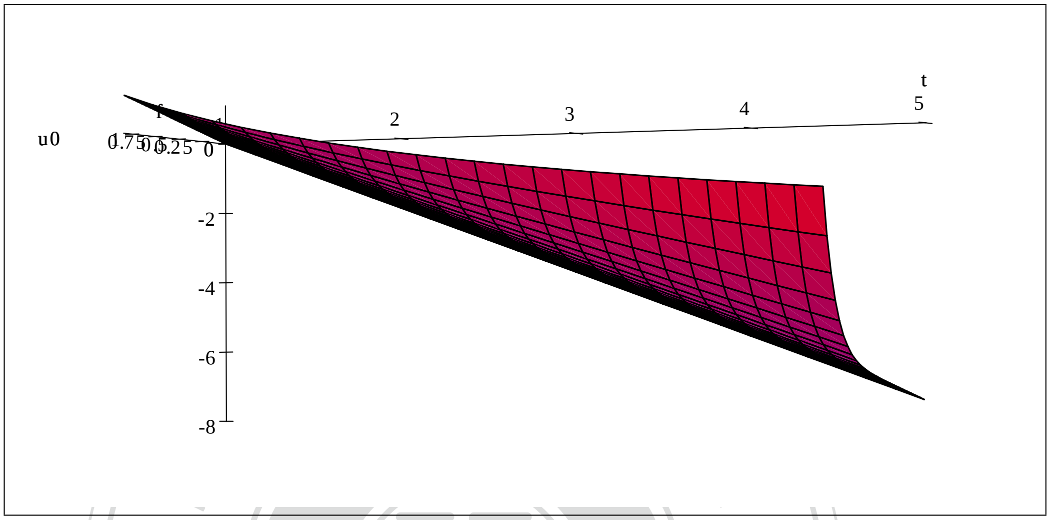

Figure 35: Graph of $f(t, u_0)$

9. $p = 8,\ u_1 = -5,\ u(t) \leq 5 + u_0 - u_0^8 + \left(-5 + u_0^8\right) t - u_0^8 \ln t := f(t, u_0)$.

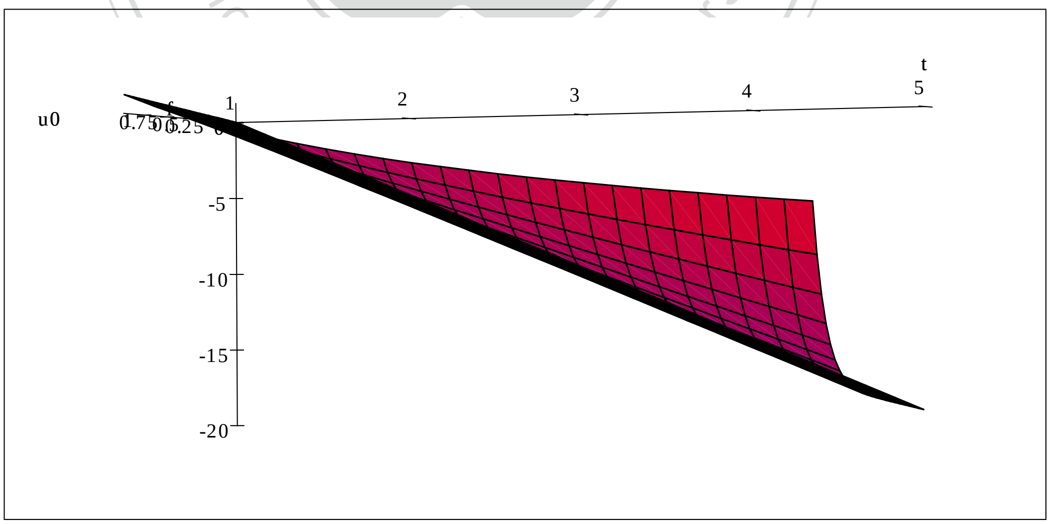

Figure 36: Graph of $f(t, u_0)$

Moreover, under the condition $E(0) \geq 0$, By lemma 2, (3.1), we have

$$v_s(s)^2 - \frac{2}{p+1} v(s)^{p+1} = E(0) + 2\int_0^s v_s(r)^2 \, dr \geq E(0).$$

Since $E(0) \geq 0$, then

$$v_s(s)^2 \geq E(0) + \frac{2}{p+1} v(s)^{p+1}$$

$$\geq \frac{2}{p+1} v(s)^{p+1}.$$

By lemma 3, (3.9), we have

$$-v_s(s) \geq \sqrt{\frac{2}{p+1}} v(s)^{\frac{p+1}{2}},$$

$$-v_s(s) v(s)^{-\frac{p+1}{2}} \geq \sqrt{\frac{2}{p+1}}$$

and

$$\frac{2}{p-1} \left(v(s)^{\frac{1-p}{2}} \right)_s \geq \sqrt{\frac{2}{p+1}}.$$

By integrating the above inequality with respect to s, we have

$$\frac{2}{p-1} \left(v(s)^{\frac{1-p}{2}} - v(0)^{\frac{1-p}{2}} \right) \geq \sqrt{\frac{2}{p+1}} s$$

and

$$v(s)^{\frac{1-p}{2}} \geq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} s \right).$$

Then, we know that

$$v(s) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} s \right)^{\frac{2}{1-p}},$$

that is,

$$u(t) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} \ln t \right)^{\frac{2}{1-p}}. \blacksquare$$

Graphs of positive solution u of $(*)$ under $u_1 < 0$, $0 < u_0 < (-u_1)^{\frac{1}{p}}$ and $E(0) \geq 0$:

(a) Graphs for fixed u_0:

1. $u_0 = 0.25$, $f(t,p) = \left(2^{p-1} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}}$.

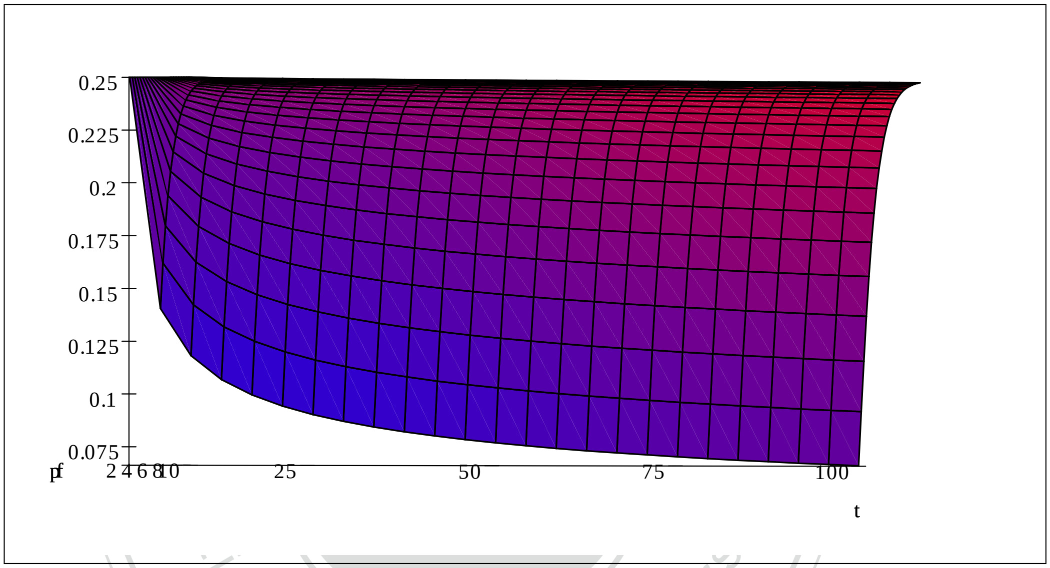

Figure 37: Graph of $f(t,p)$

2. $u_0 = 0.5,\ f(t,p) = \left(2^{\frac{p-1}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}}$.

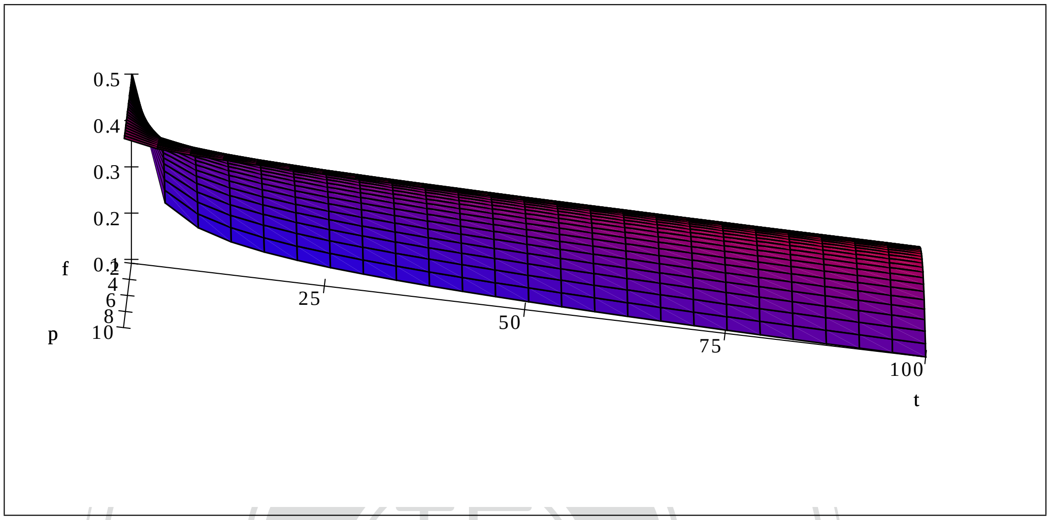

Figure 38: Graph of $f(t,p)$

3. $u_0 = 1,\ f(t,p) = \left(1 + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}}$.

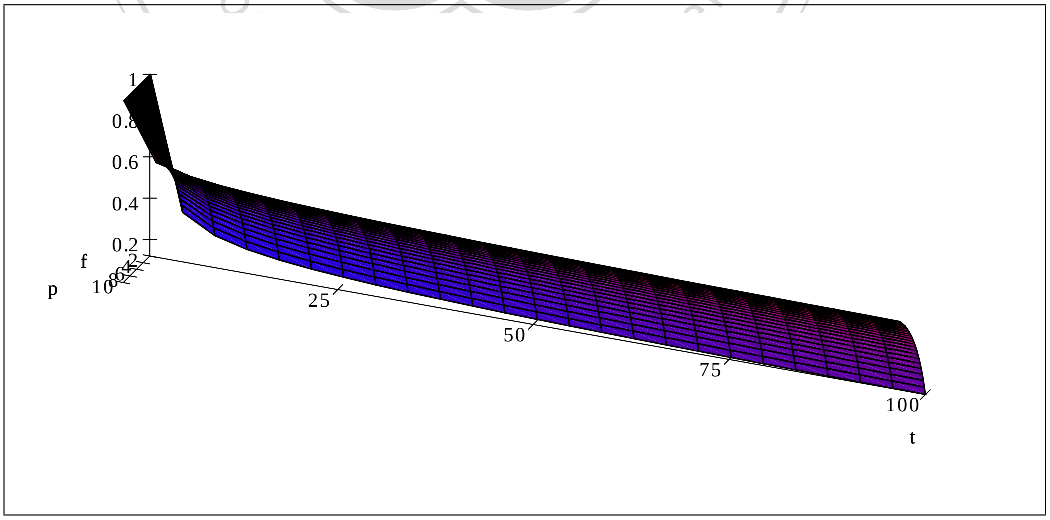

Figure 39: Graph of $f(t,p)$

(b) Graphs for fixed p**:** 1. $p = 2,\ g(u_0, t) = \left(u_0^{\frac{-1}{2}} + \sqrt{\frac{1}{6}\ln t}\right)^{-2}.$

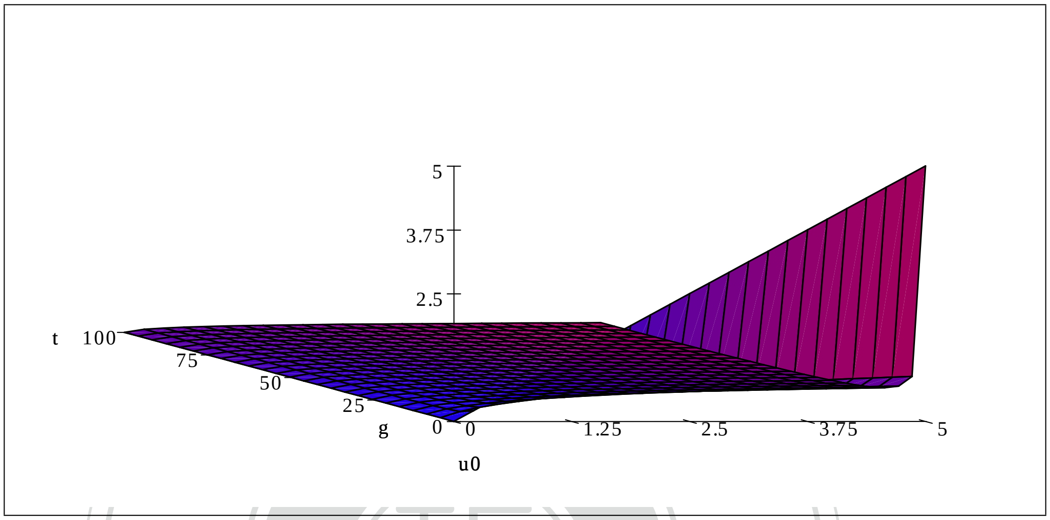

Figure 40: Graph of $g(u_0, t)$

2. $p = 3,\ g(u_0, t) = \left(u_0^{-1} + \sqrt{\frac{1}{2}\ln t}\right)^{-1}.$

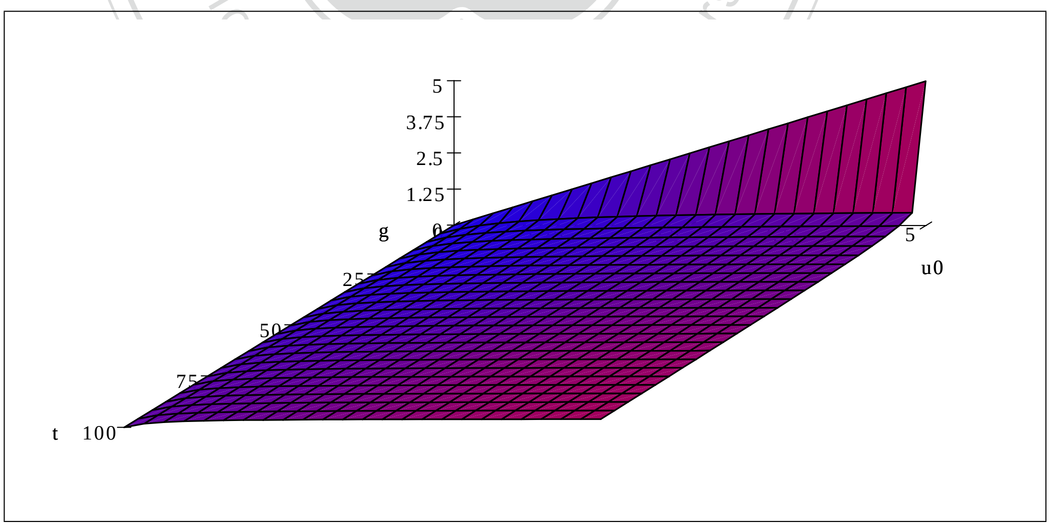

Figure 41: Graph of $g(u_0, t)$

3. $p = 5, \ g(u_0, t) = \left(u_0^{-2} + 2\sqrt{\frac{1}{3}\ln t}\right)^{\frac{-1}{2}}.$

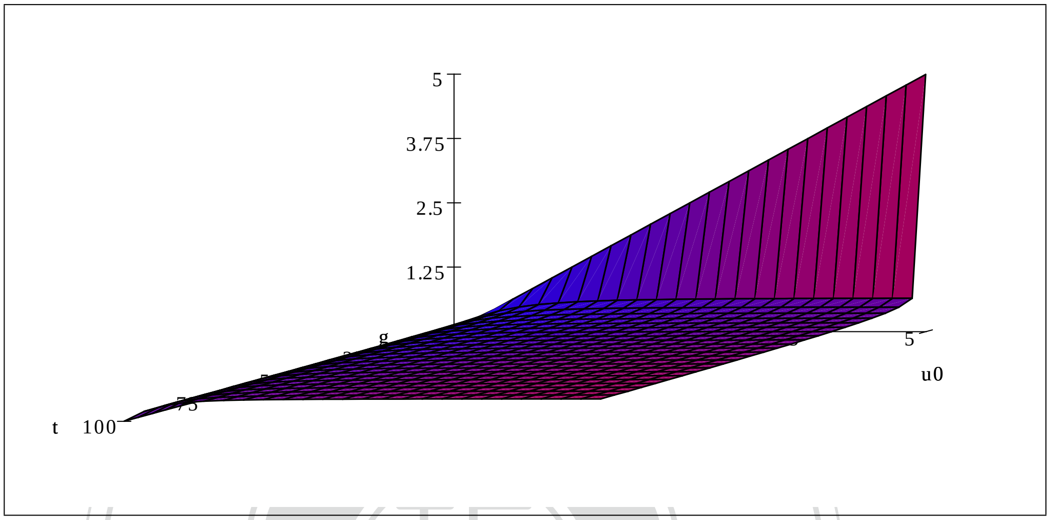

Figure 42: Graph of $g(u_0, t)$

4. $p = 7, \ g(u_0, t) = \left(u_0^{-3} + \frac{3}{2}\ln t\right)^{\frac{-1}{3}}.$

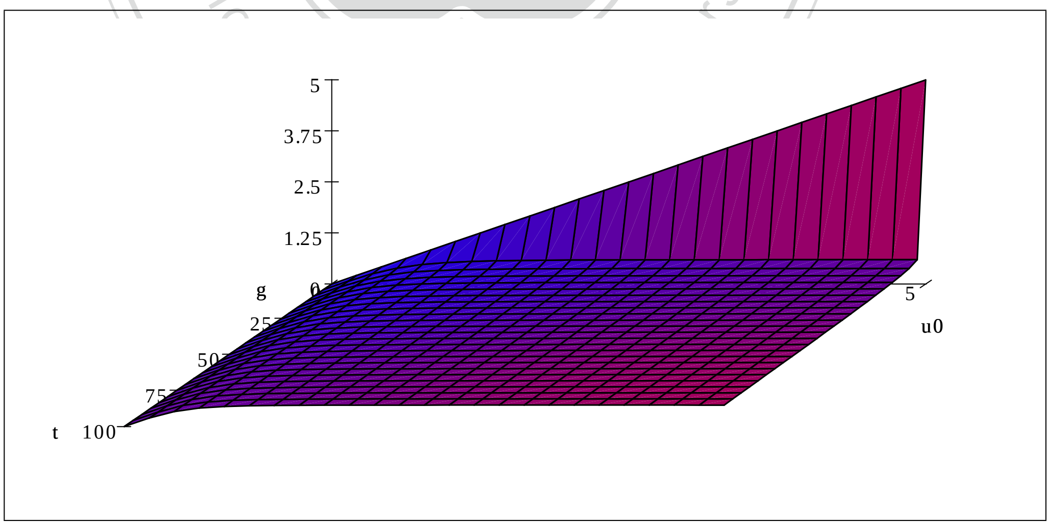

Figure 43: Graph of $g(u_0, t)$

7. Conclusions

In this paper, we have studied the behavior of positive solution u of $(*)$ under three different cases. We summarize as follows:

(a) $u_1 = 0,\ u_0 > 0$:

The life-span $T^* \leq \exp(k_1)$, where

$$k_1 := s_0 + \frac{2(p+3)}{8-\epsilon} \cdot \frac{2}{p-1} v(s_0)^{\frac{1-p}{2}}.$$

(b) $u_1 > 0,\ u_0 > 0$:

(i) $E(0) \geq 0$, the life-span $T^* \leq \exp(k_2)$, where

$$k_2 := \frac{2}{p-1}\sqrt{\frac{p+1}{2}} u_0^{\frac{1-p}{2}}.$$

(ii) $E(0) < 0$, the life-span $T^* \leq \exp(k_3)$, where

$$k_3 := \frac{2}{p-1}\frac{u_0}{u_1}.$$

(c) $u_1 < 0,\ u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$:

$$u(t) \leq (u_0 - (u_1 + u_0^p)) + (u_1 + u_0^p)\,t - u_0^p \ln t.$$

Furthermore, for $E(0) \geq 0$,

$$u(t) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}} \ln t\right)^{\frac{2}{1-p}}.$$

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