行政院國家科學委員會專題研究計畫 成果報告

有關無窮維度差分方程系統全解之研究
研究成果報告(精簡版)

計 畫 類 別 ：個別型
計 畫 編 號 ：NSC 97-2115-M-004-001-
執 行 期 間 ：97 年 08 月 01 日至 98 年 07 月 31 日
執 行 單 位 ：國立政治大學應用數學學系

計 畫 主 持 人：符聖珍

計畫參與人員：碩士班研究生-兼任助理人員：林明黎

處 理 方 式 ：本計畫可公開查詢

中 華 民 國 98 年 08 月 06 日

Oscillation and nonoscillation criteria for linear dynamic systems on time scales ⋆

Sheng-Chen Fu and Ming-Li Lin

Deparment of Mathematical Sciences, National Chengchi University, 64, S-2 Zhi-nan Road, Taipei 116, Taiwan

Abstract

In this paper we establish oscillation and nonoscillation criteria for the linear dynamic system

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma}.$$

Here we assume that p and q are nonnegative, rd-continuous functions on $\mathbb{T}$, where $\mathbb{T}$ is a time scale such that $sup\mathbb{T} = \infty$. Indeed, we extend some known oscillation theories on differential systems and difference systems to the so-called dynamic systems.

Key words: Oscillation; Linear dynamic systems; Time scale

1 Introduction

Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of $\mathbb{R}$, which is unbounded above. This paper is concerned with the linear dynamic system

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma}, \tag{1}$$

where p and q are nonnegative, rd-continuous functions on $\mathbb{T}$.

The global existence and uniqueness of solutions of (1) can be easily verified by applying Theorem 5.8 of [12]. We say that a solution (u, v) of (1) is nonoscillatory if both u and v are either eventually positive or eventually

⋆ This work is partially supported by National Science Council of the Republic of China under the contract 97-2115-M-004-001.
Corresponding author: Sheng-Chen Fu.
Email addresses: fu@nccu.edu.tw (Sheng-Chen Fu); 96751010@nccu.edu.tw (Ming-Li Lin).

negative. Otherwise, it is oscillatory. System (1) is called oscillatory if all its solutions are oscillatiory. Otherwise, it is nonoscillatory.

Oscillation for system (1) has received a lot of attention by many researchers. When $\mathbb{T} = \mathbb{R}$, system (1) is equivalent to the linear differential system

$$u^{'} = pv, \qquad v^{'} = -qu. \tag{2}$$

The oscillatory property of system (2) has been extensively studied, see for example [1], [2], [3], [4], [10] and the references cited therein. When $\mathbb{T} = \mathbb{Z}$, system (1) is equivalent to the linear difference system

$$\Delta x_n = p_n y_n, \qquad \Delta y_{n-1} = -q_n x_n. \tag{3}$$

For papers dealing with oscillatory property of system (3), the reader is referred to [5], [6], [7], [8] and the references cited therein. When $p(t) \neq 0$ for all t $\in \mathbb{T}$, system (1) can be reduced to a single dynamic equation

$$(\frac{1}{p}u^{\Delta})^{\Delta} + qu^{\sigma} = 0,$$

which has been studied by many authors (see, for example, [9], [12], and the references cited therein).

Since there are few works about oscillation of dynamic systems on time scales (see [15]), motivated by [4] and [8], in the present paper we investigate oscillatory property for system (1).

The remaider of this paper is organized as follows. Section 2 contains some basic definitions and the necessary results about time scales. In Section 3, we present our main results, which include some oscillation and nonoscillation criteria for system (1). In Section 4, we provide some useful lemmas. In Section 5, we prove the main results. Finally, in Section 6, we give several examples to illustrate the applicability of the obtained results.

2 Preliminary

For completeness, we state some fundamental definitions and results concerning time scales that we will use in the sequel. More details can be found in [11], [12], and [13] . In this section, we assume that $\mathbb{T}$ is an arbitrary time scale.

Definition 2.1 *The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by*

$$\sigma(t) := inf\{s \in \mathbb{T} | s > t\}, \qquad \rho(t) := sup\{s \in \mathbb{T} | s < t\}$$

are called the forward and backward jump operators respectively. In this definition, we put $inf\varnothing = sup\mathbb{T}$ *and* $sup\varnothing = inf\mathbb{T}$. *Graininess function* $\mu : \mathbb{T} \to [0,\infty)$ *is defined by* $\mu(t) := \sigma(t) - t$.

Definition 2.2 *A point* t *in* $\mathbb{T}$ *is said to be right-dense if* $t < sup\mathbb{T}$ *and* $\sigma(t) = t$, *and left-dense if* $t > inf\mathbb{T}$ *and* $\rho(t) = t$. *Let* $\bar{\mathbb{T}} = \mathbb{T}\cup\{sup\mathbb{T}\}\cup\{inf\mathbb{T}\}$. *We call* ∞ *left-dense if* $\infty \in \bar{\mathbb{T}}$, *and* $-\infty$ *right-dense if* $-\infty \in \bar{\mathbb{T}}$.

Note that for any left-dense $t_0 \in \mathbb{T}$ and any $\varepsilon > 0$, $L_\varepsilon(t_0) = \{t \in \mathbb{T} | 0 < t_0 - t < \varepsilon\}$ is nonempty. If $\infty \in \bar{\mathbb{T}}$, $L_\varepsilon(\infty) = \{t \in \mathbb{T} | t > \frac{1}{\varepsilon}\}$ is nonempty.

Definition 2.3 *If a function* f *maps* $\mathbb{T}$ *into* $\mathbb{R}$, *we define* f^σ *by* $f^\sigma(t) = f(\sigma(t))$ *which maps* $\mathbb{T}$ *into* $\mathbb{R}$.

Definition 2.4 *Let*

$$\mathbb{T}^{\mathcal{K}} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \text{otherwise }. \end{cases} \tag{4}$$

A function $f : \mathbb{T} \to \mathbb{R}$ *is called (delta) differentiable at* $t \in \mathbb{T}^{\mathcal{K}}$ *if*

$$\lim_{s\to t} \frac{f^\sigma(t) - f(s)}{\sigma(t) - s}$$

exists, saying $f^\Delta(t)$, *where* $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ *for some* $\delta > 0$. *In this case, we call* $f^\Delta(t)$ *the (delta) derivative of* f *at* t.

Definition 2.5 *A function on* $\mathbb{T}$ *is called rd-continuous if it is continuous at all right-dense points in* $\mathbb{T}$ *and its left-sided limits exist at all left-dense points in* $\mathbb{T}$. *The set of all rd-continuous functions on* $\mathbb{T}$ *is denoted by* $C_{rd}(\mathbb{T})$.

Definition 2.6 *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called an antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *if* $F^\Delta(t) = f(t)$, *and we define* $\int_r^s f(t)\Delta t = F(s) - F(r)$ *for all* $r, s \in \mathbb{T}$.

Note that every rd-continuous function has an antiderivative.

Lemma 2.7 *Assume that* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^{\mathcal{K}}$ *and let* $\alpha \in \mathbb{R}$ *be a constant. Then the following statements are valid:*

(1) $f^\sigma(t) = f(t) + \mu(t) f^\Delta(t)$.

(2) $f + g$ *is differentiable at* t *and*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(3) αf *is differentiable at* t *and*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(4) fg is differentiable at t and

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=g^{\Delta}(t)f(t)+g^{\sigma}(t)f^{\Delta}(t).$$

(5) If $f(t)f^{\sigma}(t)\neq 0$, then $1/f$ is differentiable at t and

$$(\frac{1}{f})^{\Delta}(t)=-\frac{f^{\Delta}(t)}{f(t)f^{\sigma}(t)}.$$

(6) If $f(t)f^{\sigma}(t)\neq 0$, then g/f is differentiable at t and

$$(\frac{g}{f})^{\Delta}(t)=\frac{f(t)g^{\Delta}(t)-f^{\Delta}(t)g(t)}{f(t)f^{\sigma}(t)}.$$

Lemma 2.8 *(1) If $f\in C_{rd}(\mathbb{T})$ and $t\in\mathbb{T}^{\mathcal{K}}$, then $\int_t^{\sigma(t)}f(\tau)\Delta\tau=\mu(t)f(t)$.*
(2) If $f^{\Delta}\geq 0$, then f is nondecreasing.

Lemma 2.9 *If a, b, $c\in\mathbb{T}$, $\alpha\in\mathbb{R}$, and f, $g\in C_{rd}$, then:*

(1) $\int_a^b[f(t)+g(t)]\Delta t=\int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t$.
(2) $\int_a^b(\alpha f)(t)\Delta t=\alpha\int_a^b f(t)\Delta t$.
(3) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t$. (Integration by Parts)

Lemma 2.10 *(Chain Rule) Assume that $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $f:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $g\circ f:\mathbb{T}\to\mathbb{R}$ is differentiable and*

$$(g\circ f)^{\Delta}(t)=\left\{\int_0^1 g'\left(f(t)+h\mu(t)f^{\Delta}(t)\right)dh\right\}f^{\Delta}(t).$$

Lemma 2.11 *(L'Hôpital's Rule) Assume that f and g are differentiable on $\mathbb{T}$. If $\lim_{t\to t_0-}g(t)=\infty$ for some left-dense $t_0\in\bar{\mathbb{T}}$, and there exists $\varepsilon>0$ such that $g(t)>0$, $g^{\Delta}(t)>0$ for all $t\in L_{\varepsilon}(t_0)$, then $\lim_{t\to t_0-}\frac{f^{\Delta}(t)}{g^{\Delta}(t)}=r\in\bar{\mathbb{R}}$ implies $\lim_{t\to t_0-}\frac{f(t)}{g(t)}=r$.*

3 Main results

Let t_0 be an arbitrary point in $\mathbb{T}$.

Theorem 3.1 *If*

$$\int_{t_0}^{\infty}p(s)\Delta s=\infty,\quad \int_{t_0}^{\infty}q(s)\Delta s=\infty, \tag{5}$$

then system (1) is oscillatory.

Theorem 3.2 *If*

$$\int_{t_0}^{\infty} p(s)\Delta s < \infty, \quad \int_{t_0}^{\infty} q(s)\Delta s < \infty, \tag{6}$$

then system (1) is nonoscillatory.

In the sequel, we are going to focus on the case

$$\int_{t_0}^{\infty} p(s)\Delta s = \infty, \quad \int_{t_0}^{\infty} q(s)\Delta s < \infty. \tag{7}$$

For convenience, we put

$$f(t) = \int_{t_0}^{t} p(s)\Delta s. \tag{8}$$

Theorem 3.3 *Suppose that*

$$\lim_{t\to\infty} \frac{\mu(t)p(t)}{f(t)} = 0 \tag{9}$$

and (7) hold. Suppose also that there exists $\lambda \in (0,1)$ *such that*

$$\int_{t_0}^{\infty} f^{\lambda}(s)q(s)\Delta s = \infty. \tag{10}$$

Then system (1) is oscillatory.

According to Theorem 3.3, we can furthermore restrict to the case:

$$\int_{t_0}^{\infty} f^{\lambda}(s)q(s)\Delta s < \infty \quad for\ all\ \lambda \in [0,1). \tag{11}$$

For convenience, we define

$$g(t,\lambda) = \begin{cases} f^{1-\lambda}(t) \int_t^{\infty} f^{\lambda}(s)q(s)\Delta s, & \text{if } \lambda < 1, \\ f^{1-\lambda}(t) \int_{t_0}^{t} f^{\lambda}(s)q(s)\Delta s, & \text{if } \lambda > 1, \end{cases} \tag{12}$$

and we set

$$g_*(\lambda) = \liminf_{t\to\infty} g(t,\lambda),$$
$$g^*(\lambda) = \limsup_{t\to\infty} g(t,\lambda).$$

Theorem 3.4 *Let (7), (9) and (11) hold. If*

$$g_*(0) > \frac{1}{4} \ or \ g_*(2) > \frac{1}{4}, \tag{13}$$

then system (1) is oscillatory.

Theorem 3.5 *Let $g_*(0) \le \frac{1}{4}$, $g_*(2) \le \frac{1}{4}$, (7), (9) and (11) hold. Suppose that there exists $\lambda \in [0,1)$ such that*

$$g^*(\lambda) > \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}\left(1 + \sqrt{1-4g_*(2)}\right). \tag{14}$$

Then system (1) is oscillatory.

Corollary 3.6 *Let $g_*(0) \le \frac{1}{4}$, $g_*(2) \le \frac{1}{4}$ and (7), (9) and (11) hold. If*

$$g^*(0) > \frac{1}{2}\left(1 + \sqrt{1-4g_*(2)}\right),$$

then system (1) is oscillatory.

Theorem 3.7 *Let $g_*(0) \le \frac{1}{4}$, $g_*(2) \le \frac{1}{4}$, and (7), (9) and (11) hold. Assume that there exists $\lambda \in [0,1)$ such that*

$$g_*(0) > \frac{\lambda(2-\lambda)}{4} \tag{15}$$

and

$$g^*(\lambda) > \frac{g_*(0)}{1-\lambda} + \frac{1}{2}\left(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\right). \tag{16}$$

Then system (1) is oscillatory.

Corollary 3.8 *Let $0 < g_*(0) \le \frac{1}{4}$, $g_*(2) \le \frac{1}{4}$, and (7), (9) and (11) hold. If*

$$g^*(0) > g_*(0) + \frac{1}{2}\left(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\right), \tag{17}$$

then system (1) is oscillatory.

Remark (i) We consider the more general first-order linear dynamic system

$$u^\Delta = au + bv, \qquad v^\Delta = cu + dv, \tag{18}$$

where a, b, c, $d \in C_{rd}$ and $b \ge 0$, $c \le 0$. Suppose that $1 + \mu a > 0$ and $1 + \mu(a+d) + \mu^2(ad - bc) > 0$. Then one can easily verify that system (18) is equivalent to system (1) with

$$p(t) = \frac{b(t)}{[e_\alpha(t,t_0)(1+\mu a)]} \ge 0, \qquad q(t) = \frac{-c(t)e_\alpha(\sigma(t),t_0)}{1+\mu a} \ge 0,$$

where $\alpha(t) = [a - d + \mu(a^2 - ad + bc)]/[1 + \mu(a+d) + \mu^2(ad - bc)]$. Hence the oscillation and nonoscillation criteria for system (18) can be obtained from Theorem 3.1-3.5, 3.7 and Corollary 3.6, 3.8.

(ii) We consider the case

$$\int_{t_0}^{\infty} p(s)\Delta s < \infty, \quad \int_{t_0}^{\infty} q(s)\Delta s = \infty. \tag{19}$$

Notice that if $(u,\, v)$ is a solution of (1) then $(\tilde{u}, \tilde{v}) = (v, -u)$ is a solution of

$$\tilde{u}^{\Delta} = -\mu pq\tilde{u} + q\tilde{v}, \qquad \tilde{v}^{\Delta} = -p\tilde{u}. \tag{20}$$

From Remark (i), we see that if $\mu^2 pq < 1$ then system (20) is equivalent to

$$\tilde{u}^{\Delta} = \tilde{p}\tilde{v}, \qquad \tilde{v}^{\Delta} = -\tilde{q}\tilde{u}^{\sigma},$$

with

$$\tilde{p} = \frac{q}{[e_{\alpha}(t,t_0)(1-\mu^2 pq)]} \geq 0, \qquad \tilde{q} = \frac{pe_{\alpha}(\sigma(t),t_0)}{1-\mu^2 pq} \geq 0.$$

Hence if (19) holds and $\mu^2 pq < 1$ then the oscillation criteria for system (1) can be obtained from Theorem 3.3-3.5, 3.7 and Corollary 3.6, 3.8 by replacing $p(t)$ and $q(t)$ by $\tilde{p}(t)$ and $\tilde{q}(t)$ respectively.

4 Some auxiliary lemmas

In this section we establish some lemmas which will be needed to prove our main results. Hereafter $[t_0,\infty)_{\mathbb{T}}$ represents an interval on $\mathbb{T}$, that is, $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$. For convenience, we let

$$w(t) = \frac{v(t)}{u(t)}, \ r = \liminf_{t\to\infty} f(t)w(t), \ \textit{and}\ R = \limsup_{t\to\infty} f(t)w(t). \tag{21}$$

Lemma 4.1 *Let (7) hold. If (u,v) is a nonoscillatory solution of (1), then uv is eventually positive.*

Proof. Without loss of generality, we may assume that

$$u(t) > 0 \quad \textit{for all}\ t \in [t_0,\infty)_{\mathbb{T}}. \tag{22}$$

From (1) we have $v^{\Delta} \leq 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Hence $v(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Otherwise, there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $v(t) < 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. In this case, $u^{\Delta}(t) \leq 0$. Then $\int_{t_1}^{t} u^{\Delta}(s)\Delta s = \int_{t_1}^{t} p(s)v(s)\Delta s$ implies

$$u(t) = u(t_1) + \int_{t_1}^{t} p(s)v(s)\Delta s \leq u(t_1) + v(t_1)\int_{t_1}^{t} p(s)\Delta s.$$

Letting $t \to \infty$ and using (7), we get $\lim_{t\to\infty} u(t) = -\infty$, and this contradicts (22). $\square$

Lemma 4.2 *Let (7) hold. If (u, v) is a nonoscillatory solution of (1), then we have eventually*

$$0 \le fw^{\sigma} \le fw \le 1.$$

Proof. By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^{\Delta}(t) \ge 0$, and $v^{\Delta}(t) \le 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then we have $w(t) > 0$ and

$$w^{\Delta}(t) \le -q(t) - p(t)w(t)w^{\sigma}(t), \tag{23}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Hence we obtain

$$w^{\Delta}(t) \le 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{24}$$

It follows from (23) and (24) that

$$w^{\Delta}(t) \le -q(t) - p(t)(w^{\sigma}(t))^2, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{25}$$

Rewrite (25) as

$$q(t) \le -w^{\Delta}(t) - p(t)(w^{\sigma}(t))^2, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{26}$$

By (23) we get on $[t_0, \infty)_{\mathbb{T}}$,

$$\frac{-w^{\Delta}(t)}{w(t)w^{\sigma}(t)} \ge \frac{q(t)}{w(t)w^{\sigma}(t)} + p(t) \ge p(t),$$

and therefore

$$\int_{t_0}^{t} \frac{-w^{\Delta}(s)}{w(s)w^{\sigma}(s)} \Delta s \ge \int_{t_0}^{t} p(s)\Delta s = f(t). \tag{27}$$

Notice that

$$\int_{t_0}^{t} \frac{-w^{\Delta}(s)}{w(s)w^{\sigma}(s)} \Delta s = \int_{t_0}^{t} \left(\frac{1}{w(s)}\right)^{\Delta} \Delta s = \frac{1}{w(t)} - \frac{1}{w(t_0)} \le \frac{1}{w(t)}. \tag{28}$$

Combining (27) and (28), we deduce that

$$w(t)f(t) \le 1, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{29}$$

By (24) and (29), we have

$$0 \le f(t)w^{\sigma}(t) \le f(t)w(t) \le 1, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{30}$$

□

Lemma 4.3 *Assume that (7) and (9) hold and let $\lambda \in [0, 1)$. Then for any $\varepsilon > 0$ there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\int_{t}^{\infty} \frac{\left[(f^{\lambda})^{\Delta}(s)\right]^2}{p(s)f^{\lambda}(s)} \Delta s \le \frac{\lambda^2}{1-\lambda}(1+\varepsilon)^{2-\lambda} f^{\lambda-1}(t) \tag{31}$$

and

$$\int_t^\infty p(s)f^{\lambda-2}(s)\Delta s \le \frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda} f^{\lambda-1}(t), \tag{32}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$.

Proof. For $s \in [t_0, \infty)_{\mathbb{T}}$, by Lemma 2.10, we have

$$(f^\lambda)^\Delta(s) \le \lambda f^{\lambda-1}(s)p(s) \tag{33}$$

and

$$(f^{\lambda-1})^\Delta(s) \le (\lambda-1)f^\sigma(s)^{\lambda-2}p(s). \tag{34}$$

From (33) and (34), we get

$$\frac{\left[(f^\lambda)^\Delta(s)\right]^2}{p(s)f^\lambda(s)} \le \lambda^2 p(s)f^{\lambda-2}(s) \le \frac{-\lambda^2}{1-\lambda}\left(\frac{f(s)}{f^\sigma(s)}\right)^{\lambda-2}(f^{\lambda-1})^\Delta(s). \tag{35}$$

Let $\varepsilon > 0$ be given. From (9), there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\frac{u(s)p(s)}{f(s)} < \varepsilon, \tag{36}$$

for all $s \in [t_1, \infty)_{\mathbb{T}}$. Then for any $s \in [t_1, \infty)_{\mathbb{T}}$ we have

$$\frac{f^\sigma(s)}{f(s)} = 1 + \frac{\mu(s)p(s)}{f(s)} < 1 + \varepsilon. \tag{37}$$

Integrating (35) from t to ∞ and by (7) and (37), we have

$$\begin{aligned}\int_t^\infty \frac{\left[(f^\lambda)^\Delta(s)\right]^2}{p(s)f^\lambda(s)}\Delta s &\le \frac{-\lambda^2}{1-\lambda}(1+\varepsilon)^{2-\lambda}\int_t^\infty (f^{\lambda-1})^\Delta(s)\Delta s\\ &= \frac{\lambda^2}{1-\lambda}(1+\varepsilon)^{2-\lambda}f^{\lambda-1}(t),\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore we get (31).
With the similar process, we can also get (32). Indeed, it follows from (7), (34) and (37) that

$$\begin{aligned}\textstyle\int_t^\infty p(s)f^{\lambda-2}(s)\Delta s &= \int_t^\infty p(s)f^\sigma(s)^{\lambda-2}\left(\frac{f(s)}{f^\sigma(s)}\right)^{\lambda-2}\Delta s\\ &\le (1+\varepsilon)^{2-\lambda}\int_t^\infty p(s)f^\sigma(s)^{\lambda-2}\Delta s\\ &\le (1+\varepsilon)^{2-\lambda}\int_t^\infty \left(\frac{f^{\lambda-1}(s)}{\lambda-1}\right)^\Delta \Delta s\\ &= \frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda}f^{\lambda-1}(t),\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore (32) holds. $\square$

Lemma 4.4 *Assume that (7), (9) and (11) hold. If system (1) is nonoscillatory, then*

$$r^2 - r + g_*(0) \le 0, \tag{38}$$

$$R^2 - R + g_*(2) \le 0. \tag{39}$$

Proof. Let (u,v) be a nonoscillatory solution of (1). By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^{\Delta}(t) \ge 0$, and $v^{\Delta}(t) \le 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then by (7) and (30) we obtain that $0 \le r \le R \le 1$ and $\lim_{t\to\infty} w(t) = 0$.
First, we derive (38). If $g_*(0) = 0$, then (38) is clear. Now we suppose that $g_*(0) \ne 0$. Integrating (25) from t to ∞ and then multiplying by $f(t)$, we have

$$f(t)\int_t^{\infty} w^{\Delta}(s)\Delta s \le -f(t)\int_t^{\infty} q(s)\Delta s - f(t)\int_t^{\infty} p(s)(w^{\sigma}(s))^2\Delta s,$$

and hence

$$f(t)w(t) \ge f(t)\int_t^{\infty} q(s)\Delta s + f(t)\int_t^{\infty} p(s)(w^{\sigma}(s))^2\Delta s, \tag{40}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. We divide into two cases: $0 < r \le 1$ or $r = 0$. For the case $0 < r \le 1$, we let $0 < \varepsilon_1 < \min\{r, g_*(0)\}$. From (9) and the definitions of r and $g_*(0)$, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\frac{f}{f^{\sigma}} > 1 - \varepsilon_1,$$

$$f(t)w(t) > r - \varepsilon_1,$$

and

$$f(t)\int_t^{\infty} q(s)\Delta s > g_*(0) - \varepsilon_1, \tag{41}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since

$$\int_t^{\infty} \frac{f^{\Delta}(s)}{f(s)f^{\sigma}(s)}\Delta s = \frac{1}{f(t)}, \tag{42}$$

we obtain on $[t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
&f(t)\int_t^{\infty} p(s)(w^{\sigma}(s))^2\Delta s \\
&= f(t)\int_t^{\infty} \frac{f^{\Delta}(s)}{f(s)f^{\sigma}(s)}\frac{f(s)}{f^{\sigma}(s)}(f^{\sigma}(s)w^{\sigma}(s))^2\Delta s \\
&\ge (1-\varepsilon_1)(r-\varepsilon_1)^2 f(t)\int_t^{\infty} \frac{f^{\Delta}(s)}{f(s)f^{\sigma}(s)}\Delta s \\
&= (1-\varepsilon_1)(r-\varepsilon_1)^2.
\end{aligned} \tag{43}$$

By (40), (41) and (43), we get

$$f(t)w(t) \geq (g_*(0) - \varepsilon_1) + (1 - \varepsilon_1)(r - \varepsilon_1)^2, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{44}$$

Taking $\liminf_{t\to\infty}$ on (44) and letting $\varepsilon_1 \to 0$, we have $r \geq g_*(0) + r^2$. Hence (38) follows. For the case $r = 0$, it suffices to show that $g_*(0) \leq 0$. Let $0 < \varepsilon_1 < g_*(0)$. As above, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that (41) hold on $[t_1, \infty)_{\mathbb{T}}$. Then by (40) and (41), we get

$$f(t)w(t) \geq g_*(0) - \varepsilon_1, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{45}$$

Taking $\liminf_{t\to\infty}$ on (45) and letting $\varepsilon_1 \to 0$, we get $g_*(0) \leq 0$. Now we derive (39). If $g_*(2) = 0$, then (39) is clear. Now we suppose that $g_*(2) \neq 0$. By Lemma 2.7 (1) and Lemma 2.10, we have on $[t_0, \infty)_{\mathbb{T}}$,

$$(f^2(s))^\Delta \leq 2f^\sigma(s)p(s) = 2\left(f(s) + \mu(s)p(s)\right)p(s). \tag{46}$$

Hence, multiplying (26) by $f^2(t)$ and integrating from t_0 to t and using (46), we get on $[t_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
\textstyle\int_{t_0}^t f^2(s)q(s)\Delta s &\leq -\textstyle\int_{t_0}^t f^2(s)w^\Delta(s)\Delta s - \int_{t_0}^t f^2(s)p(s)(w^\sigma(s))^2\Delta s \\
&= -f^2(t)w(t) + f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t (f^2(s))^\Delta w^\sigma(s)\Delta s \\
&\quad - \textstyle\int_{t_0}^t f^2(s)p(s)(w^\sigma(s))^2\Delta s \\
&\leq -f^2(t)w(t) + f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s \\
&\quad + \textstyle\int_{t_0}^t 2f(s)p(s)w^\sigma(s)\Delta s - \int_{t_0}^t f^2(s)p(s)(w^\sigma(s))^2\Delta s \\
&\leq -f^2(t)w(t) + f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s \\
&\quad + \textstyle\int_{t_0}^t f(s)p(s)w^\sigma(s)[2 - f(s)w^\sigma(s)]\Delta s.
\end{aligned}$$

Dividing this inequality by $f(t)$ and rearranging, we get on $[t_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
f(t)w(t) \leq &-\frac{1}{f(t)} \textstyle\int_{t_0}^t f^2(s)q(s)\Delta s \\
&+\frac{1}{f(t)} \left[f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s\right] \\
&+\frac{1}{f(t)} \textstyle\int_{t_0}^t f(s)p(s)w^\sigma(s)\left[2 - f(s)w^\sigma(s)\right]\Delta s.
\end{aligned} \tag{47}$$

We consider the second part of the right hand side of (47). Applying Lemma 2.11 and using (9) and (30), we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{f^2(t_0)w(t_0) + \int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s}{f(t)} \\
&= \lim_{t\to\infty} 2\mu(t)p(t)w^\sigma(t) \\
&\leq \lim_{t\to\infty} \frac{2\mu(t)p(t)}{f(t)} \\
&= 0.
\end{aligned} \tag{48}$$

Now we divide into two cases: $0 \leq R < 1$ or $R = 1$. For the case $0 \leq R < 1$, we let $0 < \varepsilon_2 < \min\{1 - R, g_*(2)\}$. From (9) and the definitions of R and $g_*(2)$, there exists $t_2 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$0 < f(t)w(t) < R + \varepsilon_2$$

and

$$\frac{1}{f(t)} \int_{t_0}^t f^2(s)q(s)\Delta s > g_*(2) - \varepsilon_2, \tag{49}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Since $0 < f(s)w^\sigma(s) \leq f(s)w(s) < R + \varepsilon_2 < 1$ on $[t_2, \infty)_{\mathbb{T}}$, we obtain that on $[t_2, \infty)_{\mathbb{T}}$,

$$\frac{1}{f(t)} \int_{t_0}^t f(s)p(s)w^\sigma(s)[2 - f(s)w^\sigma(s)]\Delta s < (R + \varepsilon_2)(2 - R - \varepsilon_2). \tag{50}$$

By (47), (49) and (50), we get on $[t_2, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
f(t)w(t) \leq\ & (-g_*(2) + \varepsilon_2) \\
& + \frac{1}{f(t)}[f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s] \\
& + (R + \varepsilon_2)(2 - R - \varepsilon_2).
\end{aligned} \tag{51}$$

Taking $\limsup_{t\to\infty}$ on (51) and letting $\varepsilon_2 \to 0$, we have $R \leq -g_*(2) + R(2 - R)$. Hence we get (39). For the case $R = 1$, we let $0 < \varepsilon_2 < g_*(2)$. As above, there exists $t_2 \in [t_0, \infty)_{\mathbb{T}}$ such that (49) holds. Since $0 < f(s)w^\sigma(s) \leq 1 = R$, we obtain that

$$\frac{1}{f(t)} \int_{t_0}^t f(s)p(s)w^\sigma(s)[2 - f(s)w^\sigma(s)]\Delta s \leq R(2 - R). \tag{52}$$

Then, using (47), (49) and (52), we get on $[t_2, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
f(t)w(t) \leq\ & (-g_*(2) + \varepsilon_2) \\
& + (\frac{1}{f(t)}[f^2(t_0)w(t_0) + \textstyle\int_{t_0}^t 2\mu(s)(p(s))^2 w^\sigma(s)\Delta s]) \\
& + R(2 - R).
\end{aligned} \tag{53}$$

Taking $\limsup_{t\to\infty}$ on (53) and letting $\varepsilon_2 \to 0$, we have $R \leq -g_*(2) + R(2-R)$. Hence we can also easily get (39). Therefore the lemma follows. □

5 Proof of the main results

Proof of Theorem 3.1. For contradiction, we assume that (1) has a nonoscillatory solution (u, v). By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^{\Delta}(t) \geq 0$, and $v^{\Delta}(t) \leq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then

$$u(t) \geq u(t_0) > 0 \quad for\ all\ t \in [t_0, \infty)_{\mathbb{T}}. \tag{54}$$

Integrating the second equation of (1) from t_0 to ∞ and using (54), we obtain

$$\int_{t_0}^{\infty} v^{\Delta}(s)\Delta s = -\int_{t_0}^{\infty} q(s)u^{\sigma}(s)\Delta s \leq -u(t_0)\int_{t_0}^{\infty} q(s)\Delta s.$$

It follows that $u(t_0)\int_{t_0}^{\infty} q(s)\Delta s \leq v(t_0) - v(\infty) \leq v(t_0) < \infty$, which contradicts (5). □

Proof of Theorem 3.2. Let $c, d > 0$ be given. From (6), there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\int_{T}^{\infty} p(s)(d + c\int_{s}^{\infty} q(\tau)\Delta\tau)\Delta s < a,$$

where $a = c/2$. Let U be the Banach space of all real-valued rd-continuous functions on $[T, \infty)_{\mathbb{T}}$ endowed with the norm $\|u\| = \sup_{t\in[T,\infty)_{\mathbb{T}}} | u(t) |$, i.e,

$$U = \{u : [T, \infty)_{\mathbb{T}} \to \mathbb{R} \mid \|u\| = \sup_{t\in[T,\infty)_{\mathbb{T}}} | u(t) | < \infty\},$$

and with the usual pointwise ordering $\leq$. First, we define a subset Ω of U as :

$$\Omega = \{u \in U \mid a \leq u(t) \leq 2a, \forall t \in [T, \infty)_{\mathbb{T}}\}.$$

Then $\inf B \in \Omega$ and $\sup B \in \Omega$ for any nonempty subset B of Ω. Second, we define an operation $F : \Omega \to U$ as :

$$(Fu)(t) = a + \int_{T}^{t} p(s)(d + \int_{s}^{\infty} q(\tau)u^{\sigma}(\tau)\Delta\tau)\Delta s, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Then F is nondecreasing. Also, if $u \in \Omega$, by the definition of F, it is clear that $(Fu)(t) \geq a$ and

$$\begin{aligned}(Fu)(t) &\leq a + \int_T^t p(s)(d + \int_s^\infty q(\tau)(2a)\Delta\tau)\Delta s\\ &= a + \int_T^t p(s)(d + c\int_s^\infty q(\tau)\Delta\tau)\Delta s\\ &\leq 2a,\end{aligned}$$

for all $t \in [T,\infty)_{\mathbb{T}}$. Hence F maps into itself. By Knaster's fixed-point theorem [14], there exists an $u \in \Omega$ such that $u = Fu$. Set

$$v(t) = d + \int_t^\infty q(k)u^\sigma(\tau)\Delta\tau,\ t \in [T,\infty)_{\mathbb{T}}.$$

It is easy to see that $v^\Delta(t) = -q(t)u^\sigma(t)$, $\lim_{t\to\infty} v(t) = d > 0$ and $u^\Delta(t) = p(t)v(t)$. Hence (u,v) is a nonoscillatory solution of (1). Therefore we conclude that system (1) is nonoscillatory. □

Proof of Theorem 3.3. For contradiction, we assume that (1) has a nonoscillatory solution (u,v). By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^\Delta(t) \geq 0$, and $v^\Delta(t) \leq 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Then by (30), we obtain that

$$0 < f(t)w^\sigma(t) \leq 1, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

and hence there exists a positive constant c such that

$$\mid f(t)w^\sigma(t)(\lambda - f(t)w^\sigma(t)) \mid< c, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{55}$$

Multiplying (26) by $f^\lambda(t)$ and then integrating from t_0 to t, we have

$$\begin{aligned}&\int_{t_0}^t f^\lambda(s)q(s)\Delta s\\ &\leq -\int_{t_0}^t f^\lambda(s)w^\Delta(s)\Delta s - \int_{t_0}^t f^\lambda(s)p(s)(w^\sigma(s))^2\Delta s\end{aligned} \tag{56}$$

Applying integration by part and Lemma 2.10, we get

$$\begin{aligned}&\int_{t_0}^t f^\lambda(s)w^\Delta(s)\Delta s\\ &= (f^\lambda w)(t) - (f^\lambda w)(t_0) - \int_{t_0}^t (f^\lambda(s))^\Delta w^\sigma(s)\Delta s\\ &\geq (f^\lambda w)(t) - (f^\lambda w)(t_0) - \int_{t_0}^t \lambda f^{\lambda-1}(s)p(s)w^\sigma(s)\Delta s.\end{aligned} \tag{57}$$

From (55), (56), (57) and Lemma 4.3, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$
\begin{aligned}
&\textstyle\int_{t_0}^{t} f^{\lambda}(s)q(s)\Delta s \\
&\le f^{\lambda}(t_0)w(t_0) + \textstyle\int_{t_0}^{t} p(s)f^{\lambda-2}(s)[f(s)w^{\sigma}(s)(\lambda - f(s)w^{\sigma}(s))]\Delta s \\
&\le f^{\lambda}(t_0)w(t_0) + c\textstyle\int_{t_0}^{t} p(s)f^{\lambda-2}(s)\Delta s \\
&\le f^{\lambda}(t_0)w(t_0) + c\textstyle\int_{t_0}^{t_1} p(s)f^{\lambda-2}(s) + \dfrac{c(1+\varepsilon)^{2-\lambda}}{1-\lambda} f^{\lambda-1}(t_0),
\end{aligned}
\tag{58}
$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore we have $\int_{t_0}^{\infty} f^{\lambda}(s)q(s)\Delta s < \infty$, which contradicts (10). □

Proof of Theorem 3.4. For contradiction, we assume that (1) is nonoscillatory. Then Lemma 4.4 implies that $g_*(0) \le r - r^2 \le \frac{1}{4}$ and $g_*(2) \le R - R^2 \le \frac{1}{4}$, which contradicts (13). □

Proof of Theorem 3.5. For contradiction, we assume that (1) has a nonoscillatory solution (u, v). By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^{\Delta}(t) \ge 0$, and $v^{\Delta}(t) \le 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then Lemma 4.4 implies

$$
R \le \frac{1}{2}\left(1 + \sqrt{1 - 4g_*(2)}\right). \tag{59}
$$

Multiplying (26) by $f^{\lambda}(t)$ and then integrating from t to ∞, we have

$$
\begin{aligned}
\int_t^{\infty} f^{\lambda}(s)q(s)\Delta s &\le -\int_t^{\infty} f^{\lambda}(s)w^{\Delta}(s)\Delta s - \int_t^{\infty} f^{\lambda}(s)p(s)(w^{\sigma}(s))^2\Delta s \\
&= -f^{\lambda}(\infty)w(\infty) + f^{\lambda}(t)w(t) + \int_t^{\infty} (f^{\lambda}(s))^{\Delta} w^{\sigma}(s)\Delta s \\
&\quad - \int_t^{\infty} f^{\lambda}(s)p(s)(w^{\sigma}(s))^2\Delta s \\
&= f^{\lambda}(t)w(t) + \frac{1}{4}\int_t^{\infty} \frac{\left[(f^{\lambda})^{\Delta}(s)\right]^2}{p(s)f^{\lambda}(s)}\Delta s \\
&\quad - \int_t^{\infty} \left[p^{1/2}(s)f^{\lambda/2}(s)w^{\sigma}(s) - \frac{1}{2}(f^{-\lambda/2}(s)p^{-1/2}(s)(f^{\lambda}(s))^{\Delta}\right]^2 \Delta s \\
&\le f^{\lambda}(t)w(t) + \frac{1}{4}\int_t^{\infty} \frac{\left[(f^{\lambda})^{\Delta}(s)\right]^2}{p(s)f^{\lambda}(s)}\Delta s.
\end{aligned}
$$

Then multiplying this inequality by $f^{1-\lambda}(t)$ and applying Lemma 4.3, we have on $[t_1, \infty)_{\mathbb{T}}$,

$$
f^{1-\lambda}(t)\int_t^{\infty} f^{\lambda}(s)q(s)\Delta s \le f(t)w(t) + \frac{\lambda^2}{4(1-\lambda)}(1+\varepsilon)^{2-\lambda}, \tag{60}
$$

where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is given in Lemma 4.3. Taking $\limsup_{t\to\infty}$ on (60), letting $\varepsilon \to 0$, and using (59), we obtain

$$g^*(\lambda) \leq R + \frac{\lambda^2}{4(1-\lambda)} \leq \frac{1}{2}\left(1 + \sqrt{1 - 4g_*(2)}\right) + \frac{\lambda^2}{4(1-\lambda)},$$

which contradicts (14). □

Proof of Corollary 3.6. By taking $\lambda = 0$ in Theorem 3.5, the proof is done. □

Proof of Theorem 3.7. For contradiction, we assume that (1) has a nonoscillatory solution (u, v). By applying Lemma 4.1 and (1), without loss of generality, we may assume that $u(t) > 0$, $v(t) > 0$, $u^\Delta(t) \geq 0$, and $v^\Delta(t) \leq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then Lemma 4.4 implies

$$r \geq \frac{1}{2}\left(1 - \sqrt{1 - 4g_*(0)}\right) \triangleq m,$$
$$R \leq \frac{1}{2}\left(1 + \sqrt{1 - 4g_*(2)}\right) \triangleq M.$$

By (15), we have $1 - 4g_*(0) < 1 - \lambda(2-\lambda) = (1-\lambda)^2$, and hence

$$m = \frac{1}{2}\left(1 - \sqrt{1 - 4g_*(0)}\right) > \frac{1}{2}\left(1 - (1-\lambda)\right) = \frac{\lambda}{2}.$$

Take $\varepsilon > 0$ such that $(1-\varepsilon)(m-\varepsilon) \geq \lambda/2$. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$m - \varepsilon < f(t)w(t) < M + \varepsilon \tag{61}$$

and

$$\frac{f}{f^\sigma} > 1 - \varepsilon \tag{62}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Multiplying (26) by $f^\lambda(s)$ and then integrating from t to ∞, we have on $[t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
&\int_t^\infty f^\lambda(s)q(s)\Delta s \\
&\leq -\int_t^\infty f^\lambda(s)w^\Delta(s)\Delta s - \int_t^\infty p(s)f^\lambda(s)(w^\sigma(s))^2\Delta s \\
&\leq f^\lambda(t)w(t) + \int_t^\infty (f^\lambda(s))^\Delta w^\sigma(s)\Delta s - \int_t^\infty p(s)f^\lambda(s)(w^\sigma(s))^2\Delta s \\
&\leq f^\lambda(t)w(t) + \int_t^\infty \lambda f^{\lambda-1}(s)p(s)w^\sigma(s)\Delta s - \int_t^\infty p(s)f^\lambda(s)(w^\sigma(s))^2\Delta s.
\end{aligned}$$

Multiplying this inequality by $f^{1-\lambda}(t)$ and using (61), (62) and Lemma 4.3, we have

$$\begin{aligned} & f^{1-\lambda}(t)\int_t^\infty f^\lambda(s)q(s)\Delta s \\ \le\ & f(t)w(t) + f^{1-\lambda}(t)\int_t^\infty p(s)f^{\lambda-2}(s)[f(s)w^\sigma(s)(\lambda - f(s)w^\sigma(s))]\Delta s \\ <\ & (M+\varepsilon) + (1-\varepsilon)(m-\varepsilon)[\lambda-(1-\varepsilon)(m-\varepsilon)]f^{1-\lambda}(t)\int_t^\infty p(s)f^{\lambda-2}(s)\Delta s \\ <\ & (M+\varepsilon) + (1-\varepsilon)(m-\varepsilon)[\lambda-(1-\varepsilon)(m-\varepsilon)]\frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda}, \end{aligned}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Taking $\limsup_{t\to\infty}$ and letting $\varepsilon \to 0$, we have

$$\begin{aligned} g^*(\lambda) &\le M + \frac{m(\lambda-m)}{1-\lambda} \\ &= \frac{g_*(0)}{1-\lambda} + \frac{1}{2}\left(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\right), \end{aligned}$$

which contradicts (16). □

Proof of Corollary 3.8. By taking $\lambda = 0$ in Theorem 3.7, the proof is done. □

6 Some examples

Example 1 Consider the system

$$u^\Delta(t) = \frac{1}{t\ln t}v(t), \qquad v^\Delta(t) = -(t+\sigma(t))u^\sigma(t),$$

where $\mathbb{T} = a^{\mathbb{N}_0}$, $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$, and $a > 1$.
Let $p(t) = \dfrac{1}{t\ln t}$ and $q(t) = t + \sigma(t)$. Since

$$\int_a^\infty p(s)\Delta s = \sum_{i=1}^\infty \frac{a-1}{i\ln a} = \frac{a-1}{\ln a}\sum_{i=1}^\infty \frac{1}{i} = \infty$$

and

$$\int_a^\infty q(s)\Delta s = \sum_{i=1}^\infty a^{2i}(a^2-1) = \infty,$$

the system is oscillatory by Theorem 3.1.

Example 2 Consider the system

$$u^\Delta(t) = \frac{1}{t^2}v(t), \qquad v^\Delta(t) = -\frac{1}{t\sigma(t)}u^\sigma(t),$$

where $\mathbb{T} = a^{\mathbb{N}_0}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $a > 1$.
Let $p(t) = \dfrac{1}{t^2}$ and $q(t) = \dfrac{1}{t\sigma(t)}$. Since

$$\int_1^\infty p(s)\Delta s = \sum_{i=0}^\infty \frac{a-1}{a^i} = a < \infty$$

and

$$\int_1^\infty q(s)\Delta s = \sum_{i=0}^\infty \frac{a-1}{a^{i+1}} = 1 < \infty,$$

the system is nonoscillatory by Theorem 3.2.

Example 3 Consider the system

$$u^\Delta(t) = tv(t), \qquad v^\Delta(t) = -\frac{1}{t^2}u^\sigma(t),$$

where $\mathbb{T} = a\mathbb{N} = \{an \mid n \in \mathbb{N}\}$ and a is a positive constant.
Let $p(t) = t$ and $q(t) = \dfrac{1}{t^2}$. Since

$$\int_a^\infty p(s)\Delta s = a^2 \sum_{i=1}^\infty i = \infty,$$

$$f(t) = \int_a^t p(s)\Delta s = a^2 \sum_{i=1}^n i = \frac{a^2 n(n+1)}{2}, \ \forall\, t = an \in \mathbb{T}.$$

$$\lim_{t\to\infty} \frac{\mu(t)p(t)}{f(t)} = \lim_{n\to\infty} \frac{2}{n+1} = 0,$$

$$\int_a^\infty q(s)\Delta s = \sum_{i=1}^\infty \frac{a}{(ai)^2} = \frac{1}{a}\sum_{i=1}^\infty \frac{1}{i^2} < \infty,$$

and

$$\int_a^\infty f^\lambda(s)q(s)\Delta s = \frac{a^{2\lambda-1}}{2^\lambda}\sum_{i=1}^\infty \frac{[i(i+1)]^\lambda}{i^2} > \frac{a^{2\lambda-2}}{2^\lambda}\sum_{i=1}^\infty i^{2\lambda-2} = \infty \; if \; \frac{1}{2} \le \lambda < 1,$$

the system is oscillatory by Theorem 3.3.

Example 4 Consider the system

$$u^\Delta(t) = v(t), \qquad v^\Delta(t) = -\frac{1}{t^2}u^\sigma(t),$$

where $\mathbb{T} = a\mathbb{N} = \{an \mid n \in \mathbb{N}\}$ and a is a positive constant.
Let $p(t) = 1$ and $q(t) = \dfrac{1}{t^2}$. We compute

$$\int_a^\infty p(s)\Delta s = \int_a^\infty 1\Delta s = \infty,$$

$$f(t) = \int_a^t 1\Delta s = \sum_{i=1}^{n} a = an, \ \forall\, t = an \in \mathbb{T},$$

and

$$\lim_{t\to\infty} \frac{\mu(t)p(t)}{f(t)} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

From Example 3, we know that

$$\int_a^\infty q(s)\Delta s < \infty.$$

Since

$$\int_a^\infty f^\lambda(s)q(s)\Delta s = a^{\lambda-1}\sum_{i=1}^{\infty} \frac{1}{i^{2-\lambda}} < \infty \ for\ all\ \lambda \in [0,1)$$

and

$$g_*(0) = \liminf_{n\to\infty} n\sum_{i=n}^{\infty}\frac{1}{i^2} > \liminf_{n\to\infty} n\int_n^\infty \frac{1}{x^2}dx = 1 > \frac{1}{4},$$

the system is oscillatory by Theorem 3.4.

Example 5 Consider the system

$$u^\Delta(t) = p(t)v(t), \qquad v^\Delta(t) = -q(t)u^\sigma(t),$$

where $\mathbb{T} = 2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$, $p(t) = 1 + \dfrac{1}{t}$ and

$$q(t) = \begin{cases} \frac{1}{2}8^{-k}, & t = 8^k \\ 0, & t \neq 8^k, \end{cases} \quad k = 1, 2, ...$$

First, we compute

$$\int_4^\infty p(s)\Delta s = \sum_{i=2}^{\infty} 2\left(1 + \frac{1}{2i}\right) = \infty,$$

$$f(t) = \int_4^t p(s)\Delta s = \sum_{i=2}^{n} 2\left(1 + \frac{1}{2i}\right) = 2n - 2 + \sum_{i=2}^{n}\frac{1}{i}, \ \forall t = 2n \in \mathbb{T}.$$

$$\lim_{t\to\infty}\frac{\mu(t)p(t)}{f(t)} = \lim_{n\to\infty}\frac{2\left(1+\frac{1}{2n}\right)}{2n-2+\sum_{i=2}^{n}\frac{1}{i}} = 0,$$

and

$$\int_4^\infty q(s)\Delta s = \sum_{k=1}^{\infty} 2q(8^k) = \sum_{k=1}^{\infty} 8^{-k} = \frac{1}{7} < \infty.$$

Since $\lambda < 1$, we have

$$\int_4^\infty f^\lambda(s)q(s)\Delta s = \sum_{k=1}^\infty 2f^\lambda(8^k)q(8^k) = \sum_{k=1}^\infty \left[\sum_{l=1}^k 2\left(1+\frac{1}{8^l}\right)\right]^\lambda 8^{-k}$$
$$< \sum_{k=1}^\infty \left[2k + \frac{2}{7}\left(1-\frac{1}{8^k}\right)\right] 8^{-k} < \infty.$$

For $t = 8^m + 2$, $m = 1, 2, ...,$

$$g(t,0) = \left(8^m + \sum_{i=2}^{8^m/2+1} \frac{1}{i}\right) \sum_{k=m+1}^\infty 8^{-k} \to \frac{1}{7},\ as\ m \to \infty.$$

For $t = 8^m$, $m = 1, 2, ...,$

$$g(t,0) = \left(8^m - 2 + \sum_{i=2}^{8^m/2} \frac{1}{i}\right) \sum_{k=m}^\infty 8^{-k} \to \frac{8}{7},\ as\ m \to \infty.$$

For $t = 8^m - 2$, $m = 1, 2, ...,$ we have

$$g(t,2) = \left(8^m - 4 + \sum_{i=2}^{8^m/2-1} \frac{1}{i}\right)^{-1} \sum_{k=1}^{m-1} \left(8^k - 2 + \sum_{l=2}^{8^k/2} \frac{1}{l}\right)^2 8^{-k} \to \frac{1}{7},\ as\ m \to \infty.$$

Therefore we get $g_*(0) \le \frac{1}{7} \le \frac{1}{4}$, $g_*(2) \le \frac{1}{7} \le \frac{1}{4}$ and $g^*(0) \ge \frac{8}{7}$. Hence the system is oscillatory by Corollary 3.6 and 3.8.

References

[1] J. D. Mirzov, On some analogues of Sturm's and Kneser's theorems for nonlinear systems, J. Math. Anal. Appl. 53(1976), No. 2, 418-425.

[2] J. D. Mirzov, On oscillation of solutions of a certain system of differential equations. (Russian) Mat. Zametki 23(1978), No. 3, 401-404.

[3] J. D. Mirzov, Asymptotic behaviour of solutions of systems of nonlinear non-autonomous ordinary differential equations. (Russian) Maikop, 1993.

[4] A. Lomtatidze and N. Partsvania, Oscillation and nonoscillation criteria for two-dimensional systems of first order linear ordinary differential equations, Georgian Math. J. 6(1999), No. 3, 285-298.

[5] J.R. Graef and E. Thandapani, Oscillation of two-dimensional difference systems, Comput. Math. Appl. 38 (1999) 157165.

[6] H.F. Huo and W.T. Li, Oscillation of certain two-dimensional nonlinear difference systems, Comput. Math. Appl. 45 (2003) 12211226.

[7] J. Jiang and X. Tang, Oscillation and asymptotic behavior of two-dimensional difference systems, J. Comput. Math. Appl. 54 (2007) 12401249.

[8] J. Jiang and X. Tang, Oscillation criteria for two-dimensional difference systems of first order difference equations, Comput. Math. 54(2007) 808-818.

[9] L. Erbe and A. Peterson, Oscillation criteria for second-order matrix dynamic equations on a time scale, Journal of Computational and Applied Mathematics 141(2002) 169185.

[10] W. T. Li, Classification schemes for nonoscillator solutions of two-dimensional nonlinear difference systems, Comput. Math. Appl. 42(2001) 341-355.

[11] R.P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, Results Math. 35(1999) 3-22.

[12] M. Bohner and A. Peterson, Dynamic Equation on Time Scales, An Introduction with Application, Birkhauser, Boston (2001).

[13] M. Bohner and A. Peterson, Advances in Dynamic Equation on Time Scales, Birkhauser, Boston (2003).

[14] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, (1991).

[15] Y. Xu and Z. Xu, Oscillation criteria for two-dimensional dynamic systems on time scales, Journal of Computational and Applied Mathematics 225 (2009) 9-19.