行政院國家科學委員會專題研究計畫 成果報告

一個反應擴散方程系統之行波解的穩定性研究
研究成果報告(精簡版)

計 畫 類 別 ：個別型
計 畫 編 號 ：NSC 98-2115-M-004-001-
執 行 期 間 ：98 年 08 月 01 日至 99 年 07 月 31 日
執 行 單 位 ：國立政治大學應用數學學系

計畫主持人：符聖珍

計畫參與人員：碩士班研究生-兼任助理人員：李宣緯

處 理 方 式 ：本計畫可公開查詢

中 華 民 國 99 年 10 月 20 日

Existence of solutions to PBVPs for first-order impulsive dynamic equations on time scales ⋆

Sheng-Chen Fu and Yi-Chang Liang

Department of Mathematical Sciences, National Chengchi University, 64, S-2 Zhi-nan Road, Taipei 116, Taiwan

Abstract

In this paper we are concernd with periodic boundary value problems for first-order impulsive dynamic equations on time scales. By using Schaefer's theorem and Banach's fixed point theorem we acquire some new existence results.

Key words: Time scale; Periodic boundary value problem; Impulsive dynamic equation
MSC: 34N05

1 Introduction

The theory of dynamic equations on time scales has received a lot of attention since it can not only unify, extend, and generalize the theories of differential equations and difference equations but also have various practical applications. For more details about this theory, we refer the readers to [1], [2], and [3]. One of the important research trends is the investigation of impulsive dynamic equations on time scales. Recently, some researchers have focused their attention on periodic boundary value problems (PBVPs for short) for first-order impulsive dynamic equations. For example, Geng, Xu, and Zhu [4] applied the method of upper and lower solutions coupled with monotone iterative techniques to derive the existence of extremal solutions and Wang [5]

⋆ This work is partially supported by National Science Council of the Republic of China under the contract 98-2115-M-004-001.
Corresponding author: Sheng-Chen Fu.
Email addresses: fu@nccu.edu.tw (Sheng-Chen Fu); 96751003@nccu.edu.tw (Yi-Chang Liang).

used the Guo-Krasnoselskii fixed point theorem to obtain some existence criteria for positive solutions. However, to the best of the authors' knowledge, there is no existence criteria for (not necessarily positive) solutions to PBVPs for first-order impulsive dynamic equations on time scales so far.

Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of $\mathbb{R}$, and let $0, T \in \mathbb{T}$. Throughout this paper, $[0,T]_{\mathbb{T}}$ represents an interval on $\mathbb{T}$, i.e., $[0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}$. Other types of intervals on $\mathbb{T}$ can be represented by a similar way. Let $J = [0, \sigma(T)]_{\mathbb{T}}$. Motivated by [6], [7], and the above works, in this paper, we are concerned with the existence of solutions to the following PBVPs for first-order impulsive dynamic equations on $\mathbb{T}$

$$x^{\Delta} + p(t)x^{\sigma} = f(t,x), \quad t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1, \ldots, m, \tag{1}$$
$$x(t_k+) - x(t_k-) = I_k(x(t_k-)), \quad k = 1, \ldots, m, \tag{2}$$
$$x(0) = x(\sigma(T)), \tag{3}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $p : J \to [0, \infty)$ is rd-continuous and regressive with $p \not\equiv 0$, and the points t_k, $k = 1, \ldots, m$, are right-dense in $\mathbb{T}$ such that $0 < t_1 < \cdots < t_m < T$. For convenience, we shall refer to (1)-(2)-(3) as (NP).

When $I_k(x) \equiv 0$ for all $k = 1, \ldots, m$, the problem (NP) can be reduced to the following PBVPs with no impulse effects

$$\begin{aligned} &x^{\Delta} + p(t)x^{\sigma} = f(t,x), \quad t \in [0,T]_{\mathbb{T}}, \\ &x(0) = x(\sigma(T)), \end{aligned}$$

which has been investigated by several researchers; see for example, [8], [9], [10], [11], and the references cited therein.

PBVPs for first-order impulsive differential equations and difference equations (i.e., the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$) have been studied; see for example, [12], [13], [6], [14], [15], [16], [17], [7], [18] for $\mathbb{T} = \mathbb{R}$ and [19] for $\mathbb{T} = \mathbb{Z}$.

Let $J_0 = [0, t_1]_{\mathbb{T}}$, $J_k = (t_k, t_{k+1}]_{\mathbb{T}}$ for $k = 1, \ldots, m-1$, and $J_m = (t_m, \sigma(T)]_{\mathbb{T}}$ and let

$$\begin{aligned} PC = \{x : J \to \mathbb{R} |\ & x_k \in C(J_k),\ \forall k = 0, \ldots, m, \text{ and both } x(t_k+) \text{ and } x(t_k-) \\ & \text{exist such that } x(t_k-) = x(t_k),\ \forall k = 1, \ldots, m\}, \end{aligned}$$

where x_k is the restriction of x to J_k for each $k = 0, \ldots, m$. We introduce the Banach space $X = \{x \in PC :\ x(0) = x(\sigma(T))\}$ with the norm $\|x\|_X = \sup_{t \in J} |x(t)|$.

Definition 1.1 *A function x is said to be a solution of (NP) if and only if $x \in PC \bigcap C^1([0,T]_{\mathbb{T}} \backslash \{t_1, t_2, ..., t_m\},\ \mathbb{R})$ and satisfies (1)-(2)-(3).*

We shall apply the well-known Banach's fixed point theorem and Schaefer's theorem to establish the existence criteria of solutions for (NP). For readers' convenience, we provide these two theorems here.

Lemma 1.2 *(Banach's fixed point theorem [20]) A contraction f of a complete metric space S has a unique fixed point in S.*

Lemma 1.3 *(Schaefer's theorem [20]) Let S ba a normed linear space, and let operator $F : S \to S$ be compact. If the set*

$$H(F) = \{x \in S : x = \mu F(x) \textit{ for some } \mu \in (0, 1)\}$$

is bounded, then F has a fixed point in S.

2 Linear problem

In this section we consider the "linear problem"

$$\begin{aligned} &x^{\Delta} + p(t)x^{\sigma} = h(t), \quad t \in [0, T]_{\mathbb{T}}, t \neq t_k, k = 1, \ldots, m, \\ &x(t_k+) - x(t_k-) = I_k(x(t_k-)), \quad k = 1, \ldots, m, \\ &x(0) = x(\sigma(T)). \end{aligned}$$

For convenience, we shall refer to this problem as (LP). Note that (LP) is not really a linear problem since the impulsive functions I_k, $k = 1, \ldots, m$, may or may not be linear.

The following two basic lemmas will be used later and their proofs can be found in [5].

Lemma 2.1 *Suppose that $h : J \to \mathbb{R}$ is rd-continuous. Then x is a solution of (LP) if and only if x is a solution of*

$$x(t) = \int_0^{\sigma(T)} G(t, s)h(s)\Delta s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k)), \quad t \in J, \tag{4}$$

where

$$G(t, s) = \begin{cases} \dfrac{e_p(s, t)e_p(\sigma(T), 0)}{e_p(\sigma(T), 0) - 1}, & 0 \leq s \leq t \leq \sigma(T), \\ \dfrac{e_p(s, t)}{e_p(\sigma(T), 0) - 1}, & 0 \leq t < s \leq \sigma(T). \end{cases}$$

Lemma 2.2 *Let $G(t, s)$ be defined as Lemma 2.1. Then*

$$0 \leqslant G(t, s) \leqslant \frac{e_p(\sigma(T), 0)}{e_p(\sigma(T), 0) - 1} \triangleq A \quad for \ all \ t, s \in J.$$

Our existence result for (LP) is as follows.

Theorem 2.3 *Suppose that there exist positive constants l_k, $k = 1, \dots, m$, such that*

$$|I_k(x) - I_k(y)| \leq l_k|x - y| \text{ for all } x, y \in \mathbb{R} \text{ and } k = 1, \dots, m.$$

If

$$A\sum_{k=1}^{m} l_k < 1,$$

then the problem (LP) has a unique solution for any $h \in PC$.

Proof. First, we define the operator $\Psi : X \to X$ by

$$\Psi x(t) = \int_0^{\sigma(T)} G(t,s)h(s)\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)),$$

so that fixed points of Ψ are solutions of (LP) and vice versa. Next, we claim that Ψ is a contraction mapping. To show this, we consider $u, v \in X$ and $t \in J$. It is easy to see that

$$\begin{aligned}
|(\Psi u)(t) - (\Psi v)(t)| &= \left|\sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)) - \sum_{k=1}^{m} G(t,t_k)I_k(v(t_k))\right| \\
&\leq \sum_{k=1}^{m} |G(t,t_k)||I_k(u(t_k)) - I_k(v(t_k))| \\
&\leq \sum_{k=1}^{m} Al_k|u(t_k) - v(t_k)| \\
&\leq \sum_{k=1}^{m} Al_k\|u - v\|,
\end{aligned}$$

and hence

$$\|\Psi u - \Psi v\| \leq A\sum_{k=1}^{m} l_k\|u - v\|.$$

This means that Ψ is a contraction mapping. Finally, applying Banach's fixed point theorem, we conclude that Ψ has a unique fixed point $x \in X$ so that (LP) has exactly one solution. □

3 Nonlinear problem

In this section we study the "nonlinear problem" (NP). It follows from Lemma 2.1 that $x \in X$ is a solution of (NP) if and only if it satisfies

$$x(t) = \int_0^{\sigma(T)} G(t,s)f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)), \quad t \in J.$$

Introduce the operator $\Phi : X \to X$ by the formula

$$\Phi x(t) = \int_0^{\sigma(T)} G(t,s) f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)), \quad t \in J.$$

Obviously, fixed points of Φ are solutions of (NP) and conversely.

Definition 3.1 *Let F be a subset of PC. We say that F is quasiequicontinuous on J if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $f \in F$ and $k = 0, \dots, m$, then*

$$|f(t) - f(\tilde{t})| < \epsilon, \ \forall t, \ \tilde{t} \in J_k \text{ and } |t - \tilde{t}| < \delta.$$

In order to show that Φ is compact, we need the following compactness criteria.

Lemma 3.2 *A set $F \subset PC$ is relatively compact on J if F is bounded and quasiequicontinuous on J.*

Proof. Let $\{x_n\}$ be a sequence in F. From assumption, we know that $\{x_n\}$ is uniformly bounded and equicontinuous on J_0. By Arzela's theorem, there is a convergent subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ on J_0. Since $\{x_n^{(1)}\}$ is uniformly bounded and equicontinuous on J_1, it follows from Arzela's theorem that there is a convergent subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ on J_1. Continuing this process, we can get a convergent subsequence $\{x_n^{(m+1)}\}$ of $\{x_n^{(m)}\}$ on J_m. It is clear that $\{x_n^{(m+1)}\}$ is a convergent subsequence of $\{x_n\}$ on J. Hence F is relatively compact. $\square$

Lemma 3.3 $\Phi : X \to X$ *is compact.*

Proof. Let D be a bounded subset of X. The continuity of f and I_k implies that there exist positive constants M and M_k such that $|f(t,x(t))| \le M$ and $|I_k(x_{t_k})| \le M_k$ for all $x \in D$, $t \in J$, and $k = 1, \dots, m$. Hence we have

$$\begin{aligned} |\Phi x(t)| &= \left| \int_0^{\sigma(T)} G(t,s) f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)) \right| \\ &\le \int_0^{\sigma(T)} |G(t,s)||f(s,x(s))|\Delta s + \sum_{k=1}^{m} |G(t,t_k)||I_k(x(t_k))| \\ &\le AM\sigma(T) + A \sum_{k=1}^{m} M_k. \end{aligned}$$

This implies that $\Phi(D)$ is bounded.

Let $x \in D$ and t, $\tilde{t} \in J_k$, where $k = 0, \ldots, m$. We have that

$$
\begin{aligned}
&|\Phi x(t) - \Phi x(\tilde{t})| \\
&\leq M \textstyle\int_0^{\tilde{t}} |G(t,s) - G(\tilde{t},s)|\Delta s + M \int_{\tilde{t}}^{t} |G(t,s) - G(\tilde{t},s)|\Delta s \\
&\qquad + M \textstyle\int_t^{\sigma(T)} |G(t,s) - G(\tilde{t},s)|\Delta s + \sum_{k=1}^{m} \left|G(t,t_k) - G(\tilde{t},t_k)\right| M_k \\
&= MA \textstyle\int_0^{\tilde{t}} \left|e_p(s,t) - e_p(s,\tilde{t})\right| \Delta s \\
&\qquad + M\eta \textstyle\int_{\tilde{t}}^{t} \left|e_p(s,t)e_p(\sigma(T),0) - e_p(s,\tilde{t})\right| \Delta s \\
&\qquad + M\eta \textstyle\int_t^{\sigma(T)} \left|e_p(s,t) - e_p(s,\tilde{t})\right| \Delta s \\
&\qquad + A \textstyle\sum_{k=1}^{j-1} \left|e_p(t_k,t) - e_p(t_k,\tilde{t})\right| M_k \\
&\qquad + \eta \textstyle\sum_{k=j}^{m} \left|e_p(t_k,t) - e_p(t_k,\tilde{t})\right| M_k,
\end{aligned}
$$

where $\eta = 1/(e_p(\sigma(T),0) - 1)$. It follows that $|\Phi x(t) - \Phi x(\tilde{t})| \to 0$ uniformly for $x \in D$ as $|t - \tilde{t}| \to 0$. So $\Phi(D)$ is quasiequicontinuous on J. By Lemma 3.2, Φ is compact. This completes the proof. □

Now we are in a position to establish the existence theorems for the problem (NP) by using fixed point theorems.

Theorem 3.4 *Suppose that there exist positive constants l_k, $k = 1, \ldots, m$, such that*

$$|I_k(u) - I_k(v)| \leq l_k |u - v| \text{ for all } u, v \in \mathbb{R},$$

and suppose also that there exists a positive constant l such that

$$|f(t,u) - f(t,v)| \leq l|u - v| \text{ for all } t \in J \text{ and } u, v \in \mathbb{R}.$$

If

$$A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right) < 1,$$

then the problem (NP) has a unique solution.

Proof. For any $u, v \in X$ and $t \in J$, we can easily get that

$$|\Phi u(t) - \Phi v(t)| \leq A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right) \|u - v\|,$$

and hence

$$\|\Phi u - \Phi v\| \leq A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right) \|u - v\|.$$

This means that Φ is a contraction mapping. By Banach's fixed point theorem, Φ has a unique fixed point which is the unique solution of (NP). This completes the proof. □

Theorem 3.5 *Suppose that there exist positive constants l and c_k, $k = 1, ..., m$, such that*

$$|f(t,x)| \leq l|x| \text{ for all } t \in J \text{ and } x \in \mathbb{R} \tag{5}$$

and

$$|I_k(x)| \leq c_k \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \dots, m. \tag{6}$$

If

$$lA\sigma(T) < 1, \tag{7}$$

then the problem (NP) has at least one solution.

Proof. Let $x \in X$ and $t \in J$. Suppose that x is a solution of $x = \mu\Phi x$ for some $\mu \in (0,1)$. Using (5) and (6), we have

$$\begin{aligned}
|x(t)| &= \left|\mu \int_0^{\sigma(T)} G(t,s) f(s,x(s))\Delta s + \mu \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k))\right| \\
&\leq \mu \int_0^{\sigma(T)} |G(t,s)||f(s,x(s))|\Delta s + \mu \sum_{k=1}^{m} |G(t,t_k)||I_k(x(t_k))| \\
&\leq \mu A l \|x\| \sigma(T) + \mu A \sum_{k=1}^{m} c_k
\end{aligned}$$

and hence

$$\|x\| \leq \mu A l \|x\| \sigma(T) + \mu A \sum_{k=1}^{m} c_k \leq A l \|x\| \sigma(T) + A \sum_{k=1}^{m} c_k.$$

Together with (7), we obtain

$$\|x\| \leq \frac{A \sum_{k=1}^{m} c_k}{1 - A l \sigma(T)}.$$

This implies that all solutions of $x = \mu\Phi x$ are uniformly bounded independent of $\mu \in (0,1)$. From Lemma 1.3, Φ has a fixed point. This completes the proof. □

Theorem 3.6 *Suppose that there exist positive constants c and c_k, $k = 1, ..., m$, such that*

$$|f(t,x)| \leq c \text{ for all } t \in J \text{ and } x \in \mathbb{R} \tag{8}$$

and

$$|I_k(x)| \leq l_k|x| \text{ for all } x \in \mathbb{R} \text{ and } k = 1, ..., m. \tag{9}$$

If

$$A \sum_{k=1}^{m} l_k < 1, \tag{10}$$

then the problem (NP) has least one solution.

Proof. Let $x \in X$ and $t \in J$. Suppose that x is a solution of $x = \mu\Phi x$ for some $\mu \in (0,1)$. Using (8) and (9), we have

$$\begin{aligned}
|x(t)| &= \left|\mu \int_0^{\sigma(T)} G(t,s) f(s,x(s))\Delta s + \mu \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k))\right| \\
&\leq \mu \int_0^{\sigma(T)} |G(t,s)||f(s,x(s))|\Delta s + \mu \sum_{k=1}^{m} |G(t,t_k)||I_k(x(t_k))| \\
&\leq \mu A c\sigma(T) + \mu A \sum_{k=1}^{m} l_k \|x\|,
\end{aligned}$$

and hence

$$\|x\| \leq \mu A c\sigma(T) + \mu A \sum_{k=1}^{m} l_k \|x\| \leq A c\sigma(T) + A \sum_{k=1}^{m} l_k \|x\|.$$

Together with (10), we obtain that

$$\|x\| \leq \frac{A c\sigma(T)}{1 - A\sum_{k=1}^{m} l_k}.$$

This implies that all solutions of $x = \mu\Phi x$ are uniformly bounded independent of $\mu \in (0,1)$. Hence it follows from Lemma 1.3 that Φ has a fixed point. So the proof is complete. □

When all impulsive functions are linear, we have the following existence result.

Theorem 3.7 *For each $k = 1, \ldots, m$, let $I_k(x) = l_k x$, where l_k is a constant. Suppose that the following conditions hold:*

(a) $|f(t,x)| \leq c$ for all $(t,x) \in J \times \mathbb{R}$, for some positive constant c,

(b) $\prod_{k=1}^{m} b_k \neq e_p(\sigma(T), 0)$, where $b_k = l_k + 1$.

Then the problem (NP) has at least one solution.

Proof. In this case, the problem (NP) can be rewritten as

$$\begin{aligned}
&x^{\triangle} + p(t)x^{\sigma} = f(t,x),\ t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1, \ldots, m, \\
&x(t_k+) = b_k x(t_k),\ k = 1, \ldots, m, \\
&x(0) = x(\sigma(T)).
\end{aligned} \tag{11}$$

We first consider the special case: $b_{k_0} = 0$ for some $1 \leq k_0 \leq m$. Let $y(t) = e_p(t,0)x(t)$. Then

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0)f(t, e_p(0,t)y(t)), \quad t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1, \ldots, m, \\ &y(t_k+) = b_k y(t_k), \quad k \neq k_0, \\ &y(t_{k_0}+) = 0, \\ &y(0) = y(\sigma(T)). \end{aligned} \tag{12}$$

We claim that the initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0)f(t, e_p(0,t)y(t)), \quad t \in J_{k_0}, \\ &y(t_{k_0}+) = 0, \end{aligned} \tag{13}$$

has at least one solution. To show this, we define an operator $L_{k_0} : C(J_{k_0}) \to C(J_{k_0})$ by

$$(L_{k_0}y)(t) = \int_{t_{k_0}}^{t} e_p(s,0)f(s, e_p(0,s)y(s))\Delta s$$

so that the fixed points of L_{k_0} are solutions to (13). Then L_{k_0} is compact. To see this, let $D \subseteq C(J_{k_0})$ be a bounded set. For any $y \in D$ and $t \in J_{k_0}$, we have

$$\begin{aligned} |(L_{k_0}y)(t)| &= \left| \int_{t_{k_0}}^{t} e_p(s,0)f(s, e_p(0,s)y(s))\Delta s \right| \\ &\leq \int_{t_{k_0}}^{t} |e_p(s,0)||f(s, e_p(0,s)y(s))|\Delta s \\ &\leq c e_p(\sigma(T), 0)(t_{k_0+1} - t_{k_0}). \end{aligned}$$

This implies that $L_{k_0}(D)$ is uniformly bounded. Also, if $t, \tilde{t} \in J_{k_0}$ and $y \in D$, then

$$|(L_{k_0}y)(t) - (L_{k_0}y)(\tilde{t})| \leq c e_p(\sigma(T), 0)|t - \tilde{t}| \to 0,$$

uniformly for $y \in D$ as $|t - \tilde{t}| \to 0$. This implies that $L_{k_0}(D)$ is equicontinuous on J_{k_0}. Hence L_{k_0} is compact.

Let $\mu \in (0,1)$. We consider the equation

$$y = \mu L_{k_0} y. \tag{14}$$

Suppose that $y \in C(J_{k_0})$ is a solution of (14). Then

$$\begin{aligned} |y(t)| &= \left|\mu \int_{t_{k_0}}^{t} e_p(s,0) f(s, e_p(0,s)y(s))\Delta s\right| \\ &\le \int_{t_{k_0}}^{t} |e_p(s,0)||f(s, e_p(0,s)y(s))|\Delta s \\ &\le c e_p(\sigma(T),0)\sigma(T), \end{aligned}$$

and hence $\|y\| \le c e_p(\sigma(T),0)\sigma(T)$. It follows that all solutions of $y = \mu L_{k_0} y$ are bounded independent of $\mu \in (0,1)$. From Lemma 1.3, L_{k_0} has a fixed point. Hence (13) has at least one solution, saying y_{k_0}, on J_{k_0}. This determines the value of $y_{k_0}(t_{k_0+1})$ that we use as the initial value for the following problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t)y(t)), \quad t \in J_{k_0+1}, \\ &y(t_{k_0+1}+) = b_{k_0+1} y_{k_0}(t_{k_0+1}). \end{aligned} \tag{15}$$

Similarly, we can get a solution y_{k_0+1} on J_{k_0+1} for (15). Continuing this process, we know that the initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t)y(t)), \quad t \in J_j, \\ &y(t_j+) = b_j y_{j-1}(t_j). \end{aligned}$$

has a solution y_j on J_j for each $j = k_0+2, \ldots, m$. Also, the initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t)y(t)), \quad t \in J_0, \\ &y(0) = y_m(\sigma(T)). \end{aligned}$$

has a solution y_0 on J_0. As before, we know that the initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t)y(t)), \quad t \in J_j, \\ &y(t_j+) = b_j y_{j-1}(t_j). \end{aligned}$$

has a solution y_j on J_j for each $j = 1, \ldots, k_0 - 1$.

Let

$$y = \begin{cases} y_0, & \text{on } J_0, \\ y_1, & \text{on } J_1, \\ \quad\vdots & \\ y_m, & \text{on } J_m. \end{cases}$$

It is easy to see that y is a solution of (12). So (NP) has at least one solution.

Now we consider the only other case: $b_k \neq 0$ for all $k = 1, \ldots, m$. Let $x(t)$ be any solution of (11). Set

$$y(t) = x(t) \prod_{0 \le t_k < t} b_k^{-1}.$$

For all $k = 1, \ldots, m$, we have

$$\begin{aligned} y(t_k+) &= b_k x(t_k) \prod_{0 \le t_i \le t_k} b_i^{-1} = x(t_k) \prod_{0 \le t_i < t_k} b_i^{-1} = y(t_k), \\ y(t_k-) &= x(t_k) \prod_{0 \le t_i < t_k} b_k^{-1} = y(t_k). \end{aligned}$$

This shows that $y(t)$ is continuous on J. Furthmore, $y(t)$ satisfies

$$\begin{aligned} & y^{\Delta}(t) + p(t) y(\sigma(t)) = F(t, y(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ & y(0) = y(\sigma(T)) \prod_{k=1}^{m} b_k, \end{aligned} \tag{16}$$

where

$$F(t, y(t)) = f(t, y(t) \prod_{0 \le t_k < t} b_k) \prod_{0 \le t_k < t} b_k^{-1}.$$

It follows that (16) has a solution if and only if the integral equation

$$y(t) = \int_0^{\sigma(T)} \tilde{G}(t, s) F(s, y(s)) \Delta s$$

is solvable. Here,

$$\tilde{G}(t, s) = \begin{cases} \eta e_p(\sigma(T), 0) e_p(s, t), & 0 \le s \le t \le \sigma(T), \\ \eta \prod_{k=1}^{m} b_k e_p(s, t), & 0 \le t < s \le \sigma(T), \end{cases}$$

where

$$\eta = \frac{1}{e_p(\sigma(T), 0) - \prod_{k=1}^{m} b_k}.$$

Define the operator $B : C(J) \to C(J)$ by

$$By = \int_0^{\sigma(T)} \tilde{G}(t, s) F(s, y(s)) \Delta s.$$

It is easy to show that B is compact. Let $\mu \in (0, 1)$ and $y \in C(J)$. Suppose that y is a solution of

$$y = \mu B y, \tag{17}$$

on J. Then

$$|y(t)| \leq \int_0^{\sigma(T)} |\tilde{G}(t,s)||F(s,y(s))|\Delta s \leq c_1 c_2 \sigma(T),$$

where

$$c_1 = \sup\left\{ c \prod_{0\leq t_k<t} |b_k|^{-1} : t \in J \right\},$$
$$c_2 = \eta e_p(\sigma(T),0) \sup\left\{ \prod_{k=1}^{m} b_k, 1 \right\}.$$

Hence $\|y\| \leq c_1c_2$. This implies that all the solutions of (17) are bounded independent of $\mu \in (0,1)$. It follows from Lemma 1.3 that B has a fixed point. Therefore (11) has at least one solution. □

Theorem 3.8 *Suppose that the following conditions hold:*

(a) $\lim\limits_{|x|\to\infty} \dfrac{f(t,x)}{x} = 0$ *uniformly for* $t \in J$,

(b) $\lim\limits_{|x|\to\infty} \dfrac{I_k(x)}{x} = 0$ *for all* $k = 1,\ldots,m$.
Then the problem (NP) has at least one solution.

Proof. Let $H\Phi = \{x \in X : x = \mu\Phi x$ for some $\mu \in (0,1)\}$. Then $H\Phi$ is bounded. Indeed, if $H\Phi$ is unbounded, then there exist sequences $\{x_n\}_{n=1}^{\infty}$ in X and $\{\mu_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $\|x_n\| \geq n$ and

$$\begin{aligned}
&x_n^{\triangle}(t) + p(t)x_n(\sigma(t)) = \mu_n f(t,x_n(t)), \quad t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1,\ldots,m,\\
&x_n(t_k+) - x_n(t_k-) = \mu_n I_k(x_n(t_k)), \quad k = 1,\ldots,m,\\
&x_n(0) = x_n(\sigma(T)).
\end{aligned}$$

Now we let $v_n = x_n/\|x_n\|$. Then $\|v_n\| = 1$ and v_n satisfies

$$\begin{aligned}
&v_n^{\triangle}(t) + p(t)v_n(\sigma(t)) = g_n(t),, \quad t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1,\ldots,m,\\
&v_n(t_k+) - v_n(t_k-) = \theta_{n,k}, \quad k = 1,\ldots,m,\\
&v_n(0) = v_n(\sigma(T)),
\end{aligned}$$

where

$$g_n(t) = \frac{\mu_n f(t,x_n(t))}{\|x_n\|} \text{ and } \theta_{n,k} = \frac{\mu_n I_k(x_n(t_k))}{\|x_n\|}.$$

By Lemma 2.1, we get

$$v_n(t) = \int_0^{\sigma(T)} G(t,s)g_n(s)\Delta s + \sum_{k=1}^{m} G(t,t_k)\theta_{n,k}, \quad t \in J.$$

From assumptions (a) and (b), we have

$$|g_n(t)| \leq \frac{|f(t, x_n(t))|}{\|x_n\|} \to 0,$$

uniformly for $t \in J$ and

$$|\theta_{n,k}| \leq \frac{|I_k(x_n(t_k))|}{\|x_n\|} \to 0 \ , \ \ k = 1, \ldots, m,$$

as $n \to \infty$, so that

$$|v_n(t)| \leq A \left\{ \int_0^{\sigma(T)} |g_n(s)| \Delta s + \sum_{k=1}^{m} |\theta_{n,k}| \right\} \to 0,$$

uniformly for $t \in J$ as $n \to \infty$. Hence $\|v_n\| \to 0$ as $n \to \infty$, which contradicts the fact that $\|v_n\|$=1. From Lemma 1.3, the problem (NP) has at least one solution. Therefore the proof is complete. □

The following corollaries can be immediately obtained from Theorem 3.8.

Corollary 3.9 *(Bounded case) Assume that the nonlinearity f is bounded and that the impulsive functions I_k, $k = 1, \ldots, m$, are bounded. Then the nonlinear problem (NP) has at least one solution.*

Corollary 3.10 *(Sublinear growth) Suppose that there exist $a \in PC$, $b \in \mathbb{R}$ and $\alpha \in [0, 1)$ such that*

$$|f(t, x)| \leq a(t) + b|x|^{\alpha} \textit{ for all } t \in J \textit{ and } x \in \mathbb{R},$$

and suppose also that there exist positive constants $a_k, b_k \in \mathbb{R}$, and $\alpha_k \in [0, 1)$ such that

$$|I_k(x)| \leq a_k + b_k|x|^{\alpha_k} \textit{ for all } x \in \mathbb{R} \textit{ and } k = 1, \ldots, m.$$

Then the problem (NP) has at least one solution.

4 Examples

Example 5.1 Let $\mathbb{T} = [0, 1] \cup \mathbb{Z}$. We consider the following PBVP on $\mathbb{T}$

$$\begin{aligned}
&x^{\Delta} + (t+1)x^{\sigma}(t) = \frac{x}{10e^t}, \quad t \in [0, 4]_{\mathbb{T}}, t \neq \frac{1}{2}, \\
&x\left(\frac{1}{2}+\right) - x\left(\frac{1}{2}-\right) = \frac{1}{4}\sin\left(x\left(\frac{1}{2}-\right)\right), \\
&x(0) = x(\sigma(4)).
\end{aligned}$$

Let

$$p(t) = t+1,\ \ f(t,x) = \frac{x}{10e^t},\ \text{ and } I(x) = \frac{1}{4}\sin x.$$

It is easy to see that

$$|f(t,u) - f(t,v)| \leq \frac{1}{10}|u-v|,\ \text{ for all } t \in [0, \sigma(4)]_{\mathbb{T}} \text{ and } u, v \in \mathbb{R},$$

and

$$|I(u) - I(v)| \leq \frac{1}{4}|u-v| \text{ for all } u, v \in \mathbb{R}.$$

Also, by a simple computation, we get $A = 360e^{\frac{3}{2}}/(360e^{\frac{3}{2}} - 1)$ and hence

$$A\left[\sigma(4)\frac{1}{10} + \frac{1}{4}\right] = \frac{3}{4}A < 1.$$

Hence by Theorem 3.4 the PBVP has at least one solution.

Example 5.2 Let $\mathbb{T} = [0, \frac{1}{2}] \cup 2^{\mathbb{N}_0}$. We consider the following PBVP on $\mathbb{T}$

$$\begin{aligned}
&x^{\Delta} + p(t)x^{\sigma}(t) = f(t,x), \quad t \in [0,4]_{\mathbb{T}}, t \neq \frac{1}{4},\\
&x(\frac{1}{4}+) - x(\frac{1}{4}-) = I(x(\frac{1}{4}-)),\\
&x(0) = x(\sigma(4)),
\end{aligned}$$

where

$$p(t) = \begin{cases} t, & t \in [0, \frac{1}{2}], \\ 1, & t \in 2^{\mathbb{N}_0}, \end{cases},\ \ f(t,x) = \frac{2\sin t}{x^2+1},\ \text{ and } I(x) = \frac{1}{24}x.$$

It is easy to see that

$$|f(t,x)| \leq 2 \text{ for all } t \in [0, \sigma(4)]_{\mathbb{T}} \text{ and } x \in \mathbb{R},$$

and

$$|I(x)| \leq \frac{1}{24}|x| \text{ for all } x \in \mathbb{R}.$$

By a simple computation, we get $A = 75e^{\frac{1}{8}}/(75e^{\frac{1}{8}} - 2)$ and so $A/24 < 1$. Then by Theorem 3.6, the PBVP has at least one solution.

Example 5.3 Let $\mathbb{T} = \mathbb{N}_0^2 \cup [6,8]$. We consider the following PBVP on $\mathbb{T}$

$$\begin{aligned}
&x^{\Delta} + p(t)x(\sigma(t)) = f(t,x), \quad t \in [0,8]_{\mathbb{T}}, t \neq 7, \\
&x(7+) - x(7-) = I(x(7-)), \\
&x(0) = x(\sigma(8)),
\end{aligned}$$

where

$$p(t) = \begin{cases} 1, & t \in \{0,1,4\}, \\ t, & t \in [6,8], \end{cases} \quad , \ f(t,x) = \frac{x}{t+18}, \text{ and } I(x) = \sin x.$$

It is easy to see that

$$|f(t,x)| \leq \frac{1}{18}|x| \text{ for all } t \in [0,\sigma(8)]_{\mathbb{T}} \text{ and } x \in \mathbb{R},$$

and

$$|I(x)| \leq 1 \text{ for all } x \in \mathbb{R}.$$

Also, by a simple computation, we get $A = 216e^{14}/(216e^{14} - 1)$ and so $A\sigma(8)/18 = A/2 < 1$. Then by Theorem 3.5, the PBVP has at least one solution.

Example 5.4 Let $\mathbb{T}$ be a time scale and let $0, T \in \mathbb{T}$. We consider the following PBVP on $\mathbb{T}$

$$\begin{aligned}
&x^{\Delta} + x^{\sigma} = e^{\frac{1}{x}} \sin t, \quad t \in [0,T]_{\mathbb{T}}, \ t \neq t_k, \ k = 1, \dots, m, \\
&x(t_k+) - x(t_k-) = x(t_k-)^{\frac{1}{2}}, \quad k = 1, \dots, m, \\
&x(0) = x(\sigma(T)),
\end{aligned}$$

where $t_k \in (0,T)_{\mathbb{T}}$ are right-dense for all $k = 1, \dots, m$. Let $f(t,x) = e^{\frac{1}{x}} \sin t$ and $I_k(x) = x^{\frac{1}{2}}$. Then it is easy to see that

$$\lim_{|x| \to \infty} \frac{f(t,x)}{x} = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \frac{I_k(x)}{x} = 0.$$

Hence it follows from Theorem 3.8 that the PBVP has least one solution .

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無研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

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請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以 100 字為限）

      □實驗失敗

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      □其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

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3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）