# 行政院國家科學委員會專題研究計畫 成果報告

## 一些 n 維半線性波方程解的暴破速度與暴破常數研究(II)
## 研究成果報告(精簡版)

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# Estimates for the Life-Span of positive Solutions of some Semilinear Wave Equations in n-dimensional $\square u - u^2 = 0$, $n \leq 3$ (II)

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## 1 1. Introduction

Haveing obtain the estimates for the life-span of positive solutions of semilinear wave equations in n-dimensional

$$\square u \; - \; u^2 = 0, n \leq 2.$$

We want to estimate the life-span and later seek for the life-span of positive solutions for the 3−dimensional semilinear wave equation

$$\square \; u = u^2 \text{ in } [0, T) \times \Omega, \Omega \subset \mathbb{R}^3 \tag{1.1}$$

with initial values $u(0, x) = u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\dot{u}(0, x) = u_1(x) \in H_0^1(\Omega)$, that is, the superlinear case. We will use the following notations:

$$\cdot := \frac{\partial}{\partial t}, \; Du := (\dot{u}, u_x), \; \square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2},$$

$$a(t) := \int_\Omega u^2(t, x)\, dx, \; E(t) := \int_\Omega \left( |Du|^2 - \frac{2}{3} u^3 \right)(t, x)\, dx.$$

For a Banach space $X$ and $0 < T \leq \infty$ we set

$$C^k(0, T, X) = \text{Space of } C^k - \text{ functions} : [0, T) \to X,$$
$$H1 := C^1\left(0, T, H_0^1(\Omega)\right) \; \cap \; C^2\left(0, T, L^2(\Omega)\right).$$

Jörgens [3] published the first exist Theorem for global solutions to the wave equation of the form

$$\square \; u + f(u) = 0 \text{ in } [0, T) \times \mathbb{R}^3, \tag{1.2}$$

for $f(u) = g\left(u^2\right) \cdot u$, his result can be applied to the equation $\square u + u^3 = 0$; and Browder [1] generalized Jörgens's result to $n > 2$. For local Lipschitz $f$, Li [10] proved the nonexistence of global Solution of the initial-boundary value problem of semilinear wave equation (1.2) in bounded domain $\Omega \subset \mathbb{R}^3$ under the assumption

$$\bar{E}(0) = \|Du\|_2^2(0) + 2 \int_\Omega f(u)(0, x)\, dx \leq 0,$$

$$\eta f(\eta) - 2(1 + 2\alpha) \int_0^\eta f(r)\, dr \leq \lambda_1 \alpha \eta^2 \quad \forall \eta \in R$$

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where $\alpha > 0$, $\lambda_1 := \sup \left\{ \|u\|_2 / \|\nabla u\|_2 : u \in H_0^1(\Omega) \right\}$ and $a'(0) > 0$. There we have a rough estimate for the life-span

$$T \le \beta_2 := 2 \left[ 1 - \left( 1 - k_2 a(0)^{-\alpha} \right)^{1/2} \right] / k_1 k_2,$$

where $k_1 := \alpha a(0)^{-\alpha-1} \sqrt{a'(0)^2 - 4E(0)\, a(0)}$, $k_2 := \left( -4\alpha^2 E(0) / k_1^2 \right)^{\alpha/1+2\alpha}$.

For $f(u) = -u^3$, there exist global solutions of (SL) for small initial data [8]; but if $E(0) < 0$ and $a'(0) > 0$ then the solutions are only local, i.e. $T < \infty$ [11].

John [4] showed the nonexistence of solutions of the initial-boundary value problem for the wave equation $\Box u = A|u|^p$, $A > 0$,

$$1 < p < 1 + \sqrt{2}, \quad \Omega = \mathbb{R}^3.$$

This problem was considered by Glassey [2] in two dimensional case $n = 2$; for $n > 3$ Sideris [15] showed the nonexistence of global solutions under the conditions

$$\|u_0\|_1 > 0 \ \text{ and } \ \|u_1\|_1 > 0.$$

According to this result Strauss [13, p.27] guessed that the solutions for the above mentioned wave equation are global for $p \ge 1 + \sqrt{2}$.

Further literature about blow up one can see [4], [5], [6], [12] and [13] and their reference.

In this paper we treat the blow-up rate, blow-up constante and the asymptotic behavior of the solution to the equation (0.1).

## 2 Definition and Fundamental Lemmas

There are many definitions of the weak solutions of the initial-boundary problems of the wave equation, we use here as following.

Definition 2.1: *For* $u \in H1$ *is called a positive weakly solution of equation* (1.1), *if*

$$\int_0^t \int_\Omega \begin{pmatrix} \dot{u}(r,x)\,\dot{\varphi}(r,x) - \nabla u(r,x) \cdot \nabla \varphi(r,x) \\ +u^2(r,x)\,\varphi(r,x) \end{pmatrix} dxdr = 0 \ \forall \varphi \in H1$$

*and*

$$\int_0^t \int_\Omega u(r,x)\,\psi(r,x)\,dxdr \ge 0$$

*for each positive* $\psi \in C_0^\infty([0,T) \times \Omega)$.

**Remark 2.1**: 1) This definition 2.1 is resulted from the multiplication with $\varphi$ to the equation (1.1) and integration in $\Omega$ from 0 to $t$.

2) From the local Lipschitz functions $u^2$, the initial-boundary value problem (1.1) possesses a unique solution in $H1$ [9]. Hereto we use the notations:

$$\frac{1}{C} := \eta_1 = \sup \left\{ \|u\|_2 \ \ / \ \ \|Du\|_2 : u \in H_0^1(\Omega) \right\},$$

$$\lambda_q \ \ = \sup \left\{ \|u\|_q \ \ / \ \ \|Du\|_2 : u \in H_0^1(\Omega) \ \cap \ L_q(\Omega) \right\},$$

for $q \geq 1$.

In this paper we need the following lemmas

Lemma 2.1: *Suppose that* $u \in H1$ *is a weakly positive solution of* $(SL)$ with $E(0) = 0$ *for* $a(0) > 0$, *then we have*:

(i) $a \in C^2(0,T)$ *and* $E(t) = E(0) \quad \forall t \in [0,T)$.
(ii) $a'(t) > 0 \quad \forall t \in [0,T)$, *provided* $a'(0) > 0$.
(iii) $a'(t) > 0 \quad \forall t \in (0,T)$, *if* $a'(0) = 0$.
(iv) *For* $a'(0) < 0$, *there exists a constant* $t_0 > 0$ *with*

$$a'(t) > 0 = a'(t_0) \quad \forall t > t_0.$$

Lemma 2.2: *Suppose that* $u$ *is a positive weakly solution in* $H1$ *of equation (1.1) with* $u(0,\cdot) = 0 = \dot{u}(0,\cdot)$ in $L^2(\Omega)$, *we have* $u \equiv 0$ *in* $H1$.

# 3 Estimates for the Life-Span of the Solutions of (1.1) under Null-Energy

In this section we focus on the case that $E(0) = 0$ and divide it into two parts
(i) $a(0) > 0, a'(0) \geq 0$ (ii) $a(0) > 0, a'(0) < 0$

## 3.1 Estimates for the Life-span of the Solutions of (1.1) under $a'(0) \geq 0$

Theorem 3.1: *Suppose that* $u \in H1$ *is a positive weakly solution of equation (1.1) with* $a'(0) \geq 0$ *and* $E(0) = 0$. *Then the Life-span of* $u$ *is finite, further*

$$T \leq \alpha_1 := k_2^{-1} \sin^{-1}\left( \frac{k_2}{k_1 a^{\frac{1}{4}}(0)} \right)$$

*with*

$$k_1 := \frac{1}{4} a^{-\frac{1}{4}}(0) \sqrt{a'(0)\, a^{-2}(0) + 4C^2}, k_2 := \frac{1}{2} C.$$

*If* $T = \alpha_1$, *then* $a(t) \to \infty, t \to T$. *Furthermore, we have also the estimate for* $a(t)$*:*

$$a(t) \geq \left(\frac{k_2}{k_1}\right)^4 \left(\sin\left(k_2\alpha_1 \; - \; k_2 t\right)\right)^{-4} \quad \forall t \in [0, T).$$

*This means that the blow-up rate of* $u$ *is* 4 *in the sin-growth.*

**Remark 3.1** 1) The Theorem 3.1 is a extension of my own Satz 2 in [9]. And the local existence and uniqueness of solutions of equation (1.1) in $H1$ are known [10].

2) For special cases:

i) For $n = 2$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (1.1) is bounded by $T \leq \alpha_1$.

ii) For $n = 3$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (1.1) is bounded

$$T \leq \alpha_2 := 2C^{-1}\sin^{-1}\left(2C\left(a'(0)^2 a(0)^{-2} + 4C^2\right)^{-\frac{1}{2}}\right).$$

If $T = \alpha_2$, then $a(t) \to \infty, t \to T$.

iii) For $a'(0) = 0$, we have $\alpha_1 = \pi C$.

iv) For $|\Omega| \to \infty$, we have also $\alpha_1 \to \dfrac{a(0)}{a'(0)}$.

As $|\Omega| \to 0$, then $\alpha_1 \to 2\sin^{-1}\left(\dfrac{1}{4C}\right)$.

## 3.2 Estimates for the Life-span of the Solutions of equation (1.1) under $a'(0) < 0$

Theorem 3.2.1: *Suppose that* $u \in H1$ *is a positive weakly solution of the initial-boundary value problem equation (1.1) with* $a(0) > 0$, $E(0) = 0$ *and* $a'(0) < 0$. *Then the life span of* $u$ *is bounded:*

$$T \leq \alpha_5 := \frac{\pi}{C} - 4a'(0)\left(\lambda_3^3\right)^2.$$

*If* $T = \alpha_5$, *then* $a(t) \to \infty, T \to \alpha_5$. *Further, we have the estimate for the blow-up rate of* $a(t)$ *in the neighborhood of* $\alpha_5$*:*

$$a(t) \geq a(t_0)\left(\sin\left(\frac{C}{2}(\alpha_5 - t)\right)\right)^{-4} \quad \forall t \in [t_0, T), t_0 \leq t_1$$

*with* $t_1 := -\frac{4}{9}\lambda_3^6 a'(0)$.

Theorem 3.2.2: *Suppose that $u$ is a positive weakly solution of equation (1.1) with $a(0) > 0$, $E(0) = 0$, and*

$$(i)\ -\frac{1}{2}r_1 a(0) < a'(0) < 0; (ii)\ \frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)} \leq e^{2r_1 t_1},$$

*where $r_1 := \sqrt{2}C$. Then the life-span of $u$ is bounded:*

$$T \leq \alpha_6 := \frac{\pi}{C} + \frac{1}{2r_1} \ln\left(\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)}\right) \leq \alpha_5.$$

*And there is a constant $t_4 > 0$ with*

$$\begin{aligned} (iii)\quad t_4 &\leq t_3 := \frac{1}{2r_1} \ln\left(\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)}\right), \\ (iv)\quad a(t) &\geq a(t_4)\left(\sin\left(\frac{1}{2}C(\alpha_6 - t)\right)\right)^{-4}. \end{aligned}$$

# 4 Estimates for the Life-Span of the Solutions of equation (1.1) under Negativ-Energy

In this chapter we suppose the energy $E(0)$ is negative and consider the following cases:

$$(i)\ a(0) > 0, a'(0) > 0\ (ii)\ a(0) > 0, a'(0) = 0\ (iii)\ a(0) > 0, a'(0) < 0.$$

## 4.1 Fundamental Lemmas

In this section we use the following lemmas and those argumentations of proof to lemmas are not true for positive energy, so under positive energy we need to seek another method to show the results.

Lemma 4.1: *Suppose that $u \in H1$ is a positive weakly solution of equation (1.1) with $a(0) > 0$ and $E(0) < 0$. Then*

*(i) for $a'(0) \geq 0$, we have $a'(t) > 0\ \ \forall t > 0$.*

*(ii) for $a'(0) < 0$, there exists a constant $t_5 > 0$ with $a'(t) > 0\quad \forall t > t_5$, $a'(t_5) = 0$ and*

$$t_5 \leq t_6 := \frac{-a'(0)}{\delta^2 - E},$$

*where $\delta$ is the positive root of the equation $2\lambda_3^3 \cdot r^3 - 3r^2 + 3E(0) = 0$.*

## 4.2 Estimates for the Life-Span of the Solutions of equation (1.1) under $E(0) < 0,\ a'(0) \geq 0$.

Theorem 4.2: *Suppose that $u \in H1$ is a positive weakly solution of equation (0.1) with $E(0) < 0$, and $a'(0) \geq 0$. Then the life-span of $u$ is bounded:*

$$T \leq \alpha_5 := k_0^{-1} k_2^{-1} \sin^{-1}\left(k_2 a(0)^{-\frac{1}{4}}\right)$$

*where*

$$k_0 := \tfrac{1}{2} a(0)^{-\frac{3}{4}} \sqrt{\tfrac{1}{4} a(0)^{-1} a'(0)^2 + \tfrac{1}{3}\left(\delta^2 - 3E(0)\right)},$$

$$k_2 := \left(\tfrac{k_1}{k_0}\right)^{\frac{1}{3}},\ k_1 := \tfrac{1}{6}\sqrt{3\delta^2 - E(0)}.$$

*Further we have*

$$a(t) \geq k_2^4 \left(\sin\left(k_0 k_2 \left(\ \alpha_5 - t\ \right)\right)\right)^{-4} \quad \forall t \in [0, T].$$

Remark 4.2: We can good estimate the rate of the singularity of $a(t)$ and the life-span of $u$, but we can not get them contemporaneously:

$$T \leq \alpha_6 := k_0^{-1} k_2^{-1} \frac{\tan^{-1}\left(k_2 a(0)^{-\frac{1}{4}}\right)}{\sqrt{1 - k_2^2 a(0)^{-\frac{1}{2}}}}$$

$$a(t) \geq k_2^4 \left\{\tan\left\{\tan^{-1}\left[\left(k_2 a(0)^{-\frac{1}{4}}\right) - k_0 k_2 \sqrt{1 - k_2^2 a(0)^{-\frac{1}{2}}}\, t\right]\right\}\right\}^{-4}$$

for each $t \in [0, T]$.

## 4.3 Blow-up set of the solution

According to the above results concerning blow-up solution, we want to seek the (set of ) blow-up point(s) and the blow-up rate and blow-up constant of the solution for the semilinear wave equation $\Box u = u^p$ with smooth initial values, for instance, $u_0, u_1$ are both in $C_0^\infty(\Omega)$ and we consider the sets

$$S := \left\{(t_0, x_0) \in \mathbb{R}^2 \ \middle|\ u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (t_0, x_0)\right\},$$

$$S_{T^*} := \left\{x_0 \in \mathbb{R} \ \middle|\ u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (T^*, x_0)\right\},$$

$$S_{T^*,\ L^q} := \left\{ x_0 \in \mathbb{R} \ \middle|\ \begin{matrix} \lim_{t\to T^*}\left(\int_{B_r(x_0)} |u|^q (t,x)\, dx\right)^{-1} = 0 \\ \lim_{t\to T^*}\left(\int_{\mathbb{R}-B_r(x_0)} |u|^q (t,x)\, dx\right)^{-1} > 0 \end{matrix} \quad \text{for each } r > 0 \right\},$$

where $B_r(x_0) = \left\{ x \in \mathbb{R} : |x - x_0| \leq r \right\}$. We call $S$, $S_{T^*}$ and $S_{T^*,\ L^q}$ the blow-up set, blow-up set at time $T^*$ and the blow-up set in the sense of $L^q$ of $u$. The problems are:

(1) What are the sets $S, S_{T^*}$ and $S_{T^*,\ L^q}$?

(2) How large are these sets?

(3) What are the blow-up rate of $u$ in the neighborhoods of $S$, $S_{T^*}$ and $S_{T^*,\ L^q}$?

(4) What are the blow-up constants of $u$ in the neighborhoods of $S$, $S_{T^*}$ and $S_{T^*,\ L^q}$?

To study the above hard problems we concentrate on the properties later.

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