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ENTIRE SOLUTIONS FOR DISCRETE REACTION-DIFFUSION EQUATIONS

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Abstract. This paper deals with a discrete reaction-diffusion equation $u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f(u(x,t))$, where $f(u) = u^2(1-u)$. Here, we prove there exist entire solutions which behave as two travelling waves coming from both sides of x-axis.

1. Introduction

In this paper, we consider the following discrete reaction-diffusion equation

$$u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f(u(x,t)), \tag{1.1}$$

which is a discrete version of the following semilinear parabolic equation

$$u_t = u_{xx} + f(u). \tag{1.2}$$

When the function $f(u)$ is such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$ and $f(u) > 0$ for any $0 < u < 1$, (1.2) is called the Fisher's equation [4] or Kolmogorov, Petrovsky and Piskunov (KPP) equation [6], and it describes the propagation of an advantageous gene within an one-dimensional habitat. When $f(u) = u^m(1-u)$, where m is an integer greater than two, it is called the mth-order Fisher's equation. In particular, it is called the Zeldovich equation if $m = 2$. For a cubic nonlinearly $f(u) = u(1-u)(u-a)$, it is called the Allen-Cahn equation ($a = 1/2$) in phase transition and also the Nagumo equation ($a \in (0,1)$) in propagation of nerve excitation. A great deal of work has been carried out to extend this equation to take into account other biological, chemical or physical factors.

A solution $u(x,t)$ of (1.1) is called a travelling wave with speed c if there exists a function $U : \mathbb{R} \to [0,1]$ such that $u(x,t) = U(x+ct)$, which connects two equilibria $u = 0, 1$. Such solution (c, U) satisfies the following travelling wave problem and it is unique up to translation

$$\begin{cases} cU'(\cdot) = U(\cdot+1) + U(\cdot-1) - 2U(\cdot) + f(U(\cdot)) \text{ on } \mathbb{R}, \\ U(-\infty) = 0,\ U(\infty) = 1,\ 0 \le U \le 1 \text{ on } \mathbb{R}. \end{cases} \tag{1.3}$$

When f is Lipschitz continuous on [0,1] with $f(0) = f(1) = 0 < f(u)$ for all $u \in (0,1)$, it has been shown in [2] that there exists $c_{min} > 0$ such that (1.3) admits a solution if and only if $c \ge c_{min}$. The existence, uniqueness and asymptotic stability of travelling waves, we refer the readers to [2, 3] and the references therein.

From the dynamical point of view, the travelling wave solution is not enough to understand the whole dynamics of a reaction-diffusion equation. Therefore, there have been many studies done recently for other types of entire solutions. For example, Chen and Guo in [2] constructed entire solutions which behave as two opposite wave fronts coming from both sides of x-axis and then annihilating in a

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finite time. Here the entire solution is meant by a solution which is defined for all $(x,t) \in \mathbb{R}^2$. Entire solutions play an important role in the whole dynamics. The study for entire solutions is crucial in the following sense: firstly, it helps us for the mathematical understanding of transient dynamics. As mentioned above, some transient dynamics can be characterized by the behavior of the past $t \approx -\infty$, even though we cannot describe the whole transient behavior. Secondly, structure of the maximal invariant set (or the global attractor) is one of the ultimate goal.

In [5], Guo and Morita studied (1.1) and (1.2) where $f(0) = f(1) = 0$, $f'(1) < 0$, and $f'(0) \neq 0$. They proved there exist entire solutions which behave as two opposite wave fronts coming from both sides of x-axis. The technique they used was to characterize the asymptotic behavior of the solutions as $t \to \pm\infty$ in terms of appropriate subsolutions and supersolutions and use the comparison argument. This argument can apply not only to a general bistable reaction-diffusion equation but also to the Fisher-KPP equation. They also extended it to a discrete diffusive Fisher-KPP equation.

In this paper, we focus on (1.1), where $f(u) = u^2(1-u)$. We note that $f'(0) = 0$ in this case. Following the method of [5], we prove the existence of entire solutions for $c = c_{min}$ in the following theorem.

Theorem 1.1. *Consider* (1.1), *where* $f(u) = u^2(1-u)$. *Let* U *be a solution of* (1.3) *with* $c = c_{min}$. *Then, for any given constants* θ_1, θ_2, *there exists an entire solution* $u(x,t)$ *of* (1.1) *such that*

$$\lim_{t\to-\infty}\{\sup_{x\geq 0}|u(x,t) - U(x+ct+\theta_1)| + \sup_{x\leq 0}|u(x,t) - U(-x+ct+\theta_2)|\} = 0. \tag{1.4}$$

2. Preliminaries

First, we define and make the notion of subsolution and supersolution of (1.1) as follows.

Definition 2.1. *A function* $\underline{u}(x,t)$ *defined on* $\mathbb{R}\times[s,S]$ *is called a subsolution of* (1.1) *if* $\underline{u}(x,t) \leq u(x,t)$ $((x,t) \in \mathbb{R}\times[s,S])$ *for any solution* $u(x,t)$ *of* (1.1) *such that* $\underline{u}(x,s) \leq u(x,s)$ $(x \in \mathbb{R})$. *We call* $\underline{u}(x,t)$ *a subsolution of* (1.1) *in* $\mathbb{R} \times (-\infty, -T]$ *for some* $T \geq 0$, *if* $\underline{u}(x,t)$ *is a subsolution of* (1.1) *defined on* $\mathbb{R} \times [s, -T]$ *for any* $s < -T$. *Similarly, a supersolution can be defined by reversing the inequalities.*

Lemma 2.2. *Let* $\phi_i(x,t)$, $i = 1, 2$, *be functions satisfying* $0 < \phi_i(x,t) < 1$ *and* $(\phi_i)_t(\cdot,t) - \phi_i(\cdot+1,t) - \phi_i(\cdot-1,t) + 2\phi_i(\cdot,t) - f(\phi_i(\cdot,t)) \leq 0$ $((x,t) \in \mathbb{R}\times(-\infty,-T])$. *Then* $\underline{u}(x,t) := \max\{\phi_1(x,t), \phi_2(x,t)\}$ *is a subsolution of* (1.1) *in* $\mathbb{R} \times (-\infty, -T]$.

Proof. Given any $s < -T$. Set $\Omega := \mathbb{R} \times [s, -T]$. Let $u(x,t)$ be a solution of (1.1) in Ω with $u(x,s) \geq \underline{u}(x,s)$ for all $x \in \mathbb{R}$. Applying the strong maximum principle (see [1]) to $\omega_i(x,t) = u(x,t) - \phi_i(x,t)$, $i = 1, 2$, we assert that $\omega_i(x,t) \geq 0$ in Ω, $i = 1, 2$.Thus $u(x,t) \geq \phi_i(x,t)$ in Ω, $i = 1, 2$, which yields the desired conclusion. □

We note that a bounded function $\phi(x,t)$ of C^2 is a subsolution of (1.1) in $\mathbb{R} \times (-\infty, -T]$ if $\phi_t(\cdot,t) - \phi(\cdot+1,t) - \phi(\cdot-1,t) + 2\phi(\cdot,t) - f(\phi(\cdot,t)) \leq 0$ in $\mathbb{R}\times(-\infty,-T]$, while it is a supersolution if $\phi_t(\cdot,t) - \phi(\cdot+1,t) - \phi(\cdot-1,t) + 2\phi(\cdot,t) - f(\phi(\cdot,t)) \geq 0$ in $\mathbb{R} \times (-\infty, -T)$ (see [1]).

From now on, we alway assume $c = c_{min}$. Let λ be the larger root of the characteristic equation

$$c\lambda - e^{\lambda} - e^{-\lambda} + 2 = 0. \tag{2.1}$$

Concerning the asymptotic behaviors of the traveling wave solution $U(x)$ near $x = \pm\infty$ in [3], we have the following estimates for $x \leq 0$:

$$ke^{\lambda x} \leq U(x) \leq Ke^{\lambda x}, \tag{2.2}$$

for some positive k,K. Also, for $x \geq 0$ we have

$$\gamma e^{-\mu x} \leq 1 - U(x) \leq \delta e^{-\mu x}, \tag{2.3}$$

for some positive γ, δ and μ is the unique positive root of

$$c\mu + e^{\mu} + e^{-\mu} - 3 = 0. \tag{2.4}$$

Moveover, there are positive numbers ψ_i $(i = 1, 2)$ such that

$$\inf_{x\leq 0} \frac{U'(x)}{U(x)} = \psi_1,\ \inf_{x\geq 0} \frac{U'(x)}{1-U(x)} = \psi_2. \tag{2.5}$$

3. Existence of entire solutions

Consider the following ordinary differential equation:

$$\dot{p}(t) = c + Ne^{\alpha p(t)},\ (t \leq 0), \tag{3.1}$$

where N, c and α are constants with c, $\alpha > 0$. We can solve this equation easily and obtain the solution as

$$p(t) = p(0) + ct - \frac{1}{\alpha} log \left\{1 + \frac{N}{c} e^{\alpha p(0)}(1 - e^{c\alpha t})\right\}. \tag{3.2}$$

If $N > 0$, it is clear that the solution $p(t)$ is monotone increasing. Let

$$\omega := p(0) - \frac{1}{\alpha} log \left(1 + \frac{N}{c} e^{\alpha p(0)}\right) . \tag{3.3}$$

Then we obtain

$$0 < p(t) - ct - \omega \leq R_0 e^{c\alpha t},\ (t \leq 0), \tag{3.4}$$

for some positive constant R_0. Now, we have the following lemma.

Lemma 3.1. *Let $p(t)$ be the solution of (2.6) with $p(0) < 0$, $\alpha = \lambda$, $N > \max\{K^2/(\psi_1 k), 2K/(\psi_2\gamma)\}$ and let ω be defined by (2.8). Suppose that $\lambda \geq \mu$. Then*

$$\overline{u}(x, t) := U(x + p(t)) + U(-x + p(t)) \tag{3.5}$$

and

$$\underline{u}(x, t) := \max\{U(x + ct + \omega), U(-x + ct + \omega)\} \tag{3.6}$$

are a supersolution and a subsolution of (1.1) for $t \leq 0$, respectively.

Proof. First, by Lemma 2.2, we see that $\underline{u}(x,t) := \max\{U(x+ct+\omega), U(-x+ct+\omega)\}$ is a subsolution of (1.1) for $t \le 0$. Next, we prove that $\overline{u}(x,t)$ is a supersolution. Let $U(x+p(t)) = U_1$, $U(-x+p(t)) = U_2$. Set $\mathcal{N}[\nu](x,t) := \nu_t(x,t) - \nu(x+1,t) - \nu(x-1,t) + 2\nu(x,t) - f(\nu(x,t))$. By a simple computation, we have

$$\mathcal{N}[\overline{u}] = (U_1' + U_2')(Ne^{\lambda p} - G(x,t)), \tag{3.7}$$

where

$$G(x,t) := \frac{U_1U_2(2 - 3U_1 - 3U_2)}{U_1' + U_2'}. \tag{3.8}$$

We also see from (2.2), (2.3) and (2.5) that

$$ke^{\lambda y} \le \quad U(y) \quad \le Ke^{\lambda y}, \ (y \le 0), \tag{3.9}$$

$$\psi_1 ke^{\lambda y} \le \quad \psi_1 U(y) \quad \le U'(y), \ (y \le 0), \tag{3.10}$$

$$\psi_2 \gamma e^{-\mu y} \le \quad \psi_2(1 - U(y)) \quad \le U'(y), \ (y \ge 0). \tag{3.11}$$

Note that $p(t) < 0$ for all $t \le 0$. We divide $\mathbb{R}$ into three regions to estimate $G(x,t)$.
(1) $p \le x \le -p$: Using (2.14) and (2.15), we obtain

$$\begin{aligned} G(x,t) &\le \frac{2U_1U_2}{U_1' + U_2'} \le \frac{2K^2 e^{\lambda(x+p)} e^{\lambda(-x+p)}}{\psi_1 k(e^{\lambda(x+p)} + e^{\lambda(-x+p)})} \\ &= \frac{2K^2 e^{2\lambda p}}{\psi_1 k(e^{\lambda x} + e^{-\lambda x})e^{\lambda p}} \le \frac{2K^2}{2\psi_1 k} e^{\lambda p}. \end{aligned} \tag{3.12}$$

(2) $x \le p$: It follows from (2.14)-(2.16) that

$$\begin{aligned} G(x,t) &\le \frac{2U_1}{U_1' + U_2'} \le \frac{2Ke^{\lambda(x+p)}}{\psi_1 ke^{\lambda(x+p)} + \psi_2 \gamma e^{-\mu(-x+p)}} \\ &= \frac{2K}{\psi_1 ke^{\lambda p} + \psi_2 \gamma e^{-(\lambda-\mu)x} e^{-\mu p}} e^{\lambda p} \\ &\le \frac{2K}{\psi_2 \gamma} e^{\lambda p}. \end{aligned} \tag{3.13}$$

(3) $-p \le x$: By the symmetry $G(-x,t) = G(x,t)$ and (2.18), we obtain

$$G(x,t) \le \frac{2K}{\psi_2 \gamma} e^{\lambda p}. \tag{3.14}$$

Hence we obtain

$$\mathcal{N}[\overline{u}] = (U_1' + U_2')(Ne^{\lambda p} - G(x,t)) \ge 0.$$

Therefore, $\overline{u}$ is a supersolution of (1.1) for $t \le 0$. This proves the lemma. □

Remark 3.2. *The assumption $\lambda \ge \mu$ in Lemma 2.3 is valid provided that $c_{min} \ge \frac{1}{2\log 2}$.*

Lemma 3.3. *Let $\overline{u}(x,t)$ and $\underline{u}(x,t)$ be the supersolution and the subsolution given in Lemma 2.3. Suppose all the assumption of Lemma 2.3 holds. Then there is a positive constant M_1 such that*

$$0 < \overline{u}(x,t) - \underline{u}(x,t) \le M_1 e^{c\lambda t} \ \ ((x,t) \in \mathbb{R} \times (-\infty, 0]). \tag{3.15}$$

Proof. Suppose that $t \leq 0$. Since $U' > 0$, we have $U(x+ct+\omega) \geq U(-x+ct+\omega)$ for $x \geq 0$. Thus $\underline{u}(x,t) = U(x+ct+\omega)$ for $x \geq 0$ and $\underline{u}(x,t) = U(-x+ct+\omega)$ for $x \leq 0$. For $x \geq 0$, we have

$$\begin{aligned} 0 \leq \overline{u}(x,t) - \underline{u}(x,t) &= U(x+p(t)) + U(-x+p(t)) - U(x+ct+\omega) \\ &\leq Ke^{\lambda(-x+p(t))} + \sup_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq M_1 e^{c\lambda t}, \end{aligned} \tag{3.16}$$

for some $M_1 > 0$. On the other hand, for $x \leq 0$, we have

$$\begin{aligned} 0 \leq \overline{u}(x,t) - \underline{u}(x,t) &= U(x+p(t)) + U(-x+p(t)) - U(-x+ct+\omega) \\ &\leq Ke^{\lambda(x+p(t))} + \sup_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq M_1 e^{c\lambda t}. \end{aligned} \tag{3.17}$$

This completes the proof. □

Following [5], we have the following proposition.

Proposition 3.4. *Under the same assumptions of Lemma* 2.3, *there is an entire solution* $u^*(x,t)$ *of* (1.1) *such that*

$$\underline{u}(x,t) \leq u^*(x,t) \leq \overline{u}(x,t) \ ((x,t) \in \mathbb{R} \times (-\infty, 0]), \tag{3.18}$$

where ω *is defined by* (2.8), $\underline{u}(x,t)$ *and* $\overline{u}(x,t)$ *are given in Lemma* 2.3.

Proof. Denote by $u(x,t;\nu_0)$ a solution to (1.1) with the initial condition $u(x,0;\nu_0(\cdot)) = \nu_0(x)$. Set

$$\nu_n(x,t) = u(x,t;\underline{u}(\cdot,-n)), \ \ n = 1,2,\dots .$$

Since $\underline{u}$ is a subsolution and $\underline{u}(x,-n-1+0) = u(x,0;\underline{u}(\cdot,-(n+1)))$, we have

$$\underline{u}(x,-n-1+t) \leq u(x,t;\underline{u}(\cdot,-(n+1))).$$

By taking $t = 1$, we obtain

$$\nu_n(x,0) = \underline{u}(x,-n) \leq u(x,1;\underline{u}(\cdot,-(n+1))) = \nu_{n+1}(x,1).$$

Thus the maximum principle yields

$$\nu_n(x,n) \leq \nu_{n+1}(x,n+1),$$

which implies $\{\nu_n(\cdot,n)\}$ is monotone increasing. On the other hand, since $\nu_n(x,n) \leq \overline{u}(x,0)$, there is a function ν^* such that ν_n converges uniformly to ν^*. Therefore, $u^*(x,t) := u(x,t;\nu^*)$ is a solution for all $t \geq 0$.

Next, we show that $u^*(x,t)$ is defined for all $t \leq 0$. Given $T \geq 0$, there is an integer n_1 such that $n_1 > T$. Then, for $n \geq n_1$, we have

$$u(x,-T;\nu_n) = u(x,-T;u(x,n;\underline{u}(\cdot,-n))) = u(x,n-T;\underline{u}(\cdot,-n)).$$

Set

$$w_n(x) = u(x,n-T;\underline{u}(\cdot,-n)). \tag{3.19}$$

Then $\nu_n(x,n) = u(x,T;w_n(x,t))$ and

$$w_{n+1}(x) = u(x,n+1-T;\underline{u}(\cdot,-(n+1))) \geq u(x,n-T;\underline{u}(\cdot,-n)) = w_n(x).$$

This implies the sequence $\{w_n\}$ is monotone increasing. Applying the same argument, there is a function ν_T to which w_n converges uniformly. We see that

$$\nu^* = \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} u(x,T;w_n(x,t)) = u(x,T;\nu_T).$$

Thus we obtain

$$\nu_T = u(x, -T; \nu^*).$$

Since $T > 0$ is arbitrary, we conclude that $u^*(x,t) := u(x,t;\nu^*)$ is defined for all $t \in \mathbb{R}$.

Finally, we show that (2.23) holds. From above, we have

$$u^*(x, -T) = u(x, -T; \nu^*) = \nu_T = \lim_{n\to\infty} \omega_n \tag{3.20}$$

Since $\underline{u}$ is a subsolution and $\overline{u}(x, -n) \geq u(x, 0; \underline{u}(\cdot, -n)) = \underline{u}(x, -n)$, we have

$$\overline{u}(x, -n+t) \geq u(x, t; \underline{u}(\cdot, -n)) \geq \underline{u}(x, -n+t) \ \forall (x,t) \in \mathbb{R} \times [0, n].$$

By taking $t = n - T$, we obtain

$$\overline{u}(x, -T) \geq \omega_n = u(x, n-T; \underline{u}(\cdot, -n)) \geq \underline{u}(x, -T). \tag{3.21}$$

Hence, it follows from (2.25) and (2.26) that $\underline{u}(x, -T) \leq u^*(x, -T) \leq \overline{u}(x, -T)$. Since $T > 0$ is arbitrary, (2.23) holds. This proves the proposition. □

Remark 3.5. *By virtue of the condition $\lambda \geq \mu$ we can check that the supersolution $\overline{u}(x,t)$, defined for $t \leq 0$, is bounded by 1 for large $|t|$. In fact, we may assume that $K < 1/2$ in the condition (2.2) by shifting appropriately. Then*

$$U(x+p(t)) + U(-x+p(t)) \leq K(e^{\lambda x} + e^{-\lambda x})e^{\lambda p} \ (p \leq x \leq -p),$$

while

$$\begin{aligned} U(x+p) + U(-x+p) \quad &\leq 1 - \gamma e^{-\mu(x+p)} + Ke{-}\lambda(x-p) \\ &\leq 1 - (\gamma - Ke^{(\lambda+\mu)p} e^{-(\lambda-\mu)x})e^{-\mu(x+p)} \ (-p \leq x), \end{aligned}$$

$$\begin{aligned} U(x+p) + U(-x+p) \quad &\leq Ke^{\lambda(x+p)} + 1 - \gamma e^{\mu(x-p)} \\ &\leq 1 - (\gamma - Ke^{(\lambda+\mu)p} e^{(\lambda-\mu)x})e^{\mu(x-p)} \ (x \leq p). \end{aligned}$$

This implies $\overline{u}(x,t) \leq 1$ for $t < -T$ with a large $T > 0$. Hence, by the strong maximum principle, we can assert that the solution $u(x,t)$ of Proposition 2.6 satisfies $0 < u(x,t) < 1$ for all $(x,t) \in \mathbb{R}^2$.

Proposition 3.6. *Let $u(x,t)$ be an entire solution constructed in Proposition 2.6. Under the same assumptions of Lemma 2.3 and Proposition 2.6, there is a positive number M_1 such that for $t \leq 0$,*

$$\begin{aligned} &0 \leq \sup\nolimits_{x\geq 0} \{u(x,t) - U(x+ct+\omega)\} \\ &\quad + \sup\nolimits_{x\leq 0} \{u(x,t) - U(-x+ct+\omega)\} \leq M_1 e^{c\lambda t}. \end{aligned} \tag{3.22}$$

Proof. Suppose that $t \leq 0$. For $x \geq 0$,

$$\begin{aligned} 0 \quad &\leq U(x+p(t)) + U(-x+p(t)) - U(x+ct+\omega) \\ &\leq \ Ke^{\lambda(-x+p(t))} + \sup\nolimits_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq \ Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq \tfrac{1}{2} M_1 e^{c\lambda t}, \end{aligned} \tag{3.23}$$

for some $M_1 > 0$. Combining (2.23) and (2.28), we obtain

$$0 \leq \ u(x,t) - U(x+ct+\omega) \leq \overline{u}(x,t) - U(x+ct+\omega) \leq \frac{1}{2} M_1 e^{c\lambda t}.$$

On the other hand, for $x \leq 0$, we have

$$\begin{aligned} 0 \quad &\leq U(x+p(t)) + U(-x+p(t)) - U(-x+ct+\omega) \\ &\leq \ Ke^{\lambda(x+p(t))} + \sup\nolimits_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq \ Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq \tfrac{1}{2} M_1 e^{c\lambda t}. \end{aligned} \tag{3.24}$$

Therefore it follows from (2.23) and (2.29) that

$$0 \leq u(x,t) - U(-x+ct+\omega) \leq \overline{u}(x,t) - U(-x+ct+\omega) \leq \frac{1}{2}M_1 e^{c\lambda t}.$$

Hence (2.27) holds. □

Proof of Theorem 1.1: Given arbitrary θ_1, θ_2, we consider the translation and the time-shift as

$$U(x+\xi+c(t+\tau)) = U(x+ct+\xi+c\tau),$$
$$U(-x-\xi+c(t+\tau)) = U(-x+ct-\xi+c\tau).$$

Define $\widetilde{u}(x,t) := u(x+\xi, t+\tau)$ with

$$\xi := \frac{\theta_1-\theta_2}{2},\ \tau := \frac{\theta_1+\theta_2-2\omega}{2c},$$

where $u(x,t)$ is the entire solution of Proposition 2.6. Then we easily obtain

$$\max\{U(x+ct+\theta_1), U(-x+ct+\theta_2)\}$$
$$\leq \tilde{u}(x,t) \leq \overline{u}(x+\xi, t+\tau)\ (t \leq -\tau).$$

On the other hand, (1.4) immediately follows from (2.27). Thus we complete the proof of Theorem 1.1. □

Remark 3.7. *Entire solutions can also be constructed by using traveling wave with speed $c > c_{min}$ if one can find a pair of suitable supersolution and subsolution. However, we cannot find such one. Therefore we left it as an open problem.*

References

1. X. Chen and J.-S. Guo, Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations, *Journal of Differential Equations* 184 (2002), 549-569.
2. X. Chen and J.-S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics, *Mathematische Annalen* 326 (2003), 123-146.
3. X. Chen, S.-C. Fu and J.-S. Guo, Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices, *SIAM Journal on Mathematical Analysis* 38 (2006), 233-258.
4. R.A Fisher, The advance of adavantageous genes, *Annals Eugenics* 7 (1937), 355-369.
5. J.-S. Guo and Y. Morita, Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations, *Discrete and Continuous Dynamical Systems* 12 (2005), 193-212.
6. A. Kolmogorov, I. Petrovsky, and N. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matiére et son application á un probléme biologique, *Bulletin Universite d'Etat a Moscow Series Internationale Section A* 1 (1937), 1-26.

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