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計畫主持人：江彌修

計畫參與人員：碩士班研究生-兼任助理人員：王盈心
碩士班研究生-兼任助理人員：賴興展
碩士班研究生-兼任助理人員：陳美君
碩士班研究生-兼任助理人員：張宇賢
碩士班研究生-兼任助理人員：謝曜謚

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Introduction

A CDO-squared (denoted as CDO^2) offers yield-enhancement for tranche investors while at the same aims to further reduce their credit risk exposures. Due to the synthetic nature of such products, the underlying CDOs (the inner CDO) can be created conceptually for the sole purpose of being referenced. However, it has some drawbacks. A CDO^2 is particularly sensitive to several areas: (1) the degree of overlap among referenced assets in the inner CDOs, (2) default correlation between the underlying credits, (3) location of defaults in the inner CDOs. An in-depth study on the pricing and hedging of this product is therefore essential to the credit market, and this entails careful investigations on the risk characteristics of such products.

A CDO^2 consists of a portfolio of inner CDOs. Each inner CDO in turn consists of a pool of reference credit entities. Such pool forms the fundamental layer in its product structure, and to further complicate the situation, the underlying names that are referred among the inner CDOs can in fact overlap, and can thus result in simultaneous defaults occurring. In particular, the overlapping feature induces a direct dependency among the inner CDOs, and further complicates the correlation structure among default events.

Unlike in the modeling of CDOs where the characterization of the lost distribution requires, in addition to a copula that captures the correlation structures, the knowledge of the number of defaults in the underlying pool before maturity. The modeling of CDO^2s requires us to also consider the location of the defaults; i.e. to specify the payoff of CDO^2, we need to know both the number of defaults and how these default events are distributed in the inner CDOs.

Literature Review

There are two major difficulties involved in the pricing of multi-named credit derivatives products: firstly, the characterization of a default distribution, and secondly, the correlation of the underlying credits. Li (2000) constructs the joint distribution of survival times via a copula function which inter-links the marginal distributions of the underlying credits. However, the Li-model suffers from the curse of dimensionality as it is required in the model that the joint distribution of default times be fully specified. As a direct consequence, the execution of Monte Carlo simulations becomes time-consuming as the number of underlying credits increases.

Laurent and Gregory (2003), Andersen et al. (2003) and Hull and White (2004) developed the semi-analytic approaches under the conditional independence assumption. The conditional independence assumption, pre-assumes that, conditional on a common risk fact default event are mutually independent. Such assumption overcomes the curse of dimensionality and gives rise to the market-standard factor copula approach.

In terms of characterizing the conditional loss distribution, Laurent and Gregory (2003) use the Fast Fourier Transforms to construct the conditional loss distribution via the convolution of characteristic functions. Andersen et al. (2003) proposes a recursive method to build up the loss distribution. Following the same stream of thoughts, Hull and White (2004) introduce the probability bucketing approach for generating the lose distribution. Under the conditional independence assumption, the correlations structure among defaults is reduced to an integral that can be solved numerically, and the resulting model is highly efficient in calculating the tranche greeks for hedging purposes. However, we must note that such simplification though result in the analytic tractability of the pricing model, information regarding the correlation structure among default times is actually lost, and the commonly observed contagion effect of defaults is no longer observable in the model. Whether or not there exist other means of bringing back a dynamic nature of default correlations under the conditional independence assumption remains an open question.

Research Objectives

In this research, I adapt the conditional independence framework for our modeling purposes due to the analytic tractability and efficiency in the computation of hedging parameters. In other words, I aim to develop a semi-analytic approach to price CDOs that are of compound-protection layers. My preliminary attempt which resulted in a published article (Chiang, Yueh, and Lee (2008)) was to consider the possibility of generating the loss distribution for CDO^2 without Monte Carlo simulation. The algorithm consists of two parts: firstly, the construction of conditional loss distribution for a portfolio of inner CDOs and secondly, the transformation of the above distribution to the conditional loss distribution of the mater CDO.

In this research, I have further extended this algorithm to full generality; in particular, I address the questions as to whether or not there is a general principle governing the generation of conditional loss distribution when the credit protection layer is of compound structure. Furthermore, adequate risk measures for such credit derivatives with compound protection structure need to be introduced in order that proper understanding of their risk profiles can be established. In addition, we must confront the question of how the overlapping effect among obligors can be quantified.

This research aims to contribute in three aspects: (1) to establish an efficient pricing model for CDOs with compound protection structure, (2) to provide appropriate risk measures for characterizing the risk profiles of such products, and (3) to provide proper ways for hedging the embedded risks. In addition, I proposed the following questions to be answered:

(1) How to construct the conditional distribution for CDO with compound protection layer?
   i. The technique to solve the problem of overlapping structure.
   ii. The method of handling the path-dependent structure of lower-layer.
(2) Is there a generalized framework for the valuation of nth power CDOs?
   i. The valuation of CDO^2.
   ii. The extension of CDO^2 to higher layers.
(3) What are the adequate risk measures for the multi-protection-layer CDO?
   i. The quantifying of diversification/overlapping risk.
   ii. The measure of risks between tranches.
(4) What are the coherent hedging parameters?
   i. The definition of tranche Delta, Gamma, and other Greeks.
   ii. The efficient method for calculation of hedging parameters.
(5) How to derive proper hedging strategies?

Methodology

1. Valuation of Tranche Spreads for CDO^2

Assume that all payments are made at specific time points t_j, where

$0 < t_1 < t_2 < ... < t_N = T$. The expected ($E\left[L_{t_j}\right]$) for any CDO^2 tranche at time t_j can be defined as:

$$E\left[L_{t_j}\right]=\sum_{i=0}^{L_{\max}} p_i(t_j)\cdot\max(0,\min(i(1-R),U)-L)$$

where $L_{\max}$ stands for the maximum loss of the reference pool, and *R* denotes the recovery rate; *U* and *L* are the upper, and lower attachment points of CDO² tranches respectively.

The default leg (DL), and premium leg (PL)) are:

$$DL=\sum_{j=1}^{N} B_{t_j}\cdot(E\left[L_{t_j}\right]-E\left[L_{t_{j-1}}\right])$$

$$PL=s\cdot\sum_{j=1}^{N} B_{t_j}\cdot\Delta_{t_j}\cdot\left(L_{\max}-E\left[L_{t_j}\right]\right)$$

where B_{t_j} represents the discount factor at time t_j ; *s* denotes the tranche spread per annum; Δ_{t_j} accounts for the time adjustment factor for accrual payments at any specific date t_j . The tranche spread, *s*, is then extracted by equating the above two legs

2. The Generalized Recursive Algorithm

In the following we begin with the construction of the conditional loss distribution for a portfolio of inner CDOs. Suppose a CDO² consists of n inner CDOs. Each inner CDO in turn consists of a reference pool of m underlying credit entities. The derivation of conditional loss distribution requires a extended recursive procedure that we consider in the following:

Let $H_i=\Pr(\tau_i<t\,|\,M)$ be the conditional default probability of the *i*-th underlying credit entity. To be consistent with the probability bucketing approach of Hull and White(2004), we divide the accumulated total loss into k sub-blocks of buckets: $u_{(0)},u_{(1)},\ldots,u_{(k)}$, where $u_{(0)}$ refers to a zero loss of total notional, $u_{(1)}$ refers to a one chosen unit loss, and $u_{(k)}$ refers to an accumulated loss of total notional. Note that the unit loss (basket size) can be arbitrarily chosen in accordance with the targeted problem in hand.

Our recursive algorithm proceeds by a sequential addition of the underlying credit entities into the reference which is initially empty. For example, at the beginning of recursion when the pool contains zero entities, the (conditional) probability of a zero loss is one, $\Pr(u_{(0)}) = 1$. In abbreviated terms, we have: $p_0 = 1$, where p_0 stands for the conditional probability of a zero accumulated loss.

Sequential addition of the underlying credit entities shall result in a shifting of conditional probability weights between buckets, for example the addition of the first and second credit entities shall result in an updating process as: $p_0^{new} = p_0\left(1 - H_1\right)$ and $p_1^{new} = p_1 + H_1 p_0$. The p_i^{new} refers the updated probability after the addition of the i-th credit entity. Let the loss given default of the i-th credit entity measure in chosen unit of loss be denoted by LGD_i, then the addition of the i-th credit entity shall result in a shifting of probability weights from bucket $u_{(j)}$ to bucket $u_{(j+LGD_i)}$, in abbreviated expressions, we have: $p_j^{new} = p_j\left(1 - H_i\right)$ and $p_{j+d_i}^{new} = p_{j+LGD_i} + p_j H_i$.

Now for a CDO², let $Lower_i$ and $Upper_i$ represent the upper and lower attachment points of a referenced tranche for the i-th underlying inner CDO. To generate the accumulate total loss for the master CDO entails finding the set of all possible linear combinations of losses resulted by the underlying inner CDOs. The set of all possible linear combination of loses are equivalent to the set of all possible buckets ranging by $u_{(j)}$, where $j = 1, 2, ..., k$. Note that a resulted loss in the master CDO is always less than or equal to the accumulated loss of its referenced inner CDOs. This is because that any tranches in a CDO² is in fact a certain percentage of the tranches extracted from the portfolio of its inner CDOs. For example, if the underlying portfolio of inner CDOs suffer a loss of $u_{(j)}$, the set of all possible loss of the mater CDO might encounter can range from $u_{(0)}$ to $u_{(j)}$. In addition, even in the case of a homogeneous pool of underlying credit entities being considered, the repetitive occurrence of referenced entities among the inner CDOs further complicate the set of possible loss that a master CDO shall subsequently encounter. This is known as the overlapping effect among inner CDO tranches.

To deal with the difficulties outlined above, we define a maximum loss of the master CDO for a CDO² structure as follows:

$$Loss^{\max} = \sum_{i=1}^{m} Upper_i - \sum_{j=1}^{m} Lower_j$$

which is the summation of the principals of the referenced tranches. With the maximum loss, we know that the possible losses of CDO² are $0,1,2,...,Loss^{\max}$. Let α_i be the set of all possible linear combination of default events among the underlying credit entities that results in a loss of $u_{(i)}$ in the master CDO of a CDO², $i = 0,1,2,...,L^{\max}$, and β_j be the set of all possible linear combination of default events among the underlying credit entities that results in a loss of $u_{(j)}$ in the portfolio of inner CDOs. For example, suppose

$$\beta_4 = \{\{\#1,\#2,\#3\}\{\#1,\#2,\#4\}\{\#3,\#4\}\{\#3,\#5\}\}$$

which form the set of all combinations of underlying credit entities that can result in a \$*4* unit loss in the CDO portfolio, and let

$$\alpha_1 = \{\{\#1,\#2,\#3\}\{\#1,\#2,\#4\}...\}\}$$

$$\alpha_0 = \{\{\#3,\#4\}\{\#3,\#5\}...\}\}$$

Here we see that the four subsets of β_4 can in fact belongs to α_1 or α_0. This means that of the four possible combinations in β_4, two of them bring a loss of \$1 unit loss to the CDO² while the other two shall cause zero loss to the CDO². In addition:

$$p_1^{new} = p_1 + \frac{H_1H_2H_3 + H_1H_2H_4}{H_1H_2H_3 + H_1H_2H_4 + H_3H_4 + H_3H_5} \times p_4$$

$$p_0^{new} = p_0 + \frac{H_3H_4 + H_3H_5}{H_1H_2H_3 + H_1H_2H_4 + H_3H_4 + H_3H_5} \times p_4$$

$$p_4^* = p_4 - \frac{H_1H_2H_3 + H_1H_2H_4}{H_1H_2H_3 + H_1H_2H_4 + H_3H_4 + H_3H_5} \times p_4 - \frac{H_3H_4 + H_3H_5}{H_1H_2H_3 + H_1H_2H_4 + H_3H_4 + H_3H_5} \times p_4 = 0$$

which completes our probability bucketing procedure for p_4.

In contrast to Hull and White (2004), we therefore arrive at a two stage probability bucketing procedure: For a chosen bucket size $u(k)$, the shifting of probability weights between p_i and p_i^{new} for $i = 0,1,2,...,k$ is carried out by the following:

$$p_i^{new} = p_i + \frac{\sum_{\alpha_i}\prod_{m\in\alpha_i} H_m}{\sum_{\beta_j}\prod_{n\in\beta_j} H_n} \times p_k \quad \text{and} \quad p_k^{new} = p_k - \frac{\sum_{\alpha_i}\prod_{m\in\alpha_i} H_m}{\sum_{\beta_i}\prod_{n\in\beta_j} H_x} \times p_k$$

Finally, as the probabilities of default we obtained are conditional on the factor *M* at a time t_j, we have to solve for the unconditional probabilities $p_i\left(t_j\right)$ with respect to the probability density function of the common factor M that it was conditional upon.

Numerical Results

1. Tranche Spreads for CDO²

Suppose our CDO² has only one tranche, with a maturity of 3 years, payment dates at the end of each year, recovery rate of 0.4, and risk-free interest rate is flat at 3%; let both distribution for *M* and Z_i (for all *i*) be standard normal distribution,

$p_i\left(t_j\right) = 1 - \exp(-0.01 t_j)$, and $a_i = \sqrt{0.3}$, therefore,

$$\alpha_i(t_j) = \Phi\left(\frac{\Phi^{-1}\left(1-\exp(-0.01t_j)\right) - \sqrt{0.3}M}{\sqrt{1-(\sqrt{0.3})^2}}\right).$$

1. When $t_j = 1$, $\Phi^{-1}\left(1-\exp(-0.01\times 1)\right) = -2.3282$

Default Loss	Conditional Default Loss Distribution $p_i^{\mid M}\left(t_j = 1\right)$					Conditional Default Loss Distribution $p_i\left(t_j = 1\right)$
	$M = -2$	$M = -1$	$M = 0$	$M = 1$	$M = 2$	
0	0.96095	0.99757	0.99993	1	1	0.788422
1	0.03649	0.00240	0.00007	*	*	0.002058
2	0.00225	0.00003	*	*	*	0.000103
3	0.00031	*	*	*	*	0.000013

* means the amount is smaller than 0.00001

2. When $t_j = 2$, $\Phi^{-1}\left(1-\exp(-0.01\times 2)\right) = -2.0579$

Default Loss	Conditional Default Loss Distribution $p_i^{\|M}\left(t_j=2\right)$					Conditional Default Loss Distribution $p_i\left(t_j=2\right)$
	$M=-2$	$M=-1$	$M=0$	$M=1$	$M=2$	
0	0.88887	0.98937	0.99956	0.99999	1	0.783615
1	0.09771	0.01028	0.00043	0.00001	*	0.006333
2	0.01121	0.00031	*	*	*	0.000543
3	0.00222	0.00003	*	*	*	0.000101

3. When $t_j=3$, $\Phi^{-1}\left(1-\exp(-0.01\times 3)\right)=-1.8874$

Default Loss	Conditional Default Loss Distribution $p_i^{\|M}\left(t_j=3\right)$					Conditional Default Loss Distribution $p_i\left(t_j=3\right)$
	$M=-2$	$M=-1$	$M=0$	$M=1$	$M=2$	
0	0.80846	0.97566	0.99873	0.99997	1	0.777236
1	0.15879	0.02312	0.00125	0.00003	*	0.011708
2	0.02605	0.00110	0.00001	*	*	0.001338
3	0.00670	0.00013	*	*	*	0.000314

With the calculation above, we get the expected loss at t_j,

$$EL_{t_1}=0.0013818,\quad EL_{t_1}=0.0046332,\quad EL_{t_1}=0.0091956$$

as well as the two legs (Note: $EL_{t_0}=0$),

$$DL=0.0085726,\quad PL=S\times 8.4641917$$

and finally the spread of our single-tranche CDO²,

$$S=\frac{0.0085726}{8.4641917}=0.0010128$$

which is about 1 bp.

2. Hedging Parameters: DV01 and SDV01

DV01

overlapping	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
Equity-Equity	0.03%	0.06%	0.08%	0.09%	0.10%	0.13%	0.14%	0.15%	0.15%	0.16%	0.17%
Equity-Mezzanine	0.03%	0.06%	0.08%	0.09%	0.10%	0.13%	0.14%	0.15%	0.15%	0.16%	0.17%
Equity-Senior	0.08%	0.52%	0.75%	0.84%	0.94%	1.08%	1.23%	1.28%	1.28%	1.35%	1.38%

Mezzanine-Equity	0.08%	0.13%	0.19%	0.21%	0.24%	0.29%	0.32%	0.34%	0.36%	0.38%	0.40%
Mezzanine-Mezzanine	0.08%	0.13%	0.20%	0.24%	0.27%	0.30%	0.33%	0.36%	0.38%	0.37%	0.31%
Mezzanine-Senior	0.19%	1.21%	1.61%	1.68%	1.72%	1.80%	1.73%	1.73%	1.75%	1.73%	1.76%
Senior-Equity	0.55%	1.71%	2.03%	1.91%	1.80%	1.62%	1.49%	1.37%	1.47%	1.45%	1.44%
Senior-Mezzanine	0.17%	0.70%	0.82%	0.78%	0.75%	0.69%	0.65%	0.61%	0.67%	0.68%	0.67%
Senior-Senior	0.16%	0.76%	0.91%	0.88%	0.85%	0.71%	0.63%	0.52%	0.74%	0.76%	0.75%

SDV01

Extent of Overlapping	**0%**	**10%**	**20%**	**30%**	**40%**	**50%**
Equity-Equity	0.52%	0.86%	1.21%	1.38%	1.55%	1.90%
Equity-Mezzanine	0.52%	0.86%	1.21%	1.38%	1.55%	1.90%
Equity-Senior	1.21%	3.12%	5.72%	6.99%	8.51%	12.39%
Mezzanine-Equity	1.21%	2.02%	2.82%	3.22%	3.63%	4.43%
Mezzanine-Mezzanine	1.21%	2.02%	2.65%	2.95%	3.22%	4.34%
Mezzanine-Senior	2.83%	7.28%	11.50%	12.81%	14.62%	18.51%
Senior-Equity	8.26%	9.99%	12.28%	13.93%	15.15%	16.69%
Senior-Mezzanine	2.41%	3.55%	4.74%	5.40%	6.02%	6.86%
Senior-Senior	1.76%	3.14%	4.14%	4.75%	5.24%	5.79%

Extent of Overlapping	**60%**	**70%**	**80%**	**90%**	**100%**
Equity-Equity	2.07%	2.23%	2.24%	2.42%	2.59%
Equity-Mezzanine	2.07%	2.23%	2.24%	2.42%	2.59%
Equity-Senior	14.53%	16.46%	16.31%	18.56%	20.73%
Mezzanine-Equity	4.82%	5.21%	5.06%	5.53%	6.05%
Mezzanine-Mezzanine	4.61%	4.97%	4.84%	5.02%	4.63%
Mezzanine-Senior	20.85%	22.20%	22.43%	24.03%	26.20%
Senior-Equity	17.30%	18.41%	18.48%	19.73%	21.16%
Senior-Mezzanine	7.50%	7.87%	8.17%	8.81%	9.52%
Senior-Senior	6.17%	6.20%	6.95%	7.64%	8.32%

Conclusion

In the research I have provided a valuation framework under which efficient calculation of the hedging parameters are made feasible. Upon doing so, we must consider the tradeoff of using a semi-analytic approach instead of raw simulations. In particular, the information regarding the correlation structure of defaults though now being simplified is in a way lost. Is there other means of gaining back this loss information in terms of a factor form? In addition, I have established a way in which characterization of the loss distribution with respect to each layer of protection is feasible. This generalized recursive algorithm developed under this research project is expected to be able to adapt to CDOs with compound protection layers of higher dimensionality.

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