行政院國家科學委員會補助專題研究計畫成果報告
期末報告

精算與財務方法在壽險保單定價、準備金估計、以及風險管理之運用(第 3 年)

計 畫 類 別 ： 個別型計畫
計 畫 編 號 ： NSC 99-2410-H-004-063-MY3
執 行 期 間 ： 101 年 08 月 01 日至 102 年 10 月 31 日
執 行 單 位 ： 國立政治大學風險管理與保險學系

計 畫 主 持 人 ： 蔡政憲

計畫參與人員： 博士班研究生-兼任助理人員：邱于芬
博士班研究生-兼任助理人員：宮可倫
博士班研究生-兼任助理人員：李湞靖
博士班研究生-兼任助理人員：吳恕銘
博士班研究生-兼任助理人員：吳玉鳳
博士班研究生-兼任助理人員：王朝生

報 告 附 件 ： 移地研究心得報告

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中 華 民 國 103 年 01 月 29 日

中 文 摘 要 ： 許多人壽保險商品隱含有選擇權，例如最低獲利保證、分紅、以及可提前解約的權利。如果沒有正確地估計這些選擇權的價值，保險公司的清償能力將受到影響。實務界和學術界分別發展出兩類方法來估計這些選擇權的價值。英美系國家的精算學會採用隨機模擬法（或稱精算法）：在實際測度下運用隨機模型模擬出有內含保證或選擇權的保單現值的機率分佈，再根據這個機率分佈來計算相關的的成本並提列準備金。學術文獻則大多運用財務理論中選擇權定價方法來評價內含的選擇權，稱為選擇權定價法或財務法。這個方法是在一些市場完美度的假設（例如）arbitrage-free or complete 以及風險中立的測度下進行的。

本計畫嘗試延伸比較或整合上述兩種方法的文獻。第一個嘗試是透過分析多期的保證以及解約選擇權，延伸了 Boyle and Hardy (1997)以及 Hardy (2003)。第二是嘗試延伸 Barbarin and Devolder (2005) and Gatzert and Kling (2007)。我們使用和 Barbarin and Devolder (2005) 相反的模式來整合精算與財務方法：先進行風險中立評價以計算保費，再用隨機模擬來估計經濟資本。第三部分則是延伸 Kling et al. (2007), Gatzert and Kling (2007), Gatzert (2008), and Graf, Kling, and Russ (2009)：探討數種常見的投資策略（例如 CPPI 與 TIPP）會如何影響保單的評價與保險公司的清償能力，並運用演算法來求解最適的投資策略、分紅機制、契約參數、以及資本結構等。

因著本計畫的資源與執行，已經產生了一篇期刊論文以及六篇審稿中或即將投稿的論文。未來將上述的嘗試寫成完整的文章後，應該還能發出更多的論文。

中文關鍵詞： 保險財務、模擬、人壽保險

英 文 摘 要 ： Many popular life insurance products contain option-like covenants: minimum return guarantees, participating clauses, and/or surrender options. Improper pricing, reserving, and/or hedging of these guarantees and options impair the solvency of an insurer. There are two paradigms to handle the issues. Actuarial associations in UK, US, and Canada adopted the stochastic simulation method (also called the actuarial approach) to analyze these embedded guarantees and options. The idea is to simulate the

payoff distribution of an embedded guarantee/option using stochastic models in the real-world probability measure. Insurers then estimate the expected cost of the guarantee/option and the associated reserves based on the simulated distribution. Academics on the other hand employed the machinery of option pricing for the valuation of the embedded guarantees and options. In this so-called option pricing approach or financial approach, computations take place under a risk-neutral probability measure with certain assumptions on the market (e.g., completeness and no arbitrage).

This project extends the literatures of comparing and integrating these two approaches in three aspects. Firstly, we extend Boyle and Hardy (1997) by analyzing the cliquet-style type of periodic guarantee and incorporating surrender options. Secondly, we turn the procedure proposed by Barbarin and Devolder (2005) the other way around: performing risk-neutral valuation first for policy premiums and then conducting stochastic simulation to calculate the associated economic capital. Thirdly, we extend Kling et al. (2007), Gatzert and Kling (2007), Gatzert (2008), and Graf, Kling, and Russ (2009) by analyzing how investment strategies affect the valuation of insurance policies and the solvency of insurance companies with some popular strategies, e.g., constant proportion portfolio insurance (CPPI) and time-invariant portfolio protection (TIPP). We further employ a heuristic search algorithm to solve for the optimal combination of investment strategies, surplus distribution schemes, contract parameters, and capital structure in a more comprehensive framework.

The resources and implementation of this project have produced one journal article and six working papers that are currently under review or will be submitted in the near future. We expect more papers will be produced when the above extensions are written up.

英文關鍵詞：　Insurance Finance, Simulation, Life Insurance

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計畫主持人：蔡政憲

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處理方式：二年後可公開查詢

中 華 民 國 103 年 1 月 28 日

中文摘要

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Abstract

Many popular life insurance products contain option-like covenants: minimum return guarantees, participating clauses, and/or surrender options. Improper pricing, reserving, and/or hedging of these guarantees and options impair the solvency of an insurer. There are two paradigms to handle the issues. Actuarial associations in UK, US, and Canada adopted the stochastic simulation method (also called the actuarial approach) to analyze these embedded guarantees and options. The idea is to simulate the payoff distribution of an embedded guarantee/option using stochastic models in the real-world probability measure. Insurers then estimate the expected cost of the guarantee/option and the associated reserves based on the simulated distribution. Academics on the other hand employed the machinery of option pricing for the valuation of the embedded guarantees and options. In this so-called option pricing approach or financial approach, computations take place under a risk-neutral probability measure with certain assumptions on the market (e.g., completeness and no arbitrage).

This project extends the literatures of comparing and integrating these two approaches in three aspects. Firstly, we extend Boyle and Hardy (1997) by analyzing the cliquet-style type of periodic guarantee and incorporating surrender options. Secondly, we turn the procedure proposed by Barbarin and Devolder (2005) the other way around: performing risk-neutral valuation first for policy premiums and then conducting stochastic simulation to calculate the associated economic capital. Thirdly, we extend Kling et al. (2007), Gatzert and Kling (2007), Gatzert (2008), and Graf, Kling, and Russ (2009) by analyzing how investment strategies affect the valuation of insurance policies and the solvency of insurance companies with some popular strategies, e.g., constant proportion portfolio insurance (CPPI) and time-invariant portfolio protection (TIPP). We further

employ a heuristic search algorithm to solve for the optimal combination of investment strategies, surplus distribution schemes, contract parameters, and capital structure in a more comprehensive framework.

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Keywords: Insurance Finance, Simulation, Life Insurance

報告內容

因著本計畫的資源與執行所產生的期刊論文目前為[1]：

1. 謝明華、黃雅文、郭維裕、與蔡政憲，2014，壽險準備金風險之衡量，經濟論文（已被接受）。

六篇審稿中或即將投稿的工作論文（working paper or work in progress）則是：

1. Kuo, Weiyu, Ming-Hua Hsieh, Chenghsien Tsai, and Yu-Ching Li, 2014, Generating Economics Scenarios for the Long-Term Solvency Assessment of Life Insurance Companies: The Orthogonal ARMA-GARCH Approach, to be submitted to the 2014 Annual Meeting of American Risk and Insurance Association.
2. Hsieh, Ming-Hua, Jin-Lung Peng, Chenghsien Tsai, Jennifer L. Wang, and Ko-Lun Kung, 2014, Explaining the Rate Spreads on Life Settlements, to be submitted to *Journal of Risk and Insurance* (earlier versions were presented in 2013 Risk Theory Seminar and 2013 American Risk and Insurance Association Annual Meeting).
3. Chan, Linus Fang-Shu, Cary Chi-Liang Tsai, and Chenghsien Tsai, 2014, Relational Modeling on Mortality Rates: International Tests and Hedging, to be submitted to *Insurance: Mathematics and Economics* (an earlier version was presented in 2012 International Longevity Risk and Capital Markets Solutions Conference).
4. Wang, Jennifer L., Ming-Hua Hsieh, and Chenghsien Tsai, 2014, Using Life Settlements to Hedge the Mortality Risk of Life Insurers: An Asset-Liability Management Approach, to be submitted to *Journal of Derivatives* or re-submitted to *Journal of Risk and Insurance* (earlier versions were presented in 2011 International Longevity Risk and Capital Markets

[1] 由於論文從著手進行到完成出版往往需要好幾年，可能會橫跨幾期的國科會計畫執行期間，因此個人會將論文主要進行期間的國科會計畫皆列為感謝支持的對象。畢竟論文得以完成的確是接受到這些計畫接續的支持。

Solutions Conference and 2012 American Risk and Insurance Association Annual Meeting).

5. Chan, Linus Fang-Shu, Cary Chi-Liang Tsai, and Chenghsien Tsai, 2014, Empirical Tests on a Relational Model of Mortality Rates with Applications to Internal Hedging, to be submitted to *Journal of Risk and Insurance* (an earlier version was presented in 2011 International Longevity Risk and Capital Markets Solutions Conference).
6. Hwang, Ya-Wen and Chenghsien Tsai, 2013, The Longevity Risk of Life Insurance Policies Induced by Pricing Errors, submitted to the *IUP Journal of Financial Risk Management*.

以下附上謝明華、黃雅文、郭維裕、與蔡政憲（2014）那篇期刊論文，以及 Kuo, Hsieh, Tsai, and Li (2014)，Hsieh, Peng, Tsai, Wang, and Kung (2014)，以及 Chan, Tsai, and Tsai (2014)等三篇工作論文於此結案報告中。其中謝明華等（2014）中的保單準備金計算和本計畫的保單評價實為一體之兩面。Kuo et al. (2014) 所發展的資產面模型和本計畫資產面的模擬是相關的，由本計畫進用的助理李湞靖協助完成，也因此將其列為共同作者之一。Chan, Tsai, and Tsai (2014)是個人在本計畫的補助下到美國 National Association of Insurance Commissioners 移地研究時所寫的。至於 Hsieh et al. (2014)則是由本計畫進用的助理宮可倫協助完成的，我們也將其列為共同作者之一。

壽險準備金風險之衡量*

謝明華
國立政治大學風險管理與保險學系副教授

黃雅文
逢甲大學風險管理與保險學系助理教授

郭維裕
國立政治大學國際經營與保貿易學系副教授

蔡政憲
國立政治大學風險管理與保險學系教授

關鍵詞：風險管理；人壽保險；準備金
JEL 分類代號：G22、G32

*聯繫作者：黃雅文，逢甲大學風險管理與保險學系助理教授，台中市西屯區文華路 100 號，04-24517250 分機 4108，Email: ywhwang@fcu.edu.tw。
本文承國科會補助 (NSC 96-2918-I-004-006 和 NSC 99-2410-H-004 -063 -MY3)，僅誌謝忱。

摘要

壽險準備金存續期間長之特性使得如何衡量準備金風險成為壽險公司首要之務。相較於傳統的 P 測度法與歐盟 Solvency II 之 QIS 2 所提出之 Q 測度法，本研究提出 QPQ 方法來衡量壽險準備金之風險。理論上，在 P 測度法下計算未來責任現值時，折現率須採用無風險利率加上風險溢酬，然而風險溢酬無客觀可行的方法來進行估計，因此，大部分精算文獻並未考量風險溢酬。在 Q 測度法下，是在風險中立假設下模擬市場風險因子未來可能的變化，無法描繪實際市場波動所造成準備金價值變動之風險。

QPQ 方法則可避免上述方法的缺點。在時點 $t=0$ 時，以市場經濟狀態並在 Q 測度下計算最佳估計值來作為準備金之估計值。但在準備金風險衡量方面，壽險公司需先在 P 測度下模擬時點 $t=H$ 之市場情境，再於每個模擬出來的市場情境所對應的 Q 測度下進行最佳估計，計算時點 $t=H$ 下之準備金。重覆進行此步驟後可以得到時點 $t=H$ 下準備金之分佈。再以對應於所關心的風險期間之無風險利率來折現，即可推得時點 $t=0$ 下準備金之分佈。最後透過風險衡量值，如風險值或尾端風險期望值來衡量準備金之風險。

本研究以生死合險、利變型年金與股票指數型基金為例進行模擬試算。數值結果發現 P 測度法與 Q 測度法所計算之準備金風險與 QPQ 方法的確有顯著差異。傳統 P 測度法可能會過度衡量風險，因其考量全段保障期間之風險。而 Q 測度下所計算之準備金風險低於 QPQ 方法所得之結果，顯示以 Q 測度法衡量準備金風險可能無法正確反映風險。由於準備金適足性對壽險公司清償能力有很大之影響，因此，基於保守監理考量，本研究建議壽險公司應採用 QPQ 方法來衡量準備金風險。

關鍵字：風險管理；人壽保險；準備金

1. 緒論

如何正確衡量準備金風險為壽險公司風險管理首要之務，因準備金風險顯著影響保險公司之清償能力。壽險公司承保人身風險，其保險契約存續期間可能長達七、八十年，利率些微的變動都會對準備金造成很大之影響。另一方面，人壽保險具儲蓄性質，因此壽險準備金金額龐大，為壽險公司主要負債項目。[2]

傳統/現行的精算方法是根據期望值來估計責任準備金。以終身壽險為例，被保險人死亡時保險公司給予理賠，因此精算人員以預定之死亡率計算未來預期理賠與保費收入之期望值，再以預定之保單利率 (固定值) 折現至評價點，即為該時點下該保單之責任準備金。換言之，Bowers et al. (1997) 之傳統/現行精算方法主要是考量死亡時間的不確定性，以期望值概念來衡量未來負債之現值。

過去文獻探討死亡時間、利率、解約率、死亡率不確定性以及費用率結構對準備金風險之影響，其研究方法為在隨機利率或解約率等模型下，模擬壽險保單準備金之分佈，進一步以統計量分析準備金風險 (如 Panjer and Bellhouse, 1980; Bellhouse and Panjer, 1981; Giaccotto, 1986; Beekman and Fuelling, 1990, 1992, 1993; De Schepper and Goovaerts, 1992; Frees, 1990; Parker, 1994, 1996; Marceau and Gaillardetz, 1999; Tsai, Kuo and Chen, 2002; Tsai, Chen and Chan, 2003; Tsai, Kuo and Chiang, 2009; Lin and Tzeng, 2010) 。綜合以上文獻發現，準備金分佈受利率風險之影響相當顯著，反觀死亡率風險對準備金的影響微乎其微。另外，費

[2] 我國壽險業 2006 至 2012 年責任準備金皆超過整體負債的 80%。

用率與解約的保單年度亦會影響準備金的期望值與不確定性。在長壽風險部分，Lin and Tzeng (2010) 實證結果發現，當保單保障期間越長，爲因應長壽風險所需額外增加計提準備金之比率越高。

上述文獻主要以模擬方法探討準備金之分佈，因此在計算準備金期望值的過程中，也可以產生其他動差的估計值，進一步進行風險的衡量。上述精算方法是在 P 測度下計算準備金之風險。但此方法下所計算之風險值 (Value at Risk, VaR) 期間有多長呢？以 20 年期生死合險爲例，保險公司通常在保單發行時，應用 VaR 概念估計其負債，並將計算所得之負債除以 $\sqrt{20}$ 作爲準備金之估計值，[3]但此方法下的 VaR 期間是一年嗎？除以 $\sqrt{20}$ 的涵意究竟爲何呢？此外，在 P 測度法下計算未來責任現值時，需以評價點所估計之利率期限結構再加上風險溢酬 (Risk Premium) 以進行折現，但上述精算文獻在 P 測度下估計準備金風險時，其折現

[3] 進行債券風險值計算時，是根據未來一年殖利率曲線之可能變動計算出債券一年後價格的機率分佈，因此，風險值的時段 (Time Horizon) 是一年。而在進行債券的評價 (Valuation) 時，多是以現在的殖利率曲線做折現率計算現值，或是根據現在的殖利率曲線做校準，在Q測度下模擬債券到期前的無風險利率。換言之，做評價時需要考量債券到期前利率的可能值 (對三十年到期的債券來說，就是未來這三十年間的利率) 。但在估計風險值時，關心的是未來一年殖利率曲線的可能變動，然後在每個可能的殖利率曲線情境下再做評價。整個完整的過程就是本文所說的QPQ方法，其中，Q的時段是債券的到期時間，而P的時段則是風險值的時段。在準備金的風險衡量部分，傳統在P測度下估計準備金風險值是以模擬準備金分佈的方式進行。其方式爲模擬未來所有保單年度 (假設爲 T 年) 的利率、死亡率、解約率等變數，並將未來各年度淨現金流量以所模擬的各年度利率折現回評價點，可得評價點之準備金分佈，進而求算風險。這樣的做法等於是在同一個P測度下同時進行評價與風險值估計，在方法論上是有問題的。而在這個有問題的作法中，到底時段是多少是令人困惑的。其中一個說法是鑑於估計準備金風險值的時候模擬了「未來所有年度之利率變動」以進行折現，因此認爲這些利率變動涵蓋了未來 T 年之風險，因此應該除以 $\sqrt{T}$ 進行修正。我國風險基礎資本額制度 (Risk-based Capital) 中之C2保險風險即是以這個方法進行風險係數之估算。針對上述以 $\sqrt{T}$ 進行修正的作法，本研究亦抱持著存疑的態度。因此本研究才提出一正確的準備金風險衡量方法，亦即QPQ方法。

率並未考量風險溢酬。[4]

另一方面，根據國際財務會計準則 (International Financial Reporting Standard, IFRS) 與 Solvency II 之 Quantitative Impact Study 5 (QIS 5)[5] 的發展趨勢，傾向以公平價值 (Fair Value) 來衡量未來保險契約負債之價值，主要概念是在無風險假設下，估計未來現金流量之折現值作爲準備金之最佳估計值 (Best Estimate)。亦即，在評估保險合約負債的現金流量時，將風險及不確定性反映於現金流量的計算上，計算各種可能發生的情境與機率，以及各種情境所產生之現金流量，並據以計算加權算術平均；最後，再將該現金流量以無風險利率貼現以求得保險合約之準備金。1990 年代起的諸多保險文獻，如 Grosen and Jorgensen (1997, 2000, 2002)、Babbel et al. (2002)、Barbarin and Devolder (2005)、Milevsky and Salisbury (2006) 等亦主張應該在 Q 測度下進行保險契約負債的評價。

QIS 2 建議準備金風險之量化有兩種方法[6]：(1)百分位數法 (Percentile Approach) ，以及(2)資金成本法 (Cost of Capital Rate) 。[7]QIS 2 提到採用百分位數法時，應以最佳估計法所求得之準備金分佈來計算 75 百分位數。換言之，QIS 2 建議在 Q 測度下同時評價準備金並估計風險，因此，在百分位數法下計算風險邊際時，須在 Q 測度下進行模擬。假設站在時點 $t=0$ ，欲估計 H 年之準備金風

[4] 風險溢酬之估計需要有商品之市場價格以及其波動度，而人壽保險商品無法於金融市場上取得交易價格與波動度，因此難以客觀估計其風險溢酬。

[5] 參見 European Commission (2010) 之「QIS5 Technical Specifications」報告書，頁 25-49。

[6] 參見 European Commission (2006) 之「QIS2 Quantitative Impact Study 2 Technical Specifications」報告書，頁 11-12。European Commission (2007) 之「QIS 3 Technical Specifications Part I: Instructions」也提到計算風險邊際可使用此兩種方法，頁 12。

[7] QIS5 則建議使用 Cost of Capital Rate 法衡量準備金風險，參見 European Commission (2010) 之「QIS5 Technical Specifications」報告書，頁 54。

險，其方法爲在時點$t=H$下，以 Q 測度下之模型模擬未來現金流量與相對應之無風險利率，可以得到時點$t=H$下準備金之分佈，再將其以無風險利率折現以推得$t=0$下準備金之分佈，最後透過風險衡量值來衡量準備金之風險。然而，在 Q 測度下，市場未來可能的變化是在風險中立環境下進行模擬，並無法描繪實際市場波動對準備金價值變動所造成之風險。

基於保守監理之考量，並合理反映準備金風險，本研究提出 QPQ 方法來衡量準備金風險。QPQ 方法乃應用巢狀模擬法 (Nested Simulations Approach) 來量化準備金之風險，由於準備金風險衡量分成「準備金最佳估計值」與「準備金風險邊際」兩部分，在準備金風險邊際部分需模擬外部情境與內部情境，因此共需三個步驟，而此三個步驟分別在 Q 測度、P 測度以及 Q 測度下進行，因此本研究稱爲 QPQ 方法，其架構如圖 1。假設保單期間爲T年，在時點$t=0$時，在 Q 測度下以最佳估計值法作爲準備金之估計值 (QPQ 中的第一個 Q)。但在準備金風險衡量方面，假設欲估計H年之準備金風險，則壽險公司應先在 P 測度下模擬$(0,H)$間之市場狀況情境$\tilde{S}_{H,i}$ (QPQ 中間的 P)。然後，在任一$\tilde{S}_{H,i}$下，於時點$t=H$我們可以再作準備金的評價 (在 Q 測度下)，計算時點$t=H$準備金之公平值 (QPQ 中的第二個 Q)。因爲模擬了多個市場情境，我們可以得到時點$t=H$下準備金之分佈，再透過風險衡量值 (風險值或尾端風險期望值) 來衡量準備金之風險。[8]

[8] 由於需在時點$t=0$估計風險，因此需再以$t=0$之利率期限結構 (零息債券殖利率曲線) 所對應到期日之殖利率進行折現，則可推得$t=0$下準備金之分佈，

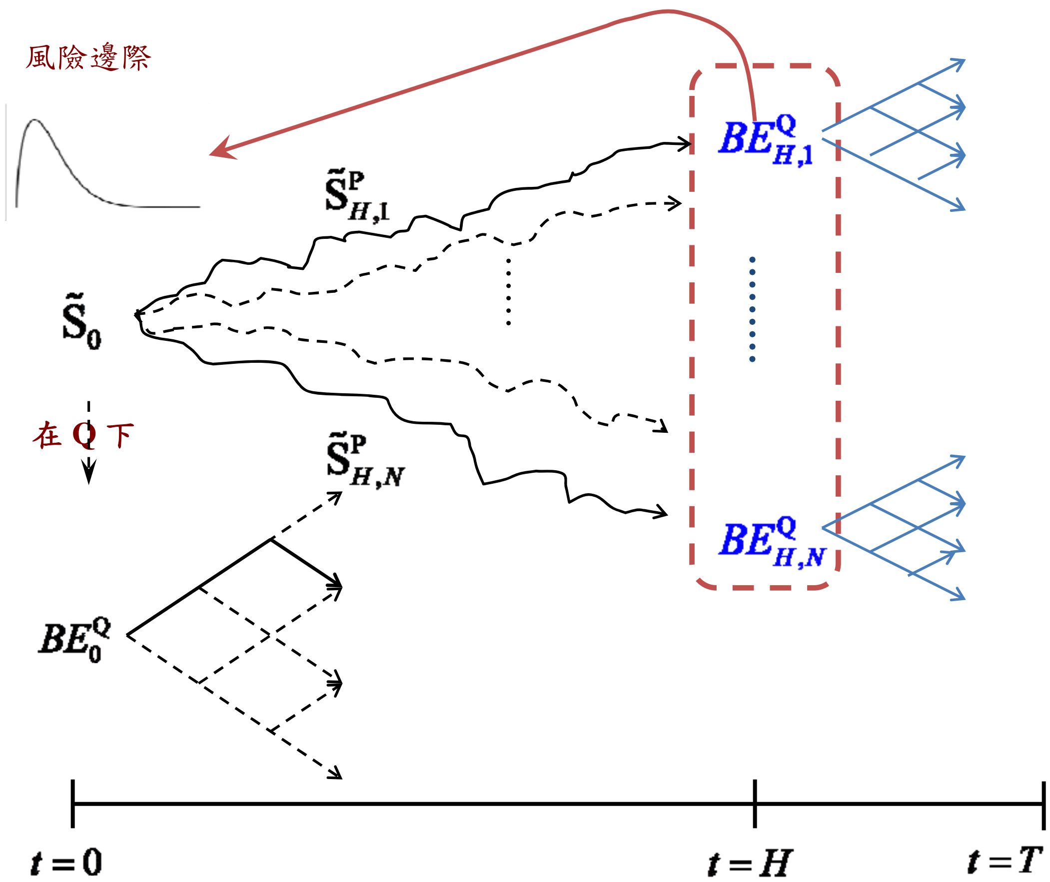

圖 1 QPQ 方法架構圖

QPQ 方法主要為解決只在一個測度下同時進行準備金評價以及衡量風險之問題。以 P 測度法為例，在同一個 P 測度下，在方法論上是有問題的，且折現率多未考慮風險溢酬。而 Q 測度法是在風險中立環境下模擬市場未來可能的變化，並無法描繪實際市場波動所造成之風險。本研究所提之 QPQ 方法則是在 P 測度下模擬市場未來可能的變化 (QPQ 中間的 P)，在每個未來的可能情境下再以 Q 測度法進行評價 (QPQ 中的第二個 Q)，即可得到未來市場狀況不確定下的價值機率分佈，最後將此機率分佈的各個值折現到評價時點來估計風險。QPQ 方法和財務工程的文獻以及最近的保險文獻一致。在選擇權風險的計算方面，McNeil et al. (2005) 介紹風險衡量方法為應用歷史資料建立風險因子在 P 測度下之模型，

並應用此模型使用蒙地卡羅法模擬風險因子，進一步在 Q 測度下求得選擇權價格在 $(t, t+1)$ 之差量分佈，最後應用風險衡量指標計算選擇權之風險。Gordy and Juneja (2010) 在衡量衍生性商品組合風險時，採用巢狀模擬法 (Nested Simulations Approach)，在 Q 測度下評價衍生性金融商品，並在 P 測度下模擬外層模擬步驟 (Outer Step) 之市場情境。在保險文獻方面，Bauer et al. (2012) 建議應以巢狀模擬法計算保險公司之清償資本額要求 (Solvency Capital Requirement)，研究中也提及需在 P 測度下模擬時點 0 到時點 1 間之市場情境。這些文獻都與 QPQ 方法概念一致。[9]

模擬結果顯示保險公司應以最佳估計法估計準備金期望值，因在 P 測度法下所使用之折現率不符合金融市場一致性評價之假設。在準備金風險衡量部分，模擬結果顯示傳統之 P 測度法可能會過度衡量風險，因其考量全段保障期間之風險。而 Q 測度下所計算之準備金風險低於 QPQ 方法所得之結果，顯示以 Q 測度衡量準備金風險可能無法正確反映風險。準備金適足性對壽險公司清償能力有很大之影響，因此，基於保守監理考量，本研究建議壽險公司應採用 QPQ 方法來衡量準備金風險。

本文架構如下：第 1 節爲緒論，說明本研究之動機，隨後介紹人壽保險未來責任價值與風險之評估方法，包含 P 測度法、Q 測度法與 QPQ 方法。第 3 節簡述評估時所用之模型，包含利率模型與股票相關資料。第 4 節爲數值分析，強調

[9] Bauer et al. (2012) 以資產的角度出發，衡量清償資本額要求，換言之，即是衡量資產扣掉負債之公平價值；Gordy and Juneja (2010) 則是探討資產的風險衡量。而本研究以負債角度出發，類似於衡量「負債扣掉資產」之公平價值與風險。

QPQ 方法估計風險與其他兩種方法之差異，最後爲結論。

2. 價值與風險評估方法

準備金係指保險公司爲因應所承擔之責任 (未來理賠之現金流出扣除未來保費之現金流入) 在今日所應提列之數值。[10]因責任準備金現值受到利率與其他風險因子之影響，其不確定性造成準備金之機率分佈，[11]學術上多以期望值 (傳統法) 或以最佳估計值 (公平評價法) 來衡量應提列之準備金。而準備金風險則係指此未來責任之不確定性。[12]

2.1 未來責任之期望值或最佳估計值

本研究中我們以時點 $t=0$ 爲例，衡量未來責任之現值 (L_0)，亦即準備金之現值。文獻與實務上之計算方法有兩種：(1) 在 P 測度法下所求得之現值稱爲期望值；(2) 在 Q 測度法下所求得之現值稱爲最佳估計值。假設未來第 y 年底現金流入 (保費收入) 爲 C_y^I，第 y 年底現金流出 (理賠支出) 爲 C_y^O。各年度現金流入與當年度存活人數有關，因此在模擬 C_y^I 時與死亡率 ($\tilde{q}$) 有關，可表示成 $C_y^I(\tilde{q})$。近年來保險公司發行的商品日益複雜，各年度現金流出除與死亡率有關外，尚可能與市場利率 ($\tilde{\mathbf{r}}$) 或金融商品報酬率 ($\tilde{\mathbf{R}}$) 有關，[13]可表示成 $C_y^O(\tilde{q},\tilde{\mathbf{r}},\tilde{\mathbf{R}})$。假設保單期間爲 T 年，則各年度之淨現金流量 CF_y 爲 $C_y^O - C_y^I$，依據各年度折現因子 v_y，

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[10] 保險契約簽訂後，保險公司向要保人收取保費，因此收取保費在先，損失理賠在後。保險公司在收取保費後，爲履行保險契約內之損失理賠責任，需逐年提存責任準備金。

[11] 利率與其他風險因子具不確定性，因此當對未來利率與其他風險因子估計不同時，會得到不同的準備金估計值。

[12] 保險法所稱之法定責任準備金係指在期望值/最佳估計值之上，再加上風險邊際，以達謹慎保守估計準備金之目的。

[13] 因保單有效期間不只一年，因此需模擬未來各年度之市場利率或金融商品報酬率，因此我們以向量符號表示 $\tilde{\mathbf{r}}$ 與 $\tilde{\mathbf{R}}$。

可計算未來責任現值 L_0 ，公式如下：

$$L_0 = \sum_{y=0}^{T} CF_y \times v_y \ .^{14} \tag{1}$$

以下介紹在 P 測度法與 Q 測度法下之模擬計算流程，並分別以 L_0^{P} 與 L_0^{Q} 代表未來責任之期望值與最佳估計值。

2.1.1 P 測度法

根據 Tsai et al. (2003)，在 P 測度法下保險公司依據市場參數所模擬之未來情境下之現金流量，並以所對應之利率折現回 $t=0$ 計算其未來責任現值 L_0^{P} 。本節介紹在 P 測度法下模擬準備金期望值之流程：

步驟 1：站在 $t=0$，收集市場利率期限結構與模擬各年度現金流出所需之金融市場商品報酬率過去歷史資料，建立在 P 測度下之模型。

步驟 2：以 $t=0$ 之市場利率期限結構 $\tilde{\mathbf{r}}_0$ 以及金融市場商品報酬率資料 R_0 爲模型起始值，可模擬出在 P 測度法下未來 N 年之利率期限結構 $\tilde{\mathbf{r}}_{t=0}^{\mathrm{P}} = \{r_{t=0,1year}^{\mathrm{P}},\ r_{t=0,2year}^{\mathrm{P}},...\}$ 與金融市場商品報酬率 $\tilde{\mathbf{R}}^{\mathrm{P}} = \{R_1^{\mathrm{P}}, R_2^{\mathrm{P}},...\}$ 。

步驟 3：以步驟 1 與 2 模擬所得之資料，可估算未來 T 年各年度之現金流出 ($C_y^O(\tilde{q},\tilde{\mathbf{r}}_t^{\mathrm{P}},\tilde{\mathbf{R}}^{\mathrm{P}})$) 與現金流入 ($C_y^I(\tilde{q})$)。在 P 測度法下，各年度折現因子 $v_y^{\mathrm{P}} = \prod_{m=1}^{m=y}(1+r_{0,m\ year}^{\mathrm{P}})^{-1}$ ， $v_0^{\mathrm{P}} = 1$ 。各年度之淨現金流量 CF_y 爲 $C_y^O - C_y^I$ ，依據各年度折現因子 v_y^{P} ，即可得該模擬情境下準備金之現值。

步驟 4：重複執行步驟 1 至步驟 3 共 N 次，即可得準備金現值之分佈 ($L_{t=0,i=1}^{\mathrm{P}}$,

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[14] 注意，本研究乃計算已收保費後之準備金，在第 3 與第 4 節的保單模擬試算裡，因假設保費爲躉繳，故未來各年度之現金流入爲 0。

$L^{\mathrm{P}}_{t=0,i=2}$， …， $L^{\mathrm{P}}_{t=0,i=N}$)，[15]再應用期望值概念計算準備金之估計值 ($L^{\mathrm{P}}_{0}=E(L^{\mathrm{P}}_{t=0,i})$)。

在 P 測度法下計算未來責任現值時，乃以評價點所估計之利率期限結構進行折現。這種做法的主要問題為高估負債金流之現值 (價格)，因其折現率未考量風險溢酬。理論上，對一個不確定的現金流之現值 (價格) 計算有兩大類可行做法，第一類為均衡 (Equilibrium) 價格之評價模型，如資本資產定價理論 (Capital Assets Pricing Model, CAPM)，此類評價模型不改變機率測度，但折現率須加上風險溢酬。第二類為無套利 (No-arbitrage) 價格之評價模型，如 Black-Scholes 之選擇權定價模型。此類評價模型不改變折現率，但須調整測度，使得不確定現金流之期望值降低。在第一類方法下，風險溢酬無客觀可行的方法來進行估計，同時，若保險商品連結金融市場標的，則此風險溢酬可能會隨著此連結商品之變動而成為一隨機變數。由於此方法在實務操作上具有困難度，[16]因而大部分精算文獻在 P 測度法下估計準備金風險時，其折現率並未考量風險溢酬。

2.1.2 Q 測度法

Babbel et al. (2002) 認為在評估保險合約負債之現金流量時，應將風險及不確定性反映於現金流量之機率測度，[17]計算各種可能發生的情境所產生之現金流

[15] 假設 $L^{\mathrm{P}}_{t=0,i=1}$, $L^{\mathrm{P}}_{t=0,i=2}$, …, $L^{\mathrm{P}}_{t=0,i=N}$ 已由小到大排序。

[16] 人壽保險商品無法於金融市場上取得交易價格與波動度，因此難以客觀估計其風險溢酬。

[17] 在 Solvency II、Swiss Solvency test，以及 IFRS 4 中，負債的價值必需符合市場一致性 (Market Consistent) 之要求。所謂市場一致性的負債價值，主要精神在於負債價值應等同於在金融市場上可取得的複製投資組合 (Replicating Portfolio) 之價值。在選擇權定價理論中 (Harrison and Kreps, 1979; Harrison and Pliska, 1981)，複製投資組合的價值可透過風險中立評價法求得。而風險中立評價法的作法是經由微妙的測度轉換將真實測度 (P-measure) 轉換成風險中立測度

量，並據以計算加權算術平均，最後，再將該現金流量以無風險利率貼現以求得保險合約之準備金，因此，保險公司必須考量符合金融市場一致性的評價假設。

在 Q 測度法下保險公司應先以金融市場所提供之利率資訊與利率模型估計無風險利率，作爲各年度現金流量之折現因子。若保單給付連結金融市場商品，則在 Q 測度法下，需以財務理論模型 (如 Black-Scholes 選擇權公式) 估計保險商品之價值。本節我們介紹在 Q 測度法下模擬準備金之最佳估計值的架構：

步驟 1：站在 $t=0$，收集市場利率期限結構 $\tilde{\mathbf{r}}_0$，由 $\tilde{\mathbf{r}}_0$ 可校準 (Calibrate; 詳細步驟請參看 3.2 節) 在 Q 測度下利率模型之參數，並進一步估計無風險利率期限結構 $\tilde{\mathbf{r}}^{\mathrm{Q}}_{t=0}=\{r^{\mathrm{Q}}_{t=0,1year},\ r^{\mathrm{Q}}_{t=0,2year},...\}$。

步驟 2：假設保險商品給付與金融市場商品報酬率有關，則以該金融商品報酬率歷史資料建構該商品報酬率之動態模型與模型參數，並以 R_0 爲起始值，可模擬出在 Q 測度下未來 T 年該商品報酬率 $\tilde{\mathbf{R}}^{\mathrm{Q}}=\{R^{\mathrm{Q}}_1,R^{\mathrm{Q}}_2,...\}$。

步驟 3： (無法以財務理論模型計算公平價值之保險商品)[18]

以蒙地卡羅模擬法計算該保險商品負債之公平價值。以步驟 1 與步驟 2 所模擬之利率期限結構 $\tilde{\mathbf{r}}^{\mathrm{Q}}_{t=0}$ 與金融商品報酬率 $\tilde{\mathbf{R}}^{\mathrm{Q}}$ 模擬未來 T 年各年度之現金流出 ($C^{O}_{y}(\tilde{q},\tilde{\mathbf{r}}^{\mathrm{Q}}_{t},\tilde{\mathbf{R}}^{\mathrm{Q}})$) 與現金流入 ($C^{I}_{y}(\tilde{q})$)。[19]在 Q 測度下，各年度

(Q-measure) 後，以無風險利率折現預期現金流量即可求得價格。因此，保險公司應以 Q 測度法來計算負債價值。

[18] 以利變型年金爲例，若帳戶價值每年依兩年期利率增加，則其負債現值無法以財務理論模型推算，此時可用蒙地卡羅模擬法進行分析。

[19] 由於本研究目的在於提出一QPQ方法以正確估計準備金風險，因此本文沒考慮解約。當解約率爲0時，在同一時點下，生死合險保單之現金流量僅與生存率與死亡率有關，因此，無論在P測度或Q測度下，其各年度現金流入與流出皆相同。而利變型年金與股票指數型年金之現金流量與市場標的 (兩年期利率與股票報酬率) 有關，因此其各年度現金流入與流出在P測度與Q測度模

折現因子 $v_y^{\mathrm{Q}}=\prod_{m=1}^{m=y}(1+r_{0,m\ year}^{\mathrm{Q}})^{-1}$ ，$v_0^{\mathrm{Q}}=1$ 。各年度之淨現金流量 CF_y 為 $C_y^O-C_y^I$ ，依據各年度折現因子 v_y^{Q} ，即可得該路徑下準備金之現值。

步驟 4：重複 N 次步驟 1 至步驟 3，即可得準備金現值之分佈 ($L_{t=0,i=1}^{\mathrm{Q}},L_{t=0,i=2}^{\mathrm{Q}}$, …, $L_{t=0,i=N}^{\mathrm{Q}}$)，再應用期望值概念計算準備金之最佳估計值 ($L_0^{\mathrm{Q}}=E(L_{t=0,i}^{\mathrm{Q}})$)。

步驟 5： (可以財務理論模型計算公平價值之保險商品)[20]

若保險商品可以財務理論模型 (選擇權) 計算公平價值，則以步驟 1 至步驟 2 求得之利率模型之參數與金融商品報酬率動態模型之參數，可計算該保險商品在 $t=0$ 負債之現值，即為期初準備金之最佳估計值 (L_0^{Q})。

2.2 未來責任風險之量化

本研究以風險衡量指標 (VaR 與 CTE) 量化未來責任之風險，其定義如下：

$$P(L\le L_{VaR(\alpha)})=1-\alpha\%, \tag{2}$$

$$L'_{VaR(\alpha)}=L_{VaR(\alpha)}-E(L), \tag{3}$$

$$L_{CTE(\alpha)}=E[L\mid L>L_{VaR(\alpha)}], \tag{4}$$

$$L'_{CTE(\alpha)}=L_{CTE(\alpha)}-E(L), \tag{5}$$

$L_{VaR(\alpha)}$ 代表準備金現值超過 $L_{VaR(\alpha)}$ 的機率為 $\alpha\%$ 。而 $L_{CTE(\alpha)}$ 則是準備金現值超過 $L_{VaR(\alpha)}$ 之期望值。責任之風險強調偏離預期期望值之差距，因此公式(3)與公式(5)

擬下之結果會有所不同。但注意，在P測度下，我們不需要也沒有做評價，我們只做風險因子的隨機模擬，所以沒有涉及現金流量。要計算現金流量都是在Q測度下。因此現金流量是否因為測度的不同而有不同是不影響我們的結果的。

[20] 以股票指數型年金為例，假設期滿時年金價值與當時股票價格有關，因此，可應用選擇權公式計算該保險商品之價值。

將未來責任風險 $L'_{VaR(\alpha)}$ 與 $L'_{CTE(\alpha)}$ 定義爲 VaR 與 CTE 再扣除未來責任期望值。

2.2.1　P 測度法

本研究以模擬法模擬準備金現值之分佈，在 2.1.1 節中，P 測度法下模擬之準備金現值分佈爲 $(L^{\mathrm{P}}_{t=0,i=1}, L^{\mathrm{P}}_{t=0,i=2}, \ldots, L^{\mathrm{P}}_{t=0,i=N})$，根據公式(2)與公式(4)之概念，$L^{\mathrm{P}}_{VaR(\alpha)} = L^{\mathrm{P}}_{t=0,i=N\times(\alpha\%)}$，$L^{\mathrm{P}}_{CTE(\alpha)} = E[L^{\mathrm{P}}_{t=0,i}]$，$i = N\times\alpha\% + 1, \ldots, N$。而未來責任之風險爲：

$$L^{\mathrm{P}}_{VaR(\alpha)}{}' = L^{\mathrm{P}}_{VaR(\alpha)} - L^{\mathrm{P}}_{0},$$

$$L^{\mathrm{P}}_{CTE(\alpha)}{}' = L^{\mathrm{P}}_{CTE(\alpha)} - L^{\mathrm{P}}_{0}.$$

2.2.2　Q 測度法

QIS 2 建議以百分位數法計算風險邊際，施行步驟如下，假設在時點 $t=0$，欲估計未來 H 年間之準備金風險，則在時點 $t=H$ 時，以 Q 測度下之模型模擬未來現金流量與相對應之無風險利率，可以得到時點 $t=H$ 下準備金之分佈，再將其以無風險利率折現以推得 $t=0$ 下準備金之分佈，再透過風險衡量值，來衡量準備金之風險。本研究以 $t=0$ 爲例，計算一年的準備金風險，因此需模擬 $t=1$ 下準備金之分佈。詳細模擬流程如下：

步驟 1：在 2.1.2 節中已建構 Q 測度下之利率模型與金融商品報酬率動態模型。以 $t=0$ 之市場利率期限結構 $\tilde{\mathbf{r}}_0$ 以及金融市場商品報酬率資料 R_0 爲模型起始值模擬 $t=1$ 之利率期限結構 $\tilde{\mathbf{r}}^{\mathrm{Q}}_{t=1}$ 與金融市場商品報酬率 $R^{\mathrm{Q}}_{t=1}$。

步驟 2：站在 $t=1$，以步驟 1 之結果，應用 2.2.1 節中計算最佳估計值方法計算

在$t=1$下準備金之最佳估計值（$L^{Q}_{t=1,i=1}$）。

步驟 3：將步驟 2 之結果以$t=0$之無風險利率（$r^{Q}_{t=0,1year}$）折現回期初（$t=0$），

$E_0(L^{Q}_{t=1,i=1})=L^{Q}_{t=1,i=1}/(1+r^{Q}_{t=0,1year})$。

步驟 4：重複 N 次步驟 1 至步驟 3，即可得準備金現值之分佈（$E_0(L^{Q}_{t=1,i=1})$，

$E_0(L^{Q}_{t=1,i=2})$, …, $E_0(L^{Q}_{t=1,i=N})$)，再應用風險衡量指標 VaR 與 CTE (公式(2)

與(4))，即可估計$L^{Q}_{VaR(\alpha)}$與$L^{Q}_{CTE(\alpha)}$。而未來責任之風險為：

$$L^{Q}_{VaR(\alpha)}{}'=L^{Q}_{VaR(\alpha)}-L^{Q}_{0},$$

$$L^{P}_{CTE(\alpha)}{}'=L^{P}_{CTE(\alpha)}-L^{P}_{0}.$$

2.2.3 QPQ 方法

此方法的概念主要是在 P 測度下模擬一年內市場環境之變化，再利用最佳估計值 (Q 測度下) 求得在金融市場變化下，一年後準備金之分佈。[21]將$t=1$之準備金分佈以時點$t=0$的利率期限結構所相對應之一年期殖利率折現至期初($t=0$)，再應用風險衡量指標，如 VaR 或 CTE，即可估計準備金之風險。詳細模擬架構如下：

步驟 1：在 2.1.1 節中已建構 P 測度下之利率模型與金融商品報酬率動態模型。

以$t=0$之市場利率期限結構$\tilde{\mathbf{r}}_0$以及金融市場商品報酬率資料R_0為模型起始值模擬$t=1$之利率期限結構$\tilde{\mathbf{r}}^{P}_{t=1}$與金融市場商品報酬率$R^{P}_{t=1}$。

步驟 2：站在$t=1$，以步驟 1 之結果，應用 2.2.1 節中計算最佳估計值方法計算

[21] 注意，這邊的步驟已是 QPQ 方法中的 P 與第二個 Q，第一個 Q 為在 Q 測度法下計算準備金在時點$t=0$之最佳估計值。

在$t=1$下準備金之最佳估計值（$L^{Q}_{t=1,i=1}$）。

步驟 3：將步驟 2 之結果以$t=0$之無風險利率（$r^{Q}_{t=0,1year}$）折現回期初（$t=0$），

$E_0(L^{Q}_{t=1,i=1})=L^{Q}_{t=1,i=1}/(1+r^{Q}_{t=0,1year})$。

步驟 4：重複N次步驟 1 至步驟 3，即可得準備金現值之分佈（$E_0(L^{Q}_{t=1,i=1})$，$E_0(L^{Q}_{t=1,i=2})$，…，$E_0(L^{Q}_{t=1,i=N})$），再應用風險衡量指標 VaR 與 CTE (公式(2)與(4))，即可估計準備金之風險。

2.2.2 節與 2.2.3 節主要差別在於模擬$t=1$準備金分佈方法不同，在 Q 測度法下是以 Q 測度所建構之利率模型與金融商品報酬率動態模型模擬，在 QPQ 方法下是以 P 測度所建構之利率模型與金融商品報酬率動態模型進行模擬。

2.3 釋例保單

為比較與分析在三種不同方法下之準備金期望值與風險衡量之差異，本研究以三張壽險保單為範例，分別為：T年期生死合險 (Endowment)、利變型年金 (Interest-Sensitive Annuity) 與股票指數型年金 (Equity-Index Annuity, EIA)，並於表 1 整理此三種保單之基本假設。

(1) T年期生死合險

假設保額F元下之年繳保費為P元，保單預定利率為d，x歲之死亡率為q_x。當被保險人於保障期間內死亡，或滿期仍存活，皆可領到F元。

(2) 利變型年金

假設累積期為T年，期初躉繳保費為SP元，若宣告利率為r_c，則累積期滿該

利變型年金之累積價值爲：$SP\times\prod_{t=0}^{T-1}(1+r_{c,t})$。保險公司以累積價值與當時之年金生命表計算每年可請領之年金金額。

(3) 股票指數型年金

本研究以點對點式 (Point-to-point) 股票指數型年金 (Hardy, 2003) 爲例，假設累積期爲T年，期初躉繳保費爲SP元，連動之股票指數價值爲S_t，若參與率爲α且保證收益爲G，則累積期滿該股票指數型年金之累積價值爲$\max\left(G, SP\left(1+\alpha\left(\frac{S_T}{S_0}-1\right)\right)\right)$。保險公司以累積價值與當時之年金生命表計算每年可請領之年金金額。

表 1: 保單基本假設[22]

T年期生死合險		利變型年金		股票指數型年金	
F	1,000,000	SP	627,708	SP	627,708
P	45,300	T	10	T	7
T	20			α	60%
d	4%	r_c	2 年期利率	G	$SP\times0.95\times(1+3\%)^7$
q_x	1989 TSO×90%			股票指數	台灣 50 指數

人身保險爲壽險公司主要經營項目，壽險保單爲主要契約類型，由於生死合險相當於定期壽險加上生存保險之組合，又可視爲終極年齡爲$x+T$歲[23]之終身壽險，因此本研究以生死合險作爲傳統型保單之範例。隨著近年來利率持續下降，壽險公司發行利變型年金商品，將傳統型商品中固定的預定利率變成連結於浮動

[22] 本研究探討之生死合險保單採用業界之假設與現金流量分析，其用台灣第三回經驗生命表 (1989 Taiwan Standard Ordinary Experience Mortality Table) 乘以 90%計價之年繳保費爲 45,300 元，換算成躉繳保費爲 627,708 元。

[23] 假設x歲投保，保障期間爲T年。

的市場利率。股票指數型年金則是將預定利率連結於股價指數，並附有保證給付機制，具備選擇權的性質。以上兩類保單主要是在利率較低的環境中，連結金融市場標的之保險商品，以吸引不同風險屬性之保險商品購買者。

3. 經濟情境模型與評價模型

3.1 經濟情境模型

根據表 1，本研究探討三張壽險保單所面臨之市場狀態變數包括利率期限結構以及股票指數報酬。在利率期限結構部分採用台灣櫃檯買賣中心所公布之殖利率資料進行估計，每年度資料有 1 個月、3 個月、6 個月、1 年、1.5 年、2 年、…、30 年期債券之利率共 62 筆資料。雖然將所有變數皆納入模型的建構，能反映出比較多的變異，但是在參數估計上卻可能面臨模型維度 (Dimension) 過高而導致估計不易/準的問題。每組利率資料 62 筆再加上股票報酬，資料龐大而無法直接使用 GARCH 模型產生變數間之共變數矩陣，因此本研究參考 Alexander (2001, 2002)、梁正德和郭維裕 (2009)，採用 Orthogonal GARCH 方法[24]來建構利率期間結構與股票指數報酬率的模型。而 Orthogonal GARCH 方法主要概念爲應用主成份分析法使風險因子正交，並進一步從所有正交因子波動中產生原來風險因子的完整共變異矩陣，以描述所欲估計之變數。

本研究收集台灣櫃檯買賣中心所公布之殖利率曲線週資料，[25]每週發佈 1 個

[24] Orthogonal GARCH 方法主要應用主成分分析法進行風險因子之估計，Hull and White (2002) 亦應用主成分分析法來建構利率期間結構。Orthogonal GARCH 方法之介紹詳見附錄 1。

[25] 利率資料來源：櫃臺買賣中心/債券交易資訊/公債/日統計/殖利率曲線。（http://www.otc.org.tw/ch/bond_trading_info/gov_bond/daily/GovBondDaily.php#）

月、3 個月、6 個月、1 年、1.5 年、2 年、…、30 年期共 62 個殖利率資料，資料收集期間爲 2006 年 1 月 6 日至 2011 年 4 月 15 日，[26]共 270 個資料點。股票指數則採台灣 50 指數週報酬率資料，其資料來源爲台灣經濟新報資料庫，資料期間與殖利率曲線相同。

首先對利率期限結構進行主成份分析 (下圖 3)，由圖 3 可知，前三個因素 (F_1、F_2 與 F_3) 爲主要決定利率期限結構之因素，因此，本研究中 $p = 3$，F_1、F_2 與 F_3 能夠解釋資料變數 (利率期限結構) 約 92.85%以上的總變異數。再將估計所得之共同因素 F_1、F_2 與 F_3 以及股票指數報酬率歷史資料進行統計分析，可得到共同因素與股票指數報酬間之相關係數矩陣如表 2。[27]其中，dF_i 表示 F_i 之差量，R_S 爲標準化後之股票指數報酬率。[28]

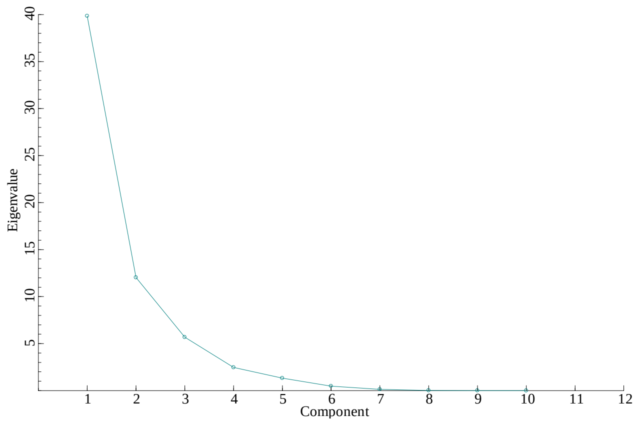

圖 3 主成份分析

[26] 櫃檯買賣中心所提供之殖利率日資料從 2006 年 1 月 2 日開始，因本研究以週資料進行模型配適，因此資料起始點從 2006 年 1 月 6 日開始。

[27] 在 Orthogonal GARCH 模型下模擬未來利率期限結構與股票指數報酬時，需先根據共同因素 (F_1、F_2、F_3 與 R_S) 的歷史資料估計相關係數矩陣。將此相關係數矩陣做 Cholesky 分解後，再乘以此四個因素動態模型中的亂數，即能將此四個因素間的相關性納入考量。

[28] 股票指數報酬率在資料期間內之平均值爲 6.212%。

表 2 相關係數矩陣

	dF_1	dF_2	dF_3	R_S
dF_1	1	-0.0002	-0.0006	0.1489
dF_2	-0.0002	1	-0.0002	0.1079
dF_3	-0.0006	-0.0002	1	0.1117
R_S	0.1489	0.1079	0.1117	1

表 3 因素分析

dF_1 ：AR(1) $F_{1,t}=a_1F_{1,t-1}+\eta_{1,t}$， $\eta_{1,t}\sim N\left(0,\sigma_{1,\eta}^2\right)$		
$a_1=-0.2315$ (0.0001)	$F_{1,0}=-0.0094$	$\sigma_{1,\eta}^2=1.0032$
dF_2 ：AR(1) $F_{2,t}=a_2F_{2,t-1}+\eta_{2,t}$， $\eta_{2,t}\sim N\left(0,\sigma_{2,\eta}^2\right)$		
$a_2=-0.1031$ (0.0920)	$F_{2,0}=-1.0905$	$\sigma_{2,\eta}^2=1.0037$
dF_3 ：AR(1)+ARCH(1) $F_{3,t}=a_3F_{3,t-1}+\eta_{3,t}$， $\eta_{3,t}\sim N\left(0,\sigma_{\eta_{3,t}}^2\right)$， $\sigma_{\eta_{3,t}}^2=\gamma_0+\gamma_1\eta_{3,t-1}^2$		
$a_3=-0.2969$ (0.0000)	$F_{3,0}=0.0651$	
$\gamma_0=0.5067$ (0.0000)	$\gamma_1=0.5823$ (0.0000)	$\eta_{3,0}=0.0560$
R_S ：GARCH(1,1) $R_{S,t}=\eta_{4,t}$， $\eta_{4,t}\sim N\left(0,\sigma_{\eta_{4,t}}^2\right)$， $\sigma_{\eta_{4,t}}^2=\beta_0+\beta_1\eta_{4,t-1}^2+\beta_2\sigma_{\eta_{4,t-1}}^2$		
$\beta_0=0.3486$	$\beta_1=0.1278$	$\beta_2=0.8420$

(0.0983)	(0.0070)	(0.0000)
$\eta_{4,0} = -2.5003$	$\sigma^2_{\eta_{4,0}} = 9.9970$	

註：括號內爲 p 值。

接下來我們建構共同因素（F_1、F_2、F_3 與 R_S）之動態變化模型，以 ARMA (p, q)-GARCH(m, n) 時間序列模型進行配適。模型配適方法第一步將模型殘差符合白噪音 (White Noise) 之時間序列模型選爲候選模型，若有多種候選模型，再根據 Akaike Information Criterion (AIC) 或 Bayesian Information Criterion (BIC) 的大小來挑選，以 AIC 或 BIC 最小者爲最適模型。[29]模型估計結果如上表 3。

3.2 評價模型

假設風險中立下，動態利率模型服從 CIR 模型 (Cox et al., 1985)，可表示爲下式：

$$dr(t) = a(b - r(t))dt - \sigma_r \sqrt{r(t)} dz_r(t), \qquad t \geq 0 .$$

a 爲利率反轉速度 (Mean Reversion Speed)，b 代表利率之平均長期水準，而 σ_r 爲利率變動之瞬間波動度，dz_r 代表隨機誤差項，注意，$2a \geq \sigma_r^2$。

CIR 模型參數配適方法爲找到最適之參數值 $\hat{a}$、$\hat{b}$ 與 $\hat{\sigma}_r$，使得預測之利率值 $\hat{r}$ 與樣本點 r 之平方誤差 (Square Error) 最小，請見下式：

$$\underset{a,b,\sigma_r}{\text{Min}} \sum_{i=1}^{N} \left(\hat{r}_i - r_i\right)^2 .$$

透過 MATLAB 程式求得 CIR 模型參數後，本研究參考 Glasserman (2003) 之方法預測未來之利率。其概念爲在已知 u 時點利率 $r(u)$ 下，$r(t)$ 會服從一時間的

[29] 相關統計資料請見附錄 2。

函數再乘以一非中央卡方隨機變數 (Noncentral Chi-square Random Variable)，詳見下式：

$$r(t)=\frac{\sigma_r(1-e^{-a(t-u)})}{4\alpha}\chi_d'^2\left(\frac{4ae^{-a(t-u)}}{\sigma_r(1-e^{-a(t-u)})}r(u)\right),\ t>u,$$

其中，自由度 $d=\frac{4ba}{\sigma_r^2}$ 。

股票價格服從幾何布朗運動模型 (Geometric Brownian Motion Model)，在 Q 測度法下，股票價格的動態過程爲 $\frac{dS(t)}{S(t)}=r_f dt+\sigma dW(t)$ ，r_f 爲無風險利率，σ 則以股票選擇權之隱含波動度估計，$W(t)$ 爲維納過程 (Wiener Process)。

本研究以 2011 年 4 月 15 日之殖利率曲線資料爲基準，取 2 年、5 年、10 年、20 年以及 30 年期殖利率共五個點配適 CIR 模型參數，在給定不同起始值下，皆得到表 4 之 CIR 模型參數校準結果。

表 4 CIR 模型參數結果

a	b	σ
0.0944	0.0268	0.0269

而股票價格動態過程之波動度資料採用台灣經濟新報資料庫下台灣 50 指數選擇權在 2011 年 4 月 15 日之隱含波動度，爲 18.34%。

4　數值分析

4.1 期望值/最佳估計值

表 5 列出三張壽險保單試算結果，以生死合險爲例，若以期望值估計準備金現值，則有低估準備金之現象，這是因爲在 P 測度下模擬所得利率較 Q 測度下高 (請參考附錄 3)。反觀股票指數型年金，其最佳估計值爲 719,695，期望值爲

924,657，這是因爲在 P 測度下，股票指數之平均報酬率爲 6.212%，然而在 Q 測度下，採用 Blake-Scholes 公式計算股票指數型年金之價值時，假設股價之報酬爲無風險利率 (1.32%)，因此，最佳估計值遠低於期望值。[30]

在利變型年金部分，$t=0$ 時準備金之期望值爲 636,913，[31]然而在 Q 測度下衡量準備金之最佳估計值爲 632,495，我們發現最佳估計值較期望值低。這是因爲在 Q 測度下，所模擬之一年期利率與兩年期利率非常接近，然而在經濟情境模型下，所模擬之一年期利率與兩年期利率差距較大，因此在 Q 測度下計算所得之期初負債 (準備金) 會較低。

表 5 期望值與最佳估計值

L_0	生死合險	利變型年金[32]	股票指數型年金
期望值（L_0^{P}）	652,546	636,913 (187)	924,657
最佳估計值（L_0^{Q}）	656,043	632,495 (23)	719,695

註：括號內之數字代表標準誤。[33]

4.2 準備金風險之量化

接下來我們在時間點 $t=0$ 衡量準備金之風險，並分別以 VaR(75)與

[30] 在 P 測度法下負債現值高達 924,657，可能原因爲股票指數之平均報酬率爲 6.212%，而折現率平均爲 1.32%。我們改以股票指數之平均報酬率 (6.212%) 爲折現率，則負債現值下降至 664,578。

[31] 依「利率變動型年金保險費率相關規範」(行政院金融監督管理委員會 2009 年 11 月 16 日金管保財字第 09802510721 號令修正發布)，利率變動型年金保險的責任準備金在累積期間以年金保單價值準備金全額提存。

[32] 本研究假設利變型年金之宣告利率爲兩年期利率 (實務上，台灣壽險公司所販售之利變型年金亦多以兩年期利率爲宣告利率)，因此，年金累積價值每年依照所公告之兩年期利率進行累積，準備金亦會隨著累積價值而增加。然而，保險公司進行準備金公平現值估計時，是以評價時點之整條殖利率曲線進行折現，而非單以兩年期利率爲折現率。因爲未來之殖利率曲線存在不確定性，故仍須估計利變型年金之準備金風險。

[33] 利變型年金係以蒙地卡羅法計算未來負債之期望值與最佳估計值，因此，本研究於表 5 中同時列出利變型年金期望值與最佳估計值計算過程之標準誤。

CTE(65)[34]衡量，並與躉繳純保費與期初準備金期望值 (最佳估計值) 相比較。表 6 至表 8 分別列出三張壽險保單在三種不同計算方式下之風險。注意，在 Q 測度法與 QPQ 方法下是先模擬 $t=1$ 之準備金分佈後再折現回 $t=0$，因此可以計算期望值。

表 6 爲生死合險之風險衡量值，在以 VaR(75)與 CTE(65)衡量準備金風險下，傳統 P 測度法之風險值最高 (8,710 與 12,921)，這是因爲傳統準備金方法是模擬保單有效期間內所有年度之現金流量與所對應之折現率，因此計算之風險並非單一年度風險值，而是全段保障期間內之風險。而 Q 測度法所得結果皆比 QPQ 方法低，根據歷史資料校準之結果，在 Q 測度法下 CIR 模型利率波動度爲 2.692%，而歷史利率資料波動度爲 9.373%，因此在 Q 測度法下模擬 $t=0$ 到 $t=1$ 之變化會較 QPQ 方法小，故風險也相對較小。另外，QPQ 方法顯示在生死合險保單下，保險公司在原先提列之準備金外，應額外計提 0.69~0.83%的比例，以因應利率變動對準備金之影響。

傳統上保險公司會以有效保單年度來調整在 P 測度法下計算之準備金風險，本研究以 $t=0$ 爲衡量時點，因此生死合險之有效保單年度爲 20 年。表 6 在 P 測度法下同時計算以 $\sqrt{T}$ 修正之風險衡量值，與 QPQ 方法相比，此方法仍然低估準備金之風險。

[34] 台灣對於投資保證的責任準備金風險衡量值標準爲 VaR(75)，而美國對於準備金之風險衡量標準爲 CTE(65)。

表 6 生死合險之風險衡量值

衡量方法	P 測度法	Q 測度法	QPQ 方法
$L_{VaR(75)}'$	8,710 (1,948)	2,195	4,526
$L_{VaR(75)}'/SP$	1.39% (0.31%)	0.35%	0.72%
$L_{VaR(75)}'/L_0$	1.33% (0.30%)	0.35%	**0.69%**
$L_{CTE(65)}'$	12,921 (2,889)	2,794	5,428
$L_{CTE(65)}'/SP$	2.06% (0.46%)	0.45%	0.86%
$L_{CTE(65)}'/L_0$	1.98% (0.43%)	0.43%	**0.83%**

註：括號內數字為以 $\sqrt{20}$ 修正後的風險衡量值。

表 7 為利變型年金之風險衡量結果，在以 VaR(75)與 CTE(65)衡量準備金風險下，傳統 P 測度法之風險最高 (4,123 與 5,681)，但同樣的問題，在傳統 P 測度法下所估算之風險並非一年度之風險。然而在 Q 測度法下，因準備金之分佈為右偏，[35]造成以 VaR(75)估計準備金風險時為負數之現象，這也再次說明以 Q 測度法估計之風險邊際並無法正確反映此類保單所承擔之風險。另外，在 QPQ

[35] 準備金期望值為 632,086，較表 5 之最佳估計值低。

方法下所估計之風險衡量值約爲躉繳保費的 0.2%，保險公司應額外計提 0.18~0.22%的比例以因應利率變動對準備金之影響。

表 7 利變型年金之風險衡量值

衡量方法	P 測度法	Q 測度法	QPQ 方法
$L_{VaR(75)}'$	4,123 (1,304)	-208[36]	1,151
$L_{VaR(75)}'/SP$	0.66% (0.21%)	0%	0.18%
$L_{VaR(75)}'/L_0$	0.65% (0.20%)	0%	**0.18%**
$L_{CTE(65)}'$	5,681 (1,796)	46	1,379
$L_{CTE(65)}'/SP$	0.90% (0.29%)	0.01%	0.22%
$L_{CTE(65)}'/L_0^{P}$	0.90% (0.28%)	0.01%	**0.22%**

註：括號內數字爲以 $\sqrt{10}$ 修正後的風險衡量值。

表 8 爲股票指數型年金之風險衡量結果，由於在 P 測度下，股票指數之報酬率爲 6.212%，因此其風險衡量值在 VaR(75)與 CTE(65)下分別高達 114,107 與 334,665，尤其是在 CTE(65)下，風險衡量值超過躉繳保費的 50%，顯示以在 P

[36] 站在 t=1 時，在 Q 測度下所模擬之一年期利率與兩年期利率間之差量較小，使得負債現值變小，甚至低於 t =0 所衡量之負債現值，因此造成 $L_{VaR(75)}'$ 爲負的情形。然而在 P 測度下模擬所得之一年期利率與兩年期利率間之差量較大，因此負債現值變高。

測度法下會有高估股票指數型年金保單之準備金風險現象。在 QPQ 方法下所估計之準備金風險約爲躉繳保費的 7%~11%，最佳估計值的 6%~9%，而 Q 測度法所衡量之風險僅爲 QPQ 方法的六成，顯示以 Q 測度法衡量準備金的風險會有低估準備金風險之情形。

表 8 股票指數型年金之風險衡量值

衡量方法	P 測度法	Q 測度法	QPQ 方法
$L_{VaR(75)}{}'$	114,107 (43,128)	26,093	45,196
$L_{VaR(75)}{}'/SP$	18.18% (6.87%)	4.16%	7.20%
$L_{VaR(75)}{}'/L_0$	12.34% (4.66%)	3.63%	**6.28%**
$L_{CTE(65)}{}'$	334,665 (126,491)	42,451	71,353
$L_{CTE(65)}{}'/SP$	53.32% (20.15%)	6.76%	11.37%
$L_{CTE(65)}{}'/L_0^{\mathrm{P}}$	36.19% (13.68%)	5.90%	**9.91%**

註：括號內數字爲以 $\sqrt{7}$ 修正後的風險衡量值。

5 結論與建議

壽險保單存續期間長，準備金龐大，因此準備金風險管理爲是壽險公司最重要之一環。文獻上多在 P 測度法下模擬準備金分佈，並衡量風險 (Tsai et al., 2003, 2009)。在 P 測度法下計算未來責任現值時，乃以評價點所估計之利率期限結構

進行折現，這種做法的主要問題爲高估負債金流之現值 (價格)。正確的方法爲折現率須加上風險溢酬，可是風險溢酬無客觀可行的方法來進行估計，因此，多數精算文獻在 P 測度法下計算未來責任現值時並未進行風險溢酬之調整。QIS 5 則提倡在 Q 測度下計算準備金之公平價值與風險邊際，換言之，在此方法下不改變折現率，但須調整測度，使得不確定現金流之期望值降低。然而在 Q 測度法下，是在風險中立環境下模擬市場未來可能的變化，無法描繪實際市場波動所造成之風險。

本研究提出一 QPQ 方法來衡量準備金之風險，此方法可解決只在一個測度下同時進行準備金評價以及衡量風險之問題與缺失。此方法的概念爲在時點t下，以 Q 測度法衡量準備金之最佳估計值，並在 P 測度下模擬一年內市場環境之變化，再利用最佳估計值 (在 Q 測度下) 求得在金融市場變化下，$t+1$下準備金之分佈。再以時點t的利率期限結構所相對應到期日之殖利率將$t+1$之準備金分佈折現至時點t，進一步應用風險衡量指標，如 VaR 或 CTE，來估計準備金之風險。QPQ 方法與 Gordy and Juneja (2010) 與 Bauer et al. (2012) 所建議之以巢狀模擬法計算衡量衍生性商品組合風險與清償資本額要求之概念相同。

模擬結果顯示，在 P 測度法下所求得之準備金期望值，以及在 Q 測度法下所求得之準備金最佳估計值差異甚大。這種做法的主要問題是使用了錯誤的折現率而高估負債金流的現值 (價格)。Babbel et al. (2002) 認爲在評估保險合約負債之現金流量時，應將風險及不確定性反映於現金流量與所對應之折現率的計算上。

因此折現率須加上風險溢酬。但正確地衡量風險溢酬在實務上執行困難，爲符合金融市場一致性的評價假設，因此 QIS 5 提倡保險公司需在 Q 測度下衡量準備金之最佳估計值。

在準備金風險衡量部分，傳統 P 測度法以模擬方法計算所得結果會過度衡量風險，因爲模擬過程所計算之風險並非單一年度風險值，而是全段保障期間內之風險。然而在 Q 測度法下可能無法正確衡量特定類型保單之準備金風險，例如利變型年金，甚至出現準備金風險爲負數之現象。數值結果發現 P 測度法與 Q 測度法所計算之準備金風險與 QPQ 方法有顯著差異。相較於 P 測度法與 Q 測度法，本研究建議保險公司以 QPQ 方法來估計之準備金之風險，以正確反映保單所承擔之風險。

QPQ 方法顯示在生死合險保單下，其保單預定利率爲 4%，然而在 Q 測度下所校準之利率模型長期平均利率水準爲 2.684%，因此相較於已計提之準備金 (最佳估計值)，保險公司應額外計提 0.69~0.83%的比例以因應利率變動對生死合險準備金之影響。在利變型年金部分，宣告利率爲兩年期定存利率，保險公司應額外計提 0.18~0.22%的比例以因應利率變動對準備金之影響。在股票指數型年金保單部分，在 P 測度下，股票指數報酬率服從 GARCH(1,1)時間序列模型，在 QPQ 方法下所估計之準備金風險約爲最佳估計值的 6%~9%。

由以上三張保單之分析結果，本研究發現準備金風險顯著。基於評價的理論以及風險的本質，本研究提出 QPQ 方法來估計準備金風險，保險公司應以正確

方式衡量準備金之風險，並計提相對應之風險資本，以避免因準備金不足而影響公司清償能力。

值得注意的是，本研究所提之 QPQ 方法爲衡量準備金風險之架構，且此架構可將選擇權概念納入其中，例如保險商品契約可能賦予保單持有人解約之權利(解約權)，因此，未來的研究可在此正確架構下 (QPQ 方法) 對保險契約中所隱含的各種選擇權進行正確的評價與風險衡量，並將解約率、死亡率不確定性與費用率結構等因素加入，以衡量上述因素對準備金風險之影響。

附錄 1：**Orthogonal GARCH 方法**

使用 Orthogonal GARCH 方法可以應用主成份分析法使風險因子正交，進一步從所有正交因子波動中產生原來風險因子的完整共變異矩陣。正交法的技巧在於將風險因子細分爲相關類別，再從細分的類別的每個主成份中產生單變量變異數預測，因爲主成份因子間並不相關，因此共變異矩陣中僅有對角線不爲零。再利用因子加權矩陣將對角線矩陣的主成份共變異矩陣轉換成完整之共變異矩陣。

(1) 因素分析 (Factor Analysis，FA)

因素分析是反映資料變數與因素 (或主成份) 間的關係，其認爲資料變數中存在無法觀察之潛在因素 (Latent Factor)，使得資料變數間存在高度相關。因此，觀察資料變數的相關程度，即可找出其潛在因素，進一步透過數學分析以少數變數來替代多變量結構。

假設可觀察之資料變數 $\mathbf{X}=(X_1\ X_2\ \dots\ X_p)'$ 與少數無法觀察得到的共同因素 (Common Factors) $\mathbf{F}=(F_1\ F_2\ \dots\ F_m)'$ 存在一線性關係：

$$\underset{(p\times 1)}{\mathbf{X}-\boldsymbol{\mu}} = \underset{(p\times m)}{\mathbf{L}}\ \underset{(m\times 1)}{\mathbf{F}} + \underset{(p\times 1)}{\boldsymbol{\varepsilon}}\ ,$$

其中，$\boldsymbol{\mu}=(\mu_1\ \mu_2\ \dots\ \mu_m)'$ 爲資料變數的期望值；$\mathbf{L}=\{l_{ij}\}$ 爲 $p\times m$ 矩陣，代表因素負荷量 (Factor Loadings)，表示潛在共同因素對資料變數的影響程度，相當於迴歸係數；$\boldsymbol{\varepsilon}=(\varepsilon_1\ \varepsilon_2\ \dots\ \varepsilon_m)'$ 爲誤差項，代表個別資料變數的獨特因素。另外，因素分析模型還需要加入下列假設：

(a) 共同因素之期望值爲 0，共變異矩陣爲一單位矩陣，即因素間彼此不相關；

(b) 誤差項之期望值爲 0，共變異爲一對角矩陣，表示誤差項間彼此不相關；

(c) 共同因素與誤差項間彼此不相關。

基於上述模型設定，可推導變數 X_i 與共同因素 F_j 之共變異即爲因素負荷量 l_{ij}，進一步推導 $\mathbf{X}$ 的共變異矩陣以及共同因素 F_j 之 GARCH 模型。

(2) 主成份分析 (Principal Component Analysis，PCA)

主成份分析是希望透過少數變數(即主成份)來解釋資料變數的大部分變異。假設有 p 個資料變數 $\mathbf{X}=(X_1\ X_2\ \ldots X_p)'$，其線性組合有 p 組，則主成份 $\mathbf{Y}=(Y_1\ Y_2\ \ldots\ Y_p)'$：

$$\mathbf{Y}=\mathbf{\Omega X},\ \ \mathbf{\Omega}=\begin{bmatrix} \mathrm{a}_1' \\ \vdots \\ \mathrm{a}_p' \end{bmatrix}=\begin{bmatrix} a_{11} & \ldots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{p1} & \ldots & a_{pp} \end{bmatrix},$$

其中，a_{ik}爲主成份Y_i對個別變數 X_k 的權重，因此主成份向量 $\mathbf{Y}$ 之變異與共變異爲：

$$Var(Y_i)=\mathbf{a}_i'\mathbf{\Sigma a}_i,\ \ i=1,2,\ldots,p.$$

$$Cov(Y_i,Y_k)=\mathbf{a}_i'\mathbf{\Sigma a}_k,\ \ i,\ k=1,2,\ldots,p.$$

所謂的主成份即爲 $\mathbf{Y}$ 中變異數極大化者，而且各主成份間須相互獨立：

$$\begin{aligned}&\underset{\mathbf{a}_i}{\text{Max}}\ \ Var(\mathbf{a}_i'\mathbf{X})\\ &\text{s.t.}\ \ \mathbf{a}_i'\mathbf{a}_i=1\quad \text{and}\quad Cov(\mathbf{a}_i'\mathbf{X},\mathbf{a}_k'\mathbf{X})=0\quad \text{for}\quad k<i.\end{aligned}$$

假設 $\mathbf{X}$ 之共變異矩陣 $\mathbf{\Sigma}$ 具有 p 組特徵值與特徵向量 $(\lambda_1,\mathbf{e}_1),(\lambda_2,\mathbf{e}_2),\ldots,(\lambda_p,\mathbf{e}_p)$，其中 $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_p$。則最適解 $\mathbf{a}_i$ 即爲 $\mathbf{\Sigma}$ 之特徵向量解 $\mathbf{e}_i$，而主成份之變異數 $Var(Y_i)$

則爲 $\mathbf{\Sigma}$ 之特徵值 λ_i。

透過主成份分析法找出的前幾個主成份能夠解釋資料變數約 80%或 90%以上的總變異數，因此可在不減少太多資訊下進行資料縮減，以這些變數代替原始的資料變數，來達到降低模型維度的目的。

附錄 2：AIC 與 BIC 統計資料

利率共同因子與股票指數報酬率	時間序列模型	AIC	BIC
dF_1	AR(1)	2.7940	2.8074
	AR(1,1)	2.8005	2.8273
dF_2	AR(1)	2.8362	2.8496
	AR(3)	2.8393	2.8527
dF_3	AR(1)+ARCH(1)	2.5443	2.5845
R_S	GARCH(1,1)	4.9905	5.0306
	GARCH(1,2)	4.9967	5.0181

附錄 3：利率期限結構

期間	1 年期	2 年期	3 年期	4 年期	5 年期
P 測度下	0.6265%	0.7844%	0.9208%	1.0403%	1.1416%
Q 測度下	0.5621%	0.6581%	0.7483%	0.8331%	0.9127%
期間	6 年期	7 年期	8 年期	9 年期	10 年期
P 測度下	1.2363%	1.3155%	1.3919%	1.4551%	1.5131%
Q 測度下	0.9876%	1.0581%	1.1243%	1.1867%	1.2455%
期間	11 年期	12 年期	13 年期	14 年期	15 年期
P 測度下	1.5647%	1.6112%	1.6517%	1.6891%	1.7192%
Q 測度下	1.3008%	1.3530%	1.4022%	1.4486%	1.4925%
期間	16 年期	17 年期	18 年期	19 年期	20 年期
P 測度下	1.7488%	1.7741%	1.7962%	1.8151%	1.8384%
Q 測度下	1.5338%	1.5730%	1.6100%	1.6450%	1.6781%

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On the way to estimate the risk of life insurance reserves*

Ming-Hua Hsieh, Associate Professor
Department of Risk Management and Insurance, National Chengchi University

Ya-Wen Hwang, Assistant Professor
Department of Risk Management and Insurance, Feng Chia University

Wei-Yu Kuo, Associate Professor
Department of International Business, National Chengchi University

Chenghsien Tsai, Professor
Department of Risk Management and Insurance, National Chengchi University

* Corresponding author, Yawen Hwang, Assistant Professor, Department of Risk Management and Insurance, Feng Chia University. 100, Wenhwa Rd., Seatwen, Taichung, Taiwan 40724, R.O.C. Tel: 04-24517250 Ext.4108. Email: ywhwang@fcu.edu.tw.
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Abstract

How to measure the risk of policy reserves is important for life insurers because policy reserves are the largest liabilities with long durations. In this paper, we propose the „QPQ“ method for determining the risk of policy reserves. We compare our approach with the traditional P-measure approach and Q-measure approach proposed by QIS 2 of Solvency II. Under the P-measure approach, the discount rate should be theothetically adjusted by risk premiums. However, it is difficult to determine the risk premiums of liabilities and thus most literatures did not consider the risk premium adjustment. Under the Q-measure approach, risk factors are simulated under the assumptions of risk nutrality. The movements of the risk factors however do not reflect the real movements of risk factors and thus can not reflect the possible real-world fluctuation of the reserves. The QPQ method can avoid the drawbacks of the above approaches.

Based on the QPQ method, life insurers use best estimate valuation to determine their reserves at time $t=0$ under Q measure. Then they generate stochastic future economic states (risk factors) from time 0 to time T under P-measure and apply the best estimate valuation to quantify their reserves at time $t=H$. For each scenario of the simulated stochastic future economic states, the reserve is again computed using best estimate valuation. The distribution of the reserve at time $t=H$ is then discounted back to time $t=0$ by the risk-free rate with maturity H. At the last step, commonly used risk measures (e.g., VaR and CTE) on the reserve distribution at time $t=0$ are used to quantify the risk margin of the reserves.

We apply the QPQ method to calculate the risk of reserves of the endowment policy, interest sensitive annuity, and and equity-indexed annuity. We find that there exist significantly differences between the QPQ approach and P-measure/Q-measure approaches. The risk of reserves is overestimated under P-measure. However, the risk margin under Q-measure is lower than that under the QPQ method and suggesting that the risk of reserves is underestimated under Q-measure. Since the adequacy of policy reserves is critical to the solvency of life insurers, we suggest life insurers adopt the QPQ method to estimate and manage the reserve risk.

Keywords: Risk Management, Life Insurance, Reserving

Work in Progress

Generating Economics Scenarios for the Long-Term Solvency Assessment of Life Insurance Companies: The Orthogonal ARMA-GARCH Approach

Weiyu Kuo
Department of International Business
Risk and Insurance Research Center
National Chengchi University

Ming-Hua Hsieh
Department of Risk Management and Insurance
Risk and Insurance Research Center
National Chengchi University

Chenghsien Tsai*
Department of Risk Management and Insurance
Risk and Insurance Research Center
National Chengchi University

Yu-Ching Li
Department of International Business
National Chengchi University

* Corresponding author. No.64, Sec.2,ZhiNan Rd., Wenshan District, Taipei City 11605, R.O.C (Taiwan); Tel: +886-2-2936-9647; Fax: +886-2-2939-3864; e-mail: ctsai@nccu.edu.tw.

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1. INTRODUCTION

Assessing the long-term solvency of a life insurer is important since the protections/promises offered by the insurer now are usually not realized until decades later. The protections will be invalid should the insurer be insolvent in the meantime. Insurance regulators and other stakeholders of life insurers thus have devised various ways to assess and maintain the solvencies of life insurers over a long run. For instance, regulators may ask insurers to establish adequate reserves to cover future liabilities and to maintain adequate capital to absorb unexpected losses.

How to assess the adequacies of reserves and capital are not easy tasks, however. The adequacies depend not only on the investment and business strategies of the insurer but also on exogenous economic conditions. For example, the low interest rate era that has persisted over the past decade in several countries threatens the solvencies of many insurers that had sold products with high pricing rates set according to the high interest rates prevailing during 1990s. Low stock returns from the end of 1990s to the beginning of 2010s aggravated the distress on the insurers' solvencies.

The actuarial professions and insurance supervisors therefore devoted resources to establishing the models that could generate possible economic scenarios of the returns on major asset classes for solvency assessment. We might trace back the modeling development to the Maturity Guarantees Working Party (1980) and the subsequent works of Wilkie (1986a; 1986b; 1987; 1992; 1995). Similar modeling was applied to the economic series / investment returns of other countries including Australia (Carter, 1991; Hua, 1994), Switzerland (Metz and Ort, 1993), and South Africa (Thomson, 1994). Starting from 1999, the American Academy of Actuaries (AAA) established three-phase economic scenario generation (ESG) models for reserve adequacy tests and the interest rate risk component (C-3 component) of the Risk-Based Capital (RBC) requirements. The Casualty Actuarial Society together with the Society of Actuaries (SOA) also commissioned an ESG project (Kevin, D'Arcy, and Gorvett, 2004) meanwhile. As the uses of ESG models became popular, private-sector companies such as Barrie and Hibbert joined the modeling development. The insurance supervisors in both North America and Europe now encourage insurers to develop their own ESG models (O'Brien, 2009).

The key issues in establishing a comprehensive ESG models include: how to deal with the large number of risk factors, how to model the dynamics of some chosen factors, and how to incorporate the relations among risk factors. Tackling the first issue essentially calls for reducing the modeling dimension, i.e., reducing the number of risk factors to be modeled. The significance of this first issue increases with the number of economic series and countries to be covered in the ESG models. With regard to the second issue, the dynamics of the chosen factors should reflect observed time-series characteristics of return volatilities (e.g., volatility clustering) as well as change patterns (e.g., autoregression). The choice of econometrics methods hinges on the number of factors to be modeled. Retaining more factors usually leads to simpler methods. The third issue, the relations among risk factors, may be coped with correlations and/or explicit functional relations. The choice depends on the model developers' views about whether the relations are from correlated random shocks or subject to common factors.

For instance, the phase-I models for the C-3 component of US RBC intended to cover the Treasury yields with 10 maturities ranging from 3 months to 30 years. To reduce the number of risk factors to be modeled, the in-charged task force assumed that the treasury curve was driven by two key rates: a long-term interest rate and the excess of a short-term rate over the long rate. The changes of these rates and the changes in the variance of the long rate[1] were then modeled to take into account mean reversion and stochastic variance.[2]

[1] The variance of the spread was assumed to be constant.

The two key rates were endogenous to each other with additional correlated random shocks. Interpolation formulas were imposed to recover the yield curve in the last stage.[3]

The C-3 phase-II models extended to cover 9 asset classes. To reduce modeling dimensions, the work group assumed that risks were driven by four stock index returns and three bond index returns. The volatilities and drifts of individual stock index returns were modeled by stochastic log volatility models. The bond index returns were assumed to be functions of Treasury yields[4] with stochastic deviations.[5,6] The stock returns and bond returns were subject to correlated random shocks.[7]

Ahlgrim, D'Arcy, and Gorvett (2004) covered the term structures of inflations and real interest rates, two stock index returns, dividend yields, real estate returns, and unemployment rates in US markets. To reduce the number of risk factors involved in the term structures, they assumed that the term structure of inflation rates followed the one-factor Vasicek (1977) model while that of interest rates followed the two-factor Vasicek model.[8] Dividend yields and real estate returns were modeled as the first-order autoregressive (AR) processes. An AR(1) process was also applied to unemployment rates with an additional term to consider the impact from inflation rates. The authors applied regime-switching models to the excess returns of stock indexes.[9] They modeled the relations among generated economic series mostly by correlated random shocks but some by functions (e.g., nominal interest rates were

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2 The number of dynamic models is therefore three.

3 Interested readers may refer to AAA's October 1999 report, Phase I Report of the American Academy of Actuaries' C-3 Subgroup of the Life Risk Based Capital Task Force to the National Association of Insurance Commissioners' Risk Based Capital Work Group.

4 The models for Treasury yields in Phase II are the same as those in Phase I.

5 The deviations were generated using normal distributions with constant standard deviations.

6 The number of dynamic models is eleven: three for bond index returns and four pairs for the four stock index return models.

7 Interested readers may refer to AAA's January 2006 report, Construction and Use of Pre-Packaged Scenarios to Support the Determination of Regulatory Risk-Based Capital Requirements for Variable Annuities and Similar Products. A joint group of AAA and SOA has been refining the interest rate and stock return models as well as updating the model parameters since then without major modeling changes.

8 The volatility terms of these risk factors were thus constant. So were other time series except the returns of stock indexes.

9 The number of dynamic models is thus eight.

functions of inflations and real interest rates).

Wilkie (1995) covered exchange rates and other economic series similar to those covered by Ahlgrim, D'Arcy, and Gorvett (2004) under a cascade framework. His fundamental variable was inflation that was modeled by an AR process and autoregressive conditional heteroscedasticity (ARCH) process. He then analyzed the univariate property of wages and further investigated the relation between wages and inflation by cointegration and vector autoregression (VAR). Other economic series were modeled similarly: univariate AR-ARCH and/or cointegration-VAR with other series, but subject to certain cascade relations.[10] He further applied univariate AR-ARCH modeling to several economic series of other countries. The relations of several economic series[11] across countries were considered using correlated residuals. Wilkie (1995) did not conduct dimension reduction when modeling because he did not consider the entire yield curve that usually involves about a dozen concerned risk factors.[12] Neither did he examine the relations of the economic series within the countries other than UK.[13]

We propose a simple but comprehensive and flexible modeling approach, called orthogonal ARMA-GARCH (autoregressive moving average – generalized autoregressive conditional heteroscedasticity) modeling in this paper, to generate large-scale economic scenarios. Many insurers are exposed to risk factors that easily reach seventy, eighty, or even a hundred. One yield curve may contain 10 or more risk factors that have significant impacts on the values of the bonds held by insurers as US RBC identified. For the insurer that hold not only treasury bonds but also corporate bonds rated as AAA, AA, A, and BBB classes, the risk factors reflecting the uncertainties about risk-free rates and credit risk spreads

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[10] For instance, Wilkie (1995) assumed that the long-term interest rate was determined by real interest rate following an AR process, inflation rate, and dividend yield that was also affected by inflation rate.

[11] They included: inflation rates, dividend yields, dividend indexes, and exchange rates.

[12] The number of UK's economic series considered in Wilkie (1995) is about ten.

[13] Indeed, the economic series such as interest rates, stock indexes, property indexes in other countries were not analyzed.

can reach fifty. The number of risk factors underlying a stock index ranges from one (when believing in a single-factor model such as the capital asset pricing model (CAPM)), three to five (Fama-French models), dozens (when treating each industry as a risk factor), to hundreds (adopted by many historical simulation methods used to calculate the value at risk (VaR) for the insurer). For the insurers having significant international investments, the number of risk factors multiplies. The ESG models that can adequately capture the risk characteristics of insurers' investments thus have to cover dozens or even hundreds of risk factors. Without an effective way to reduce the modeling dimension, model building is infeasible.

To reduce the dimension, we propose to apply factor analysis to asset class. Factor analysis addresses the problem of analyzing the structure of the relations/correlations among a large number of variables by using a much smaller number of factors/dimensions. For instance, we may apply factor analysis to condense the information contained in a yield curve into three factors. Adopting factor analysis renders three important advantages. The first crucial benefit is that the retrieved common factors can be orthogonal to each other, which enables us to bypass the obstacles in modeling the dynamics of these factors under the multi-variate framework. Secondly, adopting factor analysis allows us to model the relations among the risk factors within an asset class by common factors. This is new to the ESG related literature and makes more economic sense than using correlated random shocks. Thirdly, factor analysis renders fitness statistics (especially the percentage of variance explained). The methods/assumptions used in other papers (e.g., assuming yield curves are driven by two key rates) provides no such statistics to assess modeling risk.

After retrieving the orthogonal common factors, we model the dynamics of these factors by ARMA-GARCH processes. We are therefore afforded great flexibilities in establishing the time-series models of individual factors. We use the ARMA processes to model the dynamics of the mean/drift terms and the GARCH processes for the volatility terms, respectively. The ARMA and GARCH processes are popular in the literature and practice: adequate in fitting and forecasting, robust in estimating parameters, and easy to use. Another rationale for using GARCH is to capture the fat tails that have been identified in many papers for many financial market series, which is important for ESG models to sufficiently reflect the larger-than-normal risks through the simulated scenarios.

Next, we construct the covariance matrix of the common factors to incorporate the correlations across asset classes. We apply Cholesky decomposition to the matrix and multiply the decomposed triangle matrix to the independent random numbers generated for the ARMA-GARCH processes. In the last stage we utilize the factor loadings to recover from the simulated common factors to dozens of the initial concerned risk factors.

By combining factor analysis with ARMA-GARCH, we are capable of constructing ESG models adequately capturing the risk characteristics of numerous risk factors as demanded by life insurers as well as associated stakeholders. Our idea, albeit seemingly simple, is new to this line of literatures and has potential. It can further incorporate the risk factors of insurance liabilities and facilitate the calculation of economic capital in a unified framework.

2. THE ORTHOGONAL ARMA-GARCH APPROACH

The idea of using factor models with GARCH has been around for over two decades. For instance, Engle, Ng, and Rothschild (1990) proposed a CAPM-based framework in which the volatilities and correlations between individual asset returns were generated using the univariate GARCH variance of market returns. This is in essence a one-factor model that reduces modeling dimension from dozens to one. To tackle the difficulties in multi-variate modeling, Ding (1994) suggested the use of PCA with GARCH models. He however did not address the dimensionality issue since he retained all retrieved factors. It was Alexander and Chibumba (1996) and Alexander (2000, 2001, 2002) that advocated retaining only a few components to reduce the number of to-be-modeled risk factors and enhance the practicability. They fit GARCH (1, 1) models to all retained components.

We generalize Alexander's modeling to establish an ESG model covering distinct asset classes. The generalization is in two aspects in addition to extending to more asset classes. The ESG-generated scenarios are usually used for long-term concerns and thus should consider conditional means in addition to conditional volatilities. Secondly, fitting the dynamics of the components representing different economic series with general ARMA (p, q) - GARCH (m, n) models is more appropriate than imposing universal GARCH (1, 1) models.

To establish an orthogonal ARMA-GARCH (shortening as O-GARCH starting from here) model, we first conduct factor analysis / PCA on the risk factors of an asset class[14] to extract principal components which can represent the original set of risk factors with a minimum loss of information. Since PCA renders orthogonal components, we may apply univariate ARMA-GARCH models to them to capture the characteristics of individual components' means and volatilities. The relations among asset classes are then incorporated by the correlation matrix of all components. Our O-GARCH modeling is therefore

[14] Defining an appropriate asset class for the purpose of conducting factor analysis involves subtle considerations. Asset classes are usually defined by distinct risk types such as stock return risks (price changes and dividend yields), interest rate risk (bonds), credit risk (corporate bonds), foreign exchange rate risk, and real estate return risks (price changes and rental yields). They may also be defined by geographic areas. Researchers have to examine the characteristics of samples to determine appropriate classifications.

computationally efficient (by using factor analysis), econometrically appropriate (in identifying underlying driving factors and providing fitness statistics as well as by using general time-series models to reflect changing means and clustering volatilities), and economically sound (by using common factors in addition to random shocks to capture the relations among risk factors).

2.1 Factor Analysis

Factor analysis postulates that each observed variable (i.e., risk factors in this paper) is linearly dependent upon one or more common factors and one specific factor. Common factors are unobservable variables which influence more than one risk factor; specific factors are latent idiosyncratic variables that influence only one risk factor. Since our purpose is finding the minimum number of common factors needed to account for the maximum portion of the variance resulted from the original set of variables, we adopt the principal component analysis (PCA) method to obtain factor solutions.[15] PCA defines common factors / principal components as the linear combinations of original risk factors. Conversely, risk factors are also linear combinations of principal components. This means that principal components can account for total variances or a portion of total variances when some components are dropped off. Another advantage of using PCA is that PCA neither requires the distribution assumption about the data nor has to determine the number of common factors in advance

[15] The most widely used methods to estimate parameters are maximum likelihood and principal component (Johnson and Wichern, 2007).

(Tsay, 2005). Thirdly, the extracted principal components are orthogonal to each other.

Let $\mathbf{X}_t$ be the vector of k observed variables associated with an asset class at time t ($t = 1, 2, \ldots T$) with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. $\mathbf{X}_t$ are further assumed to be linearly dependent on m common factors $\mathbf{f}_t$ and k specific factors $\boldsymbol{\varepsilon}_t$, where $m < k$. More specifically,

$$\mathbf{X}_t - \boldsymbol{\mu} = \mathbf{L}\mathbf{f}_t + \boldsymbol{\varepsilon}_t, \tag{1}$$

where $\mathbf{L}$ is the $(k \times m)$ matrix of factor loadings.

The underlying assumptions of factor analysis are: $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$, $E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \mathbf{D}$, $E(\mathbf{f}_t) = \mathbf{0}$, $E(\mathbf{f}_t\mathbf{f}_t') = \mathbf{I}_m$, and $E(\mathbf{f}_t\boldsymbol{\varepsilon}_t') = \mathbf{0}$ where $\mathbf{I}_m$ is a $(m \times m)$ identity matrix and $\mathbf{D}$ is a diagonal matrix. Consequently, the $(k \times k)$ covariance matrix $\boldsymbol{\Sigma}$ of the observed variables can be expressed as

$$\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}' + \mathbf{D}. \tag{2}$$

We thus may regard $\mathbf{L}\mathbf{L}'$ as an approximation of the original covariance matrix, i.e.,

$$\boldsymbol{\Sigma} \approx \mathbf{L}\mathbf{L}'.$$

2.1.1 <u>Extracting factors by PCA</u>

Let $(\hat{\lambda}_1, \hat{\mathbf{e}}_1), \ldots, (\hat{\lambda}_k, \hat{\mathbf{e}}_k)$ with $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_k$ be pairs of the eigenvalues and eigenvectors of the sample covariance matrix $\hat{\boldsymbol{\Sigma}}$. $\hat{\boldsymbol{\Sigma}}$ can be decomposed by the spectral decomposition as:

$$\hat{\boldsymbol{\Sigma}} = \sum_{i=1}^{k} \hat{\lambda}_i \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i'. \tag{3}$$

Sorting common factors by eigenvalues and retaining the first m factors, we may express the matrix of estimated factor loadings by:

$$\hat{\mathbf{L}} = \left[\sqrt{\hat{\lambda}_1}\hat{\mathbf{e}}_1 \middle| \sqrt{\hat{\lambda}_2}\hat{\mathbf{e}}_2 \middle| \dots \middle| \sqrt{\hat{\lambda}_m}\hat{\mathbf{e}}_m \right].$$

The estimated specific variances are the diagonal elements of the matrix $\hat{\mathbf{\Sigma}} - \hat{\mathbf{L}}\hat{\mathbf{L}}'$. That is, $\hat{\mathbf{D}} = diag\{\hat{\sigma}_1^2, \dots, \hat{\sigma}_k^2\}$, where $\hat{\sigma}_i^2 = \hat{\sigma}_{ii}^2 - \sum_{j=1}^{m} \hat{l}_{ij}^2$, where $\hat{\sigma}_{ii}^2$ is the (i,i) th element of $\hat{\mathbf{\Sigma}}$.

$\hat{\lambda}_j$ means the contribution to the total sample variance from the *j*th factor. Since $\left(\sqrt{\hat{\lambda}_j}\hat{\mathbf{e}}_j\right)'\left(\sqrt{\hat{\lambda}_j}\hat{\mathbf{e}}_j\right) = \hat{\lambda}_j$ and the eigenvector $\hat{\mathbf{e}}_j$ has unity length, the proportion of the total sample variance $\sum_{i=1}^{k} \hat{\sigma}_{ii}^2$ explained by the *j*th factor is $\hat{\lambda}_j \Big/ \sum_{i=1}^{k} \hat{\sigma}_{ii}^2$. When one applies the spectral decomposition to normalized $\mathbf{X}_t$, the proportion of the total sample variance explained by the *j*th factor becomes $\hat{\lambda}_j / k$. We choose to analyze normalized $\mathbf{X}_t$ in the following to prevent the estimates of factor loadings from being influenced by the variables with large variances (Johnson and Wichern, 2007).

2.1.2 Selecting factors

A critical decision to be made in factor analysis is to determine how many common factors to be retained. It involves the tradeoff between model parsimony and model plausibility (Fabrigar et al., 1999). A well-known criterion is retaining the common factors with eigenvalues greater than 1. Another informal but useful guidance is to examine the scree plot which plots the eigenvalues in descending order. By looking for an “elbow” where the last substantial drop in the magnitude of the eigenvalues happens, the researcher retains the factors prior to this last substantial drop (see e.g. Fabrigar et al., 1999; Tsay, 2005; Johnson and Wichern, 2007). A bottom line for many researchers in determining the number of factors to be retained is the cumulative proportion of the total sample variance explained by the retained factors. Since ESG models should capture most variations in past series so that the generated scenarios have high probabilities to cover the to-be-realized one, a high threshold like 95% is desirable.

2.1.3 Rotating Factors

The factor loadings resulting from extracting factors represent the relations between the common factors and the risk factors. The higher the loadings are, the more representative the risk factors are on common factors. In most cases, however, the factor loadings do not provide obvious or meaningful interpretations of the relations. Factor rotation is to redistribute the variance among factors to achieve a simpler, more meaningful factor pattern.

Among several rotation approaches, we choose the popular VARIMAX orthogonal rotation (Kaiser, 1958). The first advantage of this approach is that it keeps the common factors orthogonal to each other. Secondly, it usually produces a simpler factor loading structure in which the associations between risk factors and common factors are easier to interpret than others. Another advantage is that the produced loading structure tends to be more invariant when different subsets of variables are analyzed. The popularity of VARIMAX is evident since most computer packages with factor analysis are equipped with this approach.

2.1.4 Calculating factor scores

A factor score represent a composite of all variables' loadings on the factor (Hair et al., 2010). Given any vector of observations $\mathbf{x}_t$, the tth factor score vector is given by

$$\hat{\mathbf{f}}_t = \hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\Sigma}}^{-1}(\mathbf{x}_t - \overline{\mathbf{x}}), \quad t = 1, 2, \ldots T .$$

The resulted factor scores can then be used to represent the factors in subsequent analyses.

Dimension reduction is majorly accomplished by modeling upon factor scores instead of original risk factors. The number of models to be built is reduced from k to m. More importantly, the common factors are orthogonal to each other so that we may proceed with the modeling in a uni-variate setup for each factor individually rather than under the multi-variate framework.

2.2 ARMA-GARCH Modeling

At the second stage of the O-GARCH method, the time series of the obtained orthogonal factor scores from an asset class are modeled individually. We consider the general univariate GARCH models with lagged variables in the mean equation. In other words, models like AR, ARCH or GARCH are candidates.

For a F_t, a general AR(p)-GARCH(m, s) model has the form of

$$F_t = c + \sum_{i=1}^{p} \phi_i F_{t-i} + \varepsilon_t \text{, where } \varepsilon_t \mid \Omega_{t-1} \sim N(0, \sigma_t^2),$$

and

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{m} \alpha_j \varepsilon_{t-j}^2 + \sum \ \beta \sigma_{t-1}^2, \text{ where } \omega > 0, \alpha, \beta \geq 0 .$$

This general modeling captures the possible presence of serial correlations and conditional heteroscedasticity. For the sake of stationarity, the coefficients of the lagged factors in the mean equation must sum to less than 1, and so must be the coefficients of the lagged errors and lagged conditional variances in the variance equation. We employ the maximum likelihood estimation in general; the least squares method is used only when conditional heteroscedasticity is absent.

The estimation procedure is as follows. Firstly, we test for serial correlation and use the partial autocorrelation function (PACF) determine the order of AR terms if needed. Then we test for conditional heteroscedasticity on the residuals of the mean equation. If the conditional heteroscedasticity is found, we use PACF on the squared residuals to determine the order of the variance equation. Thirdly, we use the Ljung-Box statistics to check the specification suitability. When there are several models passing the Ljung-Box test, we use the Bayesian information criterion (BIC) to select the "optimal" model. The Akaike information criterion (AIC) is used as an auxiliary.

The resulted O-GARCH models enable us to simulate risk factor (e.g., stock returns and interest rates) scenarios for the corresponding asset class. These scenarios will display essential risk properties of assets such as auto-correlations and volatility clustering. The scenarios will also reflect the relations among the risk factors within an asset class that are modeled by the retrieved common factors from that asset class. To further reflect the relations among the risk factors across asset classes,

2.3 The Covariance Matrix across Different Groups of Risk Factors

The last stage of the O-GARCH modeling involves assembling As illustrated in Alexander (2002), consider there are only two groups of m and n risk factors such as interest rates and stock returns. Let $\mathbf{P} = (P_1, P_2, \ldots P_r)$ and $\mathbf{Q} = (Q_1, Q_2, \ldots Q_s)$ are common factors extracted from group 1 and 2 separately where r and s are the number of common factors. Denote by $\mathbf{A}$ $(m \times r)$ and $\mathbf{B}$ $(n \times s)$ the matrices of factor loadings of group 1 and 2 respectively. The full-dimensional covariance matrix of the original system is given by

$$\begin{pmatrix} \mathbf{AA}' & \mathbf{ACB}' \\ (\mathbf{ACB}')' & \mathbf{BB}' \end{pmatrix}.$$

where $\mathbf{AA}'$ and $\mathbf{BB}'$ are the within-group covariance matrices of group 1 and 2 respectively and $\mathbf{ACB}'$ is the cross-group covariance matrix among group 1 and 2 in which $\mathbf{C} = \mathrm{cov}(P_r, Q_s)$ with $(r \times s)$ dimension can be estimated using O-GARCH again, now on a system of the $r+s$ common factors $P_1, P_2, \ldots P_r, Q_1, Q_2, \ldots Q_s$. Accounts need to be taken of the positive semi-definiteness of the estimated full-dimensional covariance matrix. Although $\mathbf{AA}'$ and $\mathbf{BB}'$ will always be positive semi-definite, it does not always guarantee to obtain a positive semi-definite $\mathbf{ACB}'$ (Alexander, 2008).

3. IMPLEMENTATION OF THE ORTHOGONAL GARCH APPROACH

3.1 Data Description

Four types of risk factors we examine include interest rates, stock indices, exchange rates, and real estate. They are commonly regarded as major market risks that insurance companies need to take account of and have been widely modeled in many economic scenario models, like CAS-SOA and AAA. However, previous studies haven't analyzed them altogether from an international perspective. In consideration of this, we try to analyze the risk factors of interest rates, stock indices, and exchange rates in a multinational setting with the view of insurers in Taiwan. In contrast, real estate risks are confined to Taiwanese domestic market due to regionality of real estate investing and the relatively lack of quality international real estate data.

All data except for those of real estate are obtained from Bloomberg. It is well known that time series like interest rates, exchange rates, or asset prices tend to be nonstationary. To make sure stationarity, we employ the first differenced series of our data. Based on data characteristics, we consider the change series of interest rates and the log returns of stock indices, exchange rates, and real estate.

For grouping risk factors properly, things to consider include types of investment instruments and geographic locations. In addition, a requirement of the factor model presented previously that the number of risk factors (k) should be smaller than the sample time periods (T) within each group needs to obey. Simulation performance is our concern as well. It's more possible to obtain easy to poor simulation performance if relatively low correlated risk factors are grouped together. Given these considerations, it turns out that grouping happens not only among different risk types but also within identical type.

In what follows, the variables we use as well as how we group them are presented. We also collect all relevant data information in Table 1. Note that the sample spans of each group are not probably the same. The idea that each group is treated independently in the context of O-GARCH through orthogonalization allows us not to trim data observations in the beginning of modeling until the correlation matrix of estimated factor scores is calculated.

[Insert Table 1 here]

Interest rates

Monthly zero coupon yields for the US dollar (USD), the Euro (EUR), the Australian dollar (AUD), the New Zealand dollar (NZD), the Canadian dollar (CAD), and the New Taiwan dollar (TWD) with 30 maturities between 1 year and 30 years are available. There are 180 interest rates totally considered apparently large than the sample observations we can obtain. According to geographic location and the degree of correlation, we divide interest rates into four groups. Among them, USD and CAD become the US-CA group which represents the interest rates of American areas. Similarly, the AU-NZ group formed by AUD and NZD represents the Oceania interest rates. EUR and TWD group by themselves.

Stock indices

We consider four market indices including the Dow Jones, S&P 500, Nasdaq, and

Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX), which all use month-end closing price adjusted for dividend, in terms of local currency. At first, only four equity risk factors don't raise any concerns about grouping. However, poor performance in simulation later forces us to group them. When doing simulation, we figured out that the performance of the four-index group was not better than that of groups of three-US-index as well as TAIEX. It seems sensible because the O-GARCH model performs better in a highly-correlated system. Any US stock index has lower correlation with TAIEX than other two US indices. Therefore, all four stock indices are divided into two groups – one contains three US indices and the other is TAIEX alone. Note that group containing only one risk factor, like TAIEX, does not need to perform the orthogonal factor model.

Exchange rates

The data include five monthly exchange rates, the euro, Canadian dollar, Australian dollar, New Zealand dollar, and New Taiwan dollar against the US dollar. Rather group exchange rates, we analyze them as a whole.

Real estate

We focus our attention on domestic housing prices and rents. Moreover, because of their different income characteristics, we regard them as two distinct groups. For house pricing risks, we analyze quarterly house price indices, constructed by Cathay Real Estate, for new and pre-sold houses including a Taiwan national composite index and five geographic regional indices for Taipei City, Taipei County, Taoyuan-Hsinchu, Taichung, and Tainan-Kaohsiung areas. For rent risk, taking data availability and reliability into account, we only study the quarterly average rents for Grade-A office in Taipei City which is obtained from REPro International Inc. again, since the rent risk group only contain one series, we don't perform the orthogonal factor model on it.

3.2 Descriptive Statistics

Table 2 provides some descriptive statistics for USD and CAD yields and corresponding changes at representative maturities, stock and exchange rate returns. For USD and CAD yields, the long term (more than 10 years) means are higher than short-term means, implying that both term structures of USD and CAD yields are upward sloping; the standard deviations are negatively related with maturities. The values of skewness and excess kurtosis show that both USD and CAD yield distributions tend to be near normal.

For changes in USD and CAD yields, the sample averages are negative and the standard deviations are also nearly positively proportional to maturities. The ADF test results show that yields at most maturities exhibit nonstationary but changes in yields stationary. Therefore, we examine changes in yields instead of yields for interest rate factors in the following analysis.

For stock returns, TAIEX seems to have different characteristics from three US indices. These three US stock returns of all roughly 6% are on average higher than the TAIEX return.

The standard deviations, however, of USs are smaller than TAIEX return's. This phenomenon stands up for the partition of stock returns we consider. With negative skewness and positive excess kurtosis, all four stock returns consistently tend to be left-skewed and have heavy tails.

For exchange rate returns, the sample means with relatively large standard deviations seems near zero. The skewness and kurtosis measures show that exchange rate returns are likely to be right-skewed and have heavy tails.

Whether the sample means of the risk factors considered are significantly different from zero is relevant to the factor model specification as shown in eq. (1). If the sample means of a group of risk factors are not significantly different from zero, we won't consider the mean vector $\boldsymbol{\mu}$ in eq. (1). Via hypothesis testing, we are convinced that all groups of risk factors have means significantly different from zero except for the exchange rate returns.[16] Therefore, we consider the mean vector $\boldsymbol{\mu}$ in eq. (1) for every risk factor groups except the exchange rate returns when estimating the orthogonal factor model.

[Insert Table 2 here]

3.3 Factor Analysis Results

In what follows, we focus on the sample correlation matrix in our empirical analysis. To conserve space, we only present the results of the US-CA yield change group and the FX group.

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[16] We didn't present the *t* statistics of significant tests here.

Table 3 gives the results of the eigenvalue analysis. First consider the US-CA group, a case where the orthogonal factor model is performed in a multi-country setting. At first glance, it seems reasonable that are three factors are adequate for the US-CA group since the first three eigenvalues are only eigenvalues greater than unity. However, we set another requirement for better simulation performance that the total (standardized) sample variance explained needs to be higher than 95%. Therefore, we determine that the first four common factors are needed, accounting for 96.13% of the total sample variance in the system of US-CA yield changes.

Turning now to the FX group, only the first eigenvalue is greater than one and explains just 67% of the total sample variance, which is far lower than the standard we set to have the total sample variance explained. Obeying this rule, we decide that a four-factor model which explains 97.5% of the total sample variance provides a better fit to the system of FX.

[Insert Table 3 here]

To figure out the interpretation of common factors extracted, we illustrate the rotated estimated factor loadings of the US-CA and FX groups in Figure 1 and 2, respectively. For the US-CA group, the first two factors might tend to be country-specific factors. Most maturities of USD yield changes have large loadings on factor 1, and CAD yield changes on factor 2. Factor 3 might be labeled as a short-term factor on which short-term maturities of USD and CAD yield changes have higher loadings. Factor 4 is more meaningful for USD than for CAD, especially for USD short-term maturities below 2 or 3 years. Overall, factor 1 and 4, together, might explain the USD yield changes, and factor 2 and 3 represent the CADs. For the FX group, it's more obvious that four common factors extracted might be identified as country factors. The AUD and NZD yields load highly on factor 1, and CAD, TWD, and EUR on factor 2, 3, and 4, respectively.

[Insert Figure 1 and 2 here]

3.4 Time-Series Models of Factors

Table 4 reports the estimation results of time-series models for factor scores obtained from the US-CA and FX groups. For the US-CA group, factors 1 and 2 are shown to be serial correlated and/or conditional heteroscedastic. Therefore, we model them with AR(1)-ARCH(1) and AR(4)-ARCH(6) models, respectively. Factor 3 is shown to be close to white noise, so any time-series models are needed to model it. We model factor 4 with a simple AR(2) model due to the evidence of some serial correlation. For the FX group, factor 2 and 4 are shown to be white noise as well. We employ an AR(3) model for factor 1 since it presents slight serial correlation. In the presence of serial correlation conditional heteroscedasticity, factor 3 is modeled by an AR(1)-ARCH(1) model.. With the evidence of, factor 1 follows

[Insert Table 4 here]

3.5 Correlation Matrix across Factor Groups

We calculate the correlation matrix of the common factors across groups as the initial values for simulation. The results are given in Table 5. For not losing any useful information, we calculate the correlation of any pair of factors once at a time, not of all common factors at the same time. As such, sample sizes of each correlation coefficient in the matrix are not the same. Note that for groups with one series like the TAIEX group and the house rent group the original time series are directly used to calculate the correlations.

[Insert Table 5 here]

4. SIMULATION RESULTS OF THE MODELS

With the estimated time-series parameters which capture each common factor's dynamic behaviors, the estimated factor loadings which represent the relationship between common factors and risk factors, and the correlation matrix among common factors, we then use the Monte Carlo simulation approach to simulate random movements in risk factor and generate a range of scenarios for a long time.

4.1 Simulation Process

First of all, we generate random numbers of common factors and specific factors respectively because of the orthogonality implied by the factor model used. The specific factors follow $\boldsymbol{\varepsilon}_t \sim (\mathbf{0}, \mathbf{D})$. For common factors, we want them to keep the characteristics not only of dynamics guarded by the estimated time-series models but also of the correlation among each other.

We first multiply a sequence of *IID N(0,1)* random variables by a lower triangular matrix which is obtained via the well-known Cholesky decomposition on the calculated correlation matrix of factor scores. And it ensures that the draws from this process keep the correlation we want. Then the estimated time-series model is applied to generate a sequence of random variables over time as common factors.

Once the random variables of common and specific factors are produced, we can generate the first differenced series of risk factors, and then the risk factor levels, with the factor model equation. The initial values used are the last observations of each risk factor. And we decide to generate 1000 scenarios over the next 30 years to see the long term risks facing the insurance companies.

For interest rate risk factors, we further make some restrictions and adjustments to avoid negative values of interest rate levels and unreasonable shapes of simulated yield curves. First, we limit minimum and maximum values to be 0.1 % and 25%. Second, we use the first-order autoregressive process, AR (1), in discrete time to capture the mean reversion of interest rate levels, a fundamental property that short-term rates tend to revert to a long-term value:

$$r_t = r_{t-1} + \gamma\left(r_{t-1} - \bar{r}\right) + \varepsilon_t ,$$

where r_t is the short-term rate at time *t*, $\bar{r}$ is the long term mean of r_t, and ε_t is a stochastic white noise component. The coefficient γ is the rate of mean reversion which measures the speed of adjustment, and we expect it to be negative in this model. With this process, when the rate in the previous period deviates positively (negatively) from its long-term level, the change in interest rate in the next period should be negative (positive), pushing the interest rate toward $\bar{r}$. For estimation, we first extend the interest rate sample period as far as we can obtain, and then the shortest-maturity rates are modeled as r_t and are also used to calculate the long term mean $\bar{r}$. Table 6 lists the sample information and the estimation results. As expected, the estimated values of γ for every yield are all negative.

< Insert Table 6 >

4.2 Simulation Results

For accuracy check of simulation results, we compare properties of simulated data to their historical counterparts. To save space, we only present the results of the USD, CAD yields and exchange rate returns.

The long-run mean and some percentiles of USD yields are shown in Table 7, and Table 8 is for CAD yields. As we can see, the long-run means of simulated USD and CAD yields both approximate to ones of their historical data, and so do percentiles. Extreme values, however, tend to be divergent. The maximums of simulated USD yields, for example, have more and more variation from historical values as their maturities become larger.

Adding some restrictions on extreme values of simulated interest rates when simulating makes the results distorted. Since yield changes are our objects for interest rate analysis, we also make a comparison of standard deviations for USD and CAD yield changes in Table 9 to provide another way of examining simulation results. As shown, no matter what different maturities of USD yield changes or CAD, the simulated values are very close to the historical ones, proving the accuracy of simulation results.

In Table 10, the results of exchange rate returns are present. Unlike yields, it seems that the means of simulated exchange rate returns are not close to the history data. The simulated values tend to be smaller than the historical ones. However, the simulated standard deviations are almost perfect. All of them are very close to their historical one. Overall, the properties of the simulated exchange rate returns also appear to be similar with the historical data.

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Table 1 Definition of risk factors

Risk factors	Groups	Variables	Sampled periods (obs.)
Interest rates	EUR	Include monthly zero coupon yields for the Euro (EUR) with 30 maturities between 1 year and 30 years. Changes in yield of EUR are modeled: $DIR_t = IR_t - IR_{t-1}$.	November 1991 - September 2010 (227)
	US-CA	Include monthly zero coupon yields for the US dollar (USD) and the Canadian dollar (CAD) with 30 maturities between 1 year and 30 years. Changes in yield are modeled: $DIR_t = IR_t - IR_{t-1}$.	January 1995 - September 2010 (189)
	AU-NZ	Include monthly zero coupon yields for the Australian dollar (AUD) and the New Zealand dollar (NZD) with 30 maturities between 1 year and 30 years. Changes in yield are modeled: $DIR_t = IR_t - IR_{t-1}$.	January 1995 - September 2010 (189)
	TWD	Include monthly zero coupon yields for the New Taiwan dollar (TWD) with 30 maturities between 1 year and 30 years. Changes in yield are modeled: $DIR_t = IR_t - IR_{t-1}$.	March 1999 - September 2010 (138)
Stock indices	US	Include market indices of the Dow Jones, S&P 500, and Nasdaq using month-end closing price adjusted for dividend, in terms of local currency. The monthly log returns are modeled: $RS_t = \ln(S_t) - \ln(S_{t-1})$.	November 1983 - September 2010 (323)
	TW	Include the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX). The monthly log	May 1989 - September 2010

Risk factors	Groups	Variables	Sampled periods (obs.)
		returns are modeled: $$RS_t = \ln(S_t) - \ln(S_{t-1}).$$	(257)
Exchange rates	FX	Include monthly USD spot prices with respect to EUR, CAD, AUD, NZD, and TWD. The monthly log returns are modeled: $$RFX_t = \ln(FX_t) - \ln(FX_{t-1}).$$	February 1999 - September 2010 (140)
Real estate	HP	Include quarterly house price indices for new and pre-sold houses including a Taiwan national composite index and five geographic regional indices for Taipei City, Taipei County, Taoyuan-Hsinchu, Taichung, and Tainan-Kaohsiung areas. The quarterly log returns are modeled: $$RHP_t = \ln(HP_t) - \ln(HP_{t-1}).$$	1993Q2 - 2010Q2Q (69)
	RT	Include the quarterly average rents for Grade A office in Taipei City. The quarterly log returns are modeled: $$RRT_t = \ln(RT_t) - \ln(RT_{t-1}).$$	2002Q2 - 2010Q2 (33)

Table 2 Descriptive statistics of selected risk factors [a]

	Mean	Standard Deviation	Min.	Max.	Skewness	Excess Kurtosis	ADF test [b]
Panel A: USD yields							
1y (Maturity)	3.699	2.04	7.327	0.244	-0.297	-1.351	-1.974 [2]
5y	4.524	1.505	8.012	1.299	-0.072	-0.891	-2.674 [0]
10y	5.03	1.127	7.988	2.316	0.218	-0.489	-3.993** [0]
15y	5.45	0.911	8.009	3.154	0.279	-0.269	-4.746*** [0]
20y	5.643	0.912	8.035	3.242	0.127	-0.504	-4.631*** [0]
25y	5.582	0.931	8.073	2.944	0.301	-0.222	-4.831*** [0]
30y	5.522	0.981	8.112	2.647	0.501	0.06	-4.386*** [0]
Panel B: CAD yields							
1y (Maturity)	3.871	1.77	8.959	0.396	0.134	-0.001	-2.671 [0]
5y	4.796	1.501	9.532	1.775	0.479	0.189	-4.114*** [0]
10y	5.317	1.401	9.622	2.986	0.854	0.334	-3.806** [0]
15y	5.633	1.39	9.626	3.56	0.981	0.206	-3.075 [0]
20y	5.766	1.43	9.641	3.613	0.988	0.366	-2.724 [0]
25y	5.629	1.436	9.634	3.502	1.007	0.32	-2.726 [3]
30y	5.493	1.453	9.628	3.391	1.033	0.304	-2.879 [0]
Panel C: Changes in USD yields							
1y (Maturity)	-0.037	0.256	0.66	-1.242	-1	2.973	-6.551*** [1]
5y	-0.036	0.311	0.893	-0.963	0.042	0.542	-12.356*** [0]
10y	-0.028	0.293	0.979	-1.15	0.034	1.237	-11.511*** [1]
15y	-0.025	0.297	1.07	-1.211	0.331	2.369	-12.51*** [1]
20y	-0.023	0.279	1.205	-1.07	0.599	3.443	-12.107*** [1]
25y	-0.022	0.255	0.986	-0.938	0.274	2.399	-11.797*** [1]
30y	-0.021	0.245	1.004	-0.81	0.11	2.348	-13.955*** [0]
Panel D: Changes in CAD yields							
1y (Maturity)	-0.041	0.296	0.857	-1.183	-0.462	1.108	-12.631*** [0]

5y	-0.038	0.272	0.64	-1.171	-0.382	0.967	-13.75*** [0]
10y	-0.034	0.228	0.461	-0.716	-0.238	0.178	-14.333*** [0]
15y	-0.031	0.203	0.469	-0.693	-0.221	0.42	-13.738*** [0]
20y	-0.031	0.205	0.489	-0.595	-0.258	0.417	-11.916*** [1]
25y	-0.031	0.189	0.484	-0.631	-0.325	0.376	-7.57*** [2]
30y	-0.032	0.192	0.479	-0.674	-0.31	0.495	-7.179*** [2]
Panel E: Stock returns (%)							
Dow Jones	0.674	4.57	-26.42	12.95	-1.113	4.189	-17.023*** [0]
Nasdaq	0.667	6.783	-31.79	19.87	-0.917	2.756	-16.769*** [0]
S&P500	0.602	4.566	-24.54	12.38	-1.06	3.472	-16.41*** [0]
TAIEX	0.015	9.625	-43.53	33.24	-0.289	2.623	-14.748*** [0]
Panel F: Exchange rate returns (%)							
TWD	-0.025	1.348	-3.897	3.943	-0.082	0.595	-8.992*** [0]
EUR	-0.129	3.161	-8.901	11.71	0.189	1.243	-10.492*** [0]
AUD	-0.307	3.808	-8.682	19.61	1.144	4.483	-10.453*** [0]
NZD	-0.224	3.981	-12.85	15.21	0.397	1.846	-10.98*** [0]
CAD	-0.274	2.626	-7.873	15.44	1.317	8.18	-11.375*** [0]

[a] Yields are in percentage.

[b] For yields, the test regression includes a trend and a constant term. For yield changes and return, the test regression includes a constant term. The t statistics is presented. *, **, and *** indicate significant at the 10%, 5% and 1% levels respectively. The numbers in the brackets [ ] are the optimal lags, chosen by the Bayesian information criterion.

Table 3 Portion of variance explained by the chosen factors for the US-CA yield change group and exchange rate returns

	US-CA				FX			
No. of factors	1	2	3	4	1	2	3	4
Eigenvalues	45.42	7.08	4.29	0.88	3.35	0.69	0.51	0.33
% Variance	75.7	11.8	7.16	1.47	67	13.7	10.2	6.6
Cumulative (%)	75.7	87.5	94.65	96.13	67	80.7	91	97.5

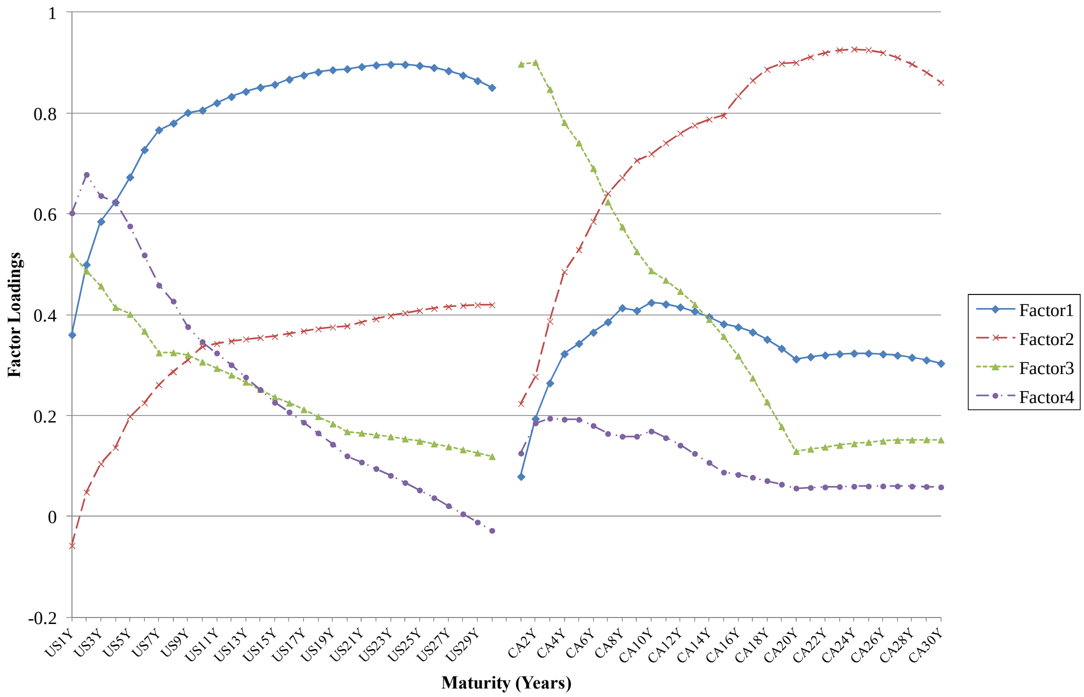

Figure 1 Rotated estimated factor loadings for the US-CA yield change group

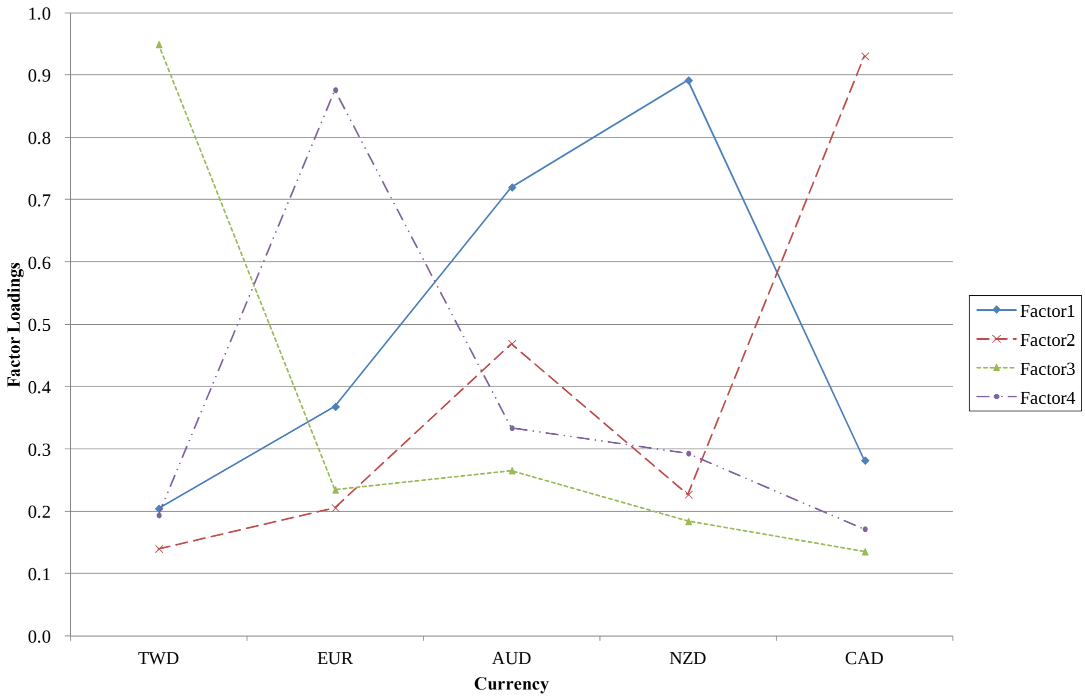

Figure 2 Rotated estimated factor loadings for the exchange rate returns

Table 4 Estimated GARCH models for factor scores of the US-CA yield changes' [a]

$$F_t = c + \sum_{i=1}^{p} \phi_i F_{t-i} + \varepsilon_t, \;\; \varepsilon_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{m} \alpha_j \varepsilon_{t-j}^2 + \sum \;\; \beta \sigma_{t-1}^2, \;\; \text{where} \;\; \omega > 0, \alpha, \beta \geq 0 \,.$$

	US-CA yield changes [b]			FX [b]	
	Factor 1	Factor 2	Factor 4	Factor 1	Factor 3
Parameter	AR(1)-ARCH(1)	AR(6)-ARCH(4)	AR(2)	AR(3)	AR(1)-ARCH(1)
ϕ_1	–0.158	—	—	—	0.316
	(-2.002)				(3.457)
ϕ_2	—	—	0.203	—	—
			(2.767)		
ϕ_3	—	0.201	—	0.259	—
		(2.277)		(3.139)	
ϕ_6	—	0.085	—	—	—
		(1.241)			
ω	0.824	0.451	—	—	0.746
	(12.825)	(5.71)			(7.065)
α_1	0.142	0.181	—	—	0.182
	(1.834)	(1.705)			(1.378)
α_2	—	0.001	—	—	—
		(0.018)			
α_3	—	0.151	—	—	—
		(1.514)			
α_4	—	0.166	—	—	—
		(1.971)			
β	—	—	—	—	—
AIC	2.801	2.7	2.816	2.782	2.754

BIC	2.853	2.822	2.851	2.803	2.817

[a] Each coefficient is reported with the associated *t*-statistic for the null hypothesis that the estimated value is equal to zero.

[b] Factor(s) not presented is(are) shown to be white noise via tests of serial correlation and conditional heteroscedasticity, so we don't model it(them) here with any AR-GARCH models.

Table 5 Correlation matrix of factors

	RS_TW	RS_US_F1	RS_ US_F2	RFX_F1	RFX_F2	RFX_F3	RFX_F4	DIR_TW_F1	DIR_TW_F2	DIR_TW_F3	DIR_TW_F4	DIR_EU_F1	DIR_EU_F2	DIR_EU_F3
RS_TW	1	0.294	0.238	-0.298	-0.256	-0.388	0.145	0.3	0.081	0.103	0.071	-0.033	0.19	-0.021
RS_US_F1	0.294	1	0	-0.311	-0.285	-0.133	0.002	0.101	-0.087	0.195	-0.136	-0.106	0.205	0.011
RS_US_F2	0.238	0	1	-0.161	-0.253	-0.245	0.113	0.212	0.054	0.025	-0.006	0.002	0.19	0.014
RFX_F1	-0.298	-0.311	-0.161	1	-0.004	0	0	-0.032	-0.094	-0.126	-0.024	0.11	-0.055	-0.191
RFX_F2	-0.256	-0.285	-0.253	-0.004	1	0.001	-0.001	-0.028	0.011	-0.105	0	0.038	-0.204	-0.132
RFX_F3	-0.388	-0.133	-0.245	0	0.001	1	0	-0.175	-0.062	-0.003	0.009	0.021	-0.12	0.013
RFX_F4	0.145	0.002	0.113	0	-0.001	0	1	0.259	0.033	0.014	-0.26	0.054	0.101	-0.058
DIR_TW_F1	0.3	0.101	0.212	-0.032	-0.028	-0.175	0.259	1	0	0	0	0.163	0.3	0.079
DIR_TW_F2	0.081	-0.087	0.054	-0.094	0.011	-0.062	0.033	0	1	0	0	-0.015	0.057	-0.048
DIR_TW_F3	0.103	0.195	0.025	-0.126	-0.105	-0.003	0.014	0	0	1	0	0.225	0.279	0.142
DIR_TW_F4	0.071	-0.136	-0.006	-0.024	0	0.009	-0.26	0	0	0	1	0.015	-0.091	-0.028
DIR_EU_F1	-0.033	-0.106	0.002	0.11	0.038	0.021	0.054	0.163	-0.015	0.225	0.015	1	0	0
DIR_EU_F2	0.19	0.205	0.19	-0.055	-0.204	-0.12	0.101	0.3	0.057	0.279	-0.091	0	1	0
DIR_EU_F3	-0.021	0.011	0.014	-0.191	-0.132	0.013	-0.058	0.079	-0.048	0.142	-0.028	0	0	1
DIR_USCA_F1	0.193	0.077	0.159	0.069	-0.06	-0.023	0.196	0.406	0.046	0.151	-0.094	0.527	0.271	0.42
DIR_USCA_F2	-0.096	-0.057	-0.153	0.159	0.023	-0.05	0.082	-0.005	0.032	0.142	-0.049	0.39	-0.04	0.241
DIR_USCA_F3	0.06	0.056	0.03	-0.025	-0.369	0.126	0.168	0.086	0.022	0.156	-0.022	0.015	0.427	0.041
DIR_USCA_F4	0.28	0.06	0.242	-0.087	0.041	-0.067	0.228	0.163	-0.005	0.098	-0.132	-0.076	0.39	-0.125
DIR_AUNZ_F1	0.088	-0.049	0.017	-0.143	0.027	-0.099	0.19	0.151	-0.05	0.108	-0.027	0.297	0.06	0.386

DIR_AUNZ_F2	0.123	-0.039	0.07	0.1	-0.122	0.008	0.222	0.31	0.05	0.088	-0.02	0.265	0.159	0.16
DIR_AUNZ_F3	0.133	0.093	0.095	-0.238	-0.373	-0.015	0.123	0.18	-0.004	0.202	-0.029	0.13	0.357	0.078
DIR_AUNZ_F4	0.255	0.083	0.054	-0.299	0.101	-0.014	0.167	0.113	0.039	0.375	-0.074	-0.051	0.319	-0.031
DIR_AUNZ_F5	0.039	-0.036	0.003	0.155	0.167	-0.101	0.104	0.14	0.001	-0.069	0.025	0.316	-0.042	0.144
RHP_F1	0.162	0.339	-0.036	-0.014	-0.139	-0.211	0.209	0.246	0.245	0.34	-0.148	-0.201	0.311	0.053
RHP_F2	0.171	-0.145	-0.003	0.131	-0.273	0.092	-0.098	0.11	0.023	-0.203	0.158	0.129	-0.072	0.066
RHP_F3	-0.08	-0.102	0.06	-0.052	-0.089	0.082	0.229	0.249	-0.161	0.141	-0.216	0.237	0.145	0.008
RHP_F4	-0.112	0.001	-0.048	-0.054	-0.096	0.069	0.034	0.005	0.002	0.096	-0.089	-0.001	-0.056	-0.299
RHP_F5	0.081	-0.145	0.212	-0.006	0.06	-0.066	-0.219	0.173	0.164	-0.229	0.207	-0.085	0.158	-0.045
RRT	-0.003	0.063	0.007	0.283	0.035	-0.11	-0.004	0.315	-0.287	-0.073	0.112	0	0.577	-0.319

Note: We calculate the correlation of any pair of factors once at a time, not of all common factors at the same time. Relevant information about sample period of each group can refer to Table 1. The original series of the TAIEX stock returns (RS_TW) and house rent returns (RRT) are directly used to calculate the correlations.

Table 5 Correlation matrix of factor scores (cont.)

	DIR_USCA _F1	DIR_USCA _F2	DIR_USCA _F3	DIR_USCA _F4	DIR_AUNZ _F1	DIR_AUNZ _F2	DIR_AUNZ _F3	DIR_AUNZ _F4	DIR_AUNZ _F5	RHP_F1	RHP_F2	RHP_F3	RHP_F4	RHP_F5	RRT
RS_TW	0.193	-0.096	0.06	0.28	0.088	0.123	0.133	0.255	0.039	0.162	0.171	-0.08	-0.112	0.081	-0.003
RS_US_F1	0.077	-0.057	0.056	0.06	-0.049	-0.039	0.093	0.083	-0.036	0.339	-0.145	-0.102	0.001	-0.145	0.063
RS_US_F2	0.159	-0.153	0.03	0.242	0.017	0.07	0.095	0.054	0.003	-0.036	-0.003	0.06	-0.048	0.212	0.007
RFX_F1	0.069	0.159	-0.025	-0.087	-0.143	0.1	-0.238	-0.299	0.155	-0.014	0.131	-0.052	-0.054	-0.006	0.283
RFX_F2	-0.06	0.023	-0.369	0.041	0.027	-0.122	-0.373	0.101	0.167	-0.139	-0.273	-0.089	-0.096	0.06	0.035
RFX_F3	-0.023	-0.05	0.126	-0.067	-0.099	0.008	-0.015	-0.014	-0.101	-0.211	0.092	0.082	0.069	-0.066	-0.11
RFX_F4	0.196	0.082	0.168	0.228	0.19	0.222	0.123	0.167	0.104	0.209	-0.098	0.229	0.034	-0.219	-0.004
DIR_TW_F1	0.406	-0.005	0.086	0.163	0.151	0.31	0.18	0.113	0.14	0.246	0.11	0.249	0.005	0.173	0.315
DIR_TW_F2	0.046	0.032	0.022	-0.005	-0.05	0.05	-0.004	0.039	0.001	0.245	0.023	-0.161	0.002	0.164	-0.287
DIR_TW_F3	0.151	0.142	0.156	0.098	0.108	0.088	0.202	0.375	-0.069	0.34	-0.203	0.141	0.096	-0.229	-0.073
DIR_TW_F4	-0.094	-0.049	-0.022	-0.132	-0.027	-0.02	-0.029	-0.074	0.025	-0.148	0.158	-0.216	-0.089	0.207	0.112
DIR_EU_F1	0.527	0.39	0.015	-0.076	0.297	0.265	0.13	-0.051	0.316	-0.201	0.129	0.237	-0.001	-0.085	0
DIR_EU_F2	0.271	-0.04	0.427	0.39	0.06	0.159	0.357	0.319	-0.042	0.311	-0.072	0.145	-0.056	0.158	0.577
DIR_EU_F3	0.42	0.241	0.041	-0.125	0.386	0.16	0.078	-0.031	0.144	0.053	0.066	0.008	-0.299	-0.045	-0.319
DIR_USCA_F1	1	0	0	0	0.137	0.301	0.177	0.013	0.301	-0.043	0.004	0.266	-0.244	0.036	0.179
DIR_USCA_F2	0	1	0	0	0.293	0.317	0.223	-0.066	0.251	-0.108	0.126	0.075	-0.158	-0.342	-0.081
DIR_USCA_F3	0	0	1	0	0.136	0.085	0.401	0.251	-0.199	0.225	-0.164	0.196	-0.057	0.089	0.292
DIR_USCA_F4	0	0	0	1	0.081	0.057	0.153	0.262	0.042	0.194	0.077	0.043	0.002	0.114	0.325

DIR_AUNZ_F1	0.137	0.293	0.136	0.081	1	0	0	0	0	-0.002	0.033	-0.03	-0.284	-0.01	0.008
DIR_AUNZ_F2	0.301	0.317	0.085	0.057	0	1	0	0	0	-0.154	0.111	0.214	-0.242	-0.163	0.331
DIR_AUNZ_F3	0.177	0.223	0.401	0.153	0	0	1	0	0	0.284	-0.089	0.208	0.174	0.288	0.18
DIR_AUNZ_F4	0.013	-0.066	0.251	0.262	0	0	0	1	0	0.197	-0.06	-0.16	0.103	-0.136	-0.23
DIR_AUNZ_F5	0.301	0.251	-0.199	0.042	0	0	0	0	1	0.046	0.166	-0.039	-0.017	-0.042	0.162
RHP_F1	-0.043	-0.108	0.225	0.194	-0.002	-0.154	0.284	0.197	0.046	1	0	0	0	0	0.242
RHP_F2	0.004	0.126	-0.164	0.077	0.033	0.111	-0.089	-0.06	0.166	0	1	0	0	0	0.128
RHP_F3	0.266	0.075	0.196	0.043	-0.03	0.214	0.208	-0.16	-0.039	0	0	1	0	0	0.37
RHP_F4	-0.244	-0.158	-0.057	0.002	-0.284	-0.242	0.174	0.103	-0.017	0	0	0	1	0	-0.087
RHP_F5	0.036	-0.342	0.089	0.114	-0.01	-0.163	0.288	-0.136	-0.042	0	0	0	0	1	0.272
RRT	0.179	-0.081	0.292	0.325	0.008	0.331	0.18	-0.23	0.162	0.242	0.128	0.37	-0.087	0.272	1

Table 6 Estimates of the mean reversion model for zero coupon yields

$$r_t = r_{t-1} + \gamma(r_{t-1} - \bar{r}) + \varepsilon_t, \quad \gamma < 0$$

where r_t = the short term yield at time t, $\bar{r}$ = the long term sample mean of r_t, and ε_t is a stochastic white noise component.

Shortest-term yield available	Sample used	γ	$\bar{r}$ (%)
EUR 1 year	October 1991－September 2010	-0.0179	3.7460
USD 1 year	April 1989－September 2010	-0.0126	4.2308
CAD 1 year	December 1994－September 2010	-0.0304	3.8707
AUD 1 year	December 1994－September 2010	-0.0592	5.5244
NZD 1 year	December 1994－September 2010	-0.0276	6.2838
TWD 1 year	March 1999－September 2010	-0.0129	2.1420

Table 7 Descriptive statistics of historical and simulated results for USD yields

	1 year	5 year	10 year	15 year	20 year	25 year
			Panel A: Historical data			
Long-run mean	4.231	5.208	5.727	6.061	6.240	6.259
Minimum	0.244	1.299	2.316	3.154	3.242	2.944
5th percentile	0.370	2.051	3.307	4.187	4.323	4.303
10th percentile	0.519	2.471	3.680	4.432	4.620	4.572
50th percentile	4.423	4.562	4.806	5.329	5.616	5.488
90th percentile	5.959	6.527	6.601	6.746	6.888	6.927
Maximum	7.327	8.012	7.988	8.009	8.035	8.074
			Panel B: Simulated data			
Long-run mean	4.189	5.155	5.677	6.015	6.198	6.219
Minimum	0.100	0.100	0.100	0.100	0.100	0.100
5th percentile	0.801	1.449	2.291	2.685	3.052	3.293
10th percentile	1.412	2.000	2.789	3.197	3.544	3.752
50th percentile	3.478	4.359	4.983	5.368	5.598	5.665
90th percentile	5.014	6.488	7.227	7.663	7.788	7.714
Maximum	8.913	11.033	12.263	12.981	12.932	12.582

Table 8 Descriptive statistics of historical and simulated results for CAD yields

	1 year	5 year	10 year	15 year	20 year	25 year
			Panel A: Historical data			
Long-run mean	3.871	4.796	5.317	5.633	5.766	5.629
Minimum	0.396	1.775	2.986	3.560	3.613	3.502
5th percentile	0.584	2.481	3.536	4.123	4.209	4.047
10th percentile	1.192	2.822	3.808	4.220	4.287	4.140
50th percentile	4.010	4.572	5.161	5.478	5.695	5.562
90th percentile	5.941	6.813	7.547	8.103	8.223	8.283
Maximum	8.959	9.532	9.622	9.626	9.641	9.635
			Panel B: Simulated data			
Long-run mean	3.847	4.780	5.308	5.626	5.760	5.624
Minimum	0.100	0.100	0.100	0.421	0.265	0.391
5th percentile	1.542	2.385	3.213	3.743	3.769	3.703
10th percentile	2.002	2.863	3.621	4.104	4.152	4.076
50th percentile	3.631	4.541	5.098	5.437	5.561	5.431
90th percentile	5.184	6.116	6.474	6.681	6.881	6.687
Maximum	9.187	10.265	11.971	12.238	13.437	12.890

Table 9 Standard deviation of historical and simulated results for USD and CAD yield changes

	1 year	5 year	10 year	15 year	20 year	25 year
			Panel A: Historical data			
USD yield changes	0.888	1.079	1.019	1.031	0.969	0.885
CAD yield changes	1.026	0.944	0.793	0.704	0.712	0.657
			Panel B: Simulated data			
USD yield changes	0.568	0.876	0.944	0.989	0.949	0.880
CAD yield changes	0.963	0.923	0.763	0.664	0.673	0.640

Table 10 Descriptive statistics of historical and simulated results for the exchange rate returns

	TWD	EUR	AUD	NZD	CAD
Panel A: Historical data					
Mean	-0.296	-1.545	-3.686	-2.690	-3.287
Standard deviation	4.687	10.985	13.236	13.842	9.128
Minimum	-3.897	-8.897	-8.683	-12.853	-7.873
5th percentile	-2.420	-5.820	-5.542	-5.864	-4.276
10th percentile	-1.738	-3.918	-4.853	-4.732	-3.126
50th percentile	-0.005	-0.010	-0.717	-0.665	-0.452
90th percentile	1.539	3.419	3.861	4.605	2.376
Maximum	3.943	11.704	19.616	15.213	15.444
Panel B: Simulated data					
Mean	-0.069	-0.070	-0.154	-0.176	-0.043
Standard deviation	4.714	11.034	12.717	13.631	9.099
Minimum	-8.804	-14.158	-16.286	-16.580	-12.161
5th percentile	-2.228	-5.249	-6.044	-6.485	-4.312
10th percentile	-1.725	-4.082	-4.710	-5.053	-3.369
50th percentile	-0.008	-0.008	-0.017	-0.014	-0.007
90th percentile	1.711	4.080	4.692	5.031	3.366
Maximum	9.305	16.644	17.160	17.247	11.958

Explaining the Rate Spreads on Life Settlements*

Ming-Hua Hsieh
Department of Risk Management and Insurance
Risk and Insurance Research Center
College of Commerce, National Chengchi University
E-mail: mhsieh@nccu.edu.tw

Jin-Lung Peng
Department of Risk Management and Insurance
Risk and Insurance Research Center
College of Commerce, National Chengchi University
E-mail: jlpeng@nccu.edu.tw

Chenghsien Tsai**
Department of Risk Management and Insurance
Risk and Insurance Research Center
College of Commerce, National Chengchi University
E-mail: ctsai@nccu.edu.tw

Jennifer L. Wang (corresponding author)
Department of Risk Management and Insurance
Risk and Insurance Research Center
College of Commerce, National Chengchi University
E-mail: jenwang@nccu.edu.tw

Ko-Lun Kung
Department of Risk Management and Insurance
College of Commerce, National Chengchi University
E-mail: kolun0126@gmail.com

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Explaining the Rate Spreads on Life Settlements

ABSTRACT

Scholars have paid attention to the determinants of rate spreads on various investment products, but not to those on life settlements yet. This study investigates the spread determinants of life settlements, which also extends the boundary of the literature on life settlements. The data on life settlements are from Coventry. We estimate the expected rate spread of a life settlement under certain death time and uncertain death time with or without considering mortality improvements. The average return of life settlement is 13% if we assume death time is certain. Considering uncertain death time would increase the average return to 27%. Additional mortality improvement would decrease the average return by 2.5%. The regression results show that the expected spreads contain the default risk premiums associated with the underlying insurers. The spreads also relate to the factors affecting the surrender tendencies of the underlying policies. All regression models product adequate adjusted R^2 and consistent results. We further infer that there are significant, positive premiums for bearing the non-systematic mortality risk of life settlements.

JEL Classification: G22

Keywords: Life Settlements, Risk Premiums, Rate Spreads

1. Introduction

The determinants of rate spreads on various investment products attract the attentions of scholars as well as practitioners. The renowned CAPM (Capital Asset Pricing Model; Sharpe, 1964) showed that the rate spreads of stocks were determined by the market risk premium and the betas of individual stocks reflecting the sensitivities of stock prices to market movements. Ross (1976) developed the arbitrage pricing theory (APT) in which the risk premiums of stocks were determined by some macroeconomic factors and the associated betas. The market risk premium might be one of the factors. Fama and French (1996) proposed a three-factor model to explain the risk premiums of stocks. The factors were size and book-to-market ratio in addition to the market risk. Jegadeesh and Titman (2001) added an additional factor: the momentum factor.

Many papers studied the determinants of yield spreads on corporate bonds, e.g., Fons (1994), Longstaff and Schwartz (1995), Duffie and Singleton (1997), Duffee (1999), Elton et al. (2001), Collin-Dufresne, Goldstein, and Martin (2001), Eom, Helwege, and Huang (2004), Longstaff, Mithal, and Neis (2005), Chen, Lesmond, and Wei (2007), Alexander (2008), and Nayak (2010). The spreads of corporate bond yields over government bond yields consist of three (or more) components: expected default loss, tax premium, and the premiums for non-default risks. Bodie, Kane, and Marcus (1993), Fons (1994), Cumby and Evans (1995), and other early papers assumed the spread is all default premium. Elton et al. (2001) showed that expected default accounted for a small fraction of the spread only. State taxes explained a substantial portion of the spread, and the remaining portion was closely related to the factors explaining the risk premiums of stocks. Using the credit default swap premium as a direct measure of the default component in corporate bond spreads, Longstaff, Mithal, and Neis (2005) found that the spread of a corporate bond was majorly due to its default risk. The non-default component was time varying and strongly related to the measures of market and bond-specific illiquidity. Chen, Lesmond, and Wei (2007) further confirmed the importance of liquidity in determining corporate bond spreads. Alexander (2008) identified inflation uncertainty as another determinant to the yield spread.

The spread determinants of other fixed-income products were studied in the literature as well. With regard to corporate loan spreads, early studies admitted the credit quality of borrowers as one determinant but rejected loan maturity (Barclay and Clifford, 1995a, 1995b; Stohs and Mauer, 1996; Amar et al., 1997; Ozkan, 2000; Steven, Nandy, and Sharpe, 2000). Gottesman and Roberts (2002) found evidence that lenders were compensated for longer maturity loans. Santos (2011) identified that losses occurred to banks also affected the spreads of the loans that were made after the losses. Menz (2012) investigated the correlation and causality between corporate governance and credit spreads. Regarding the spreads of emerging market bonds, Min (1998), Alexopoulou, Bunda, and Ferrando (2009), and Küçük (2010) showed that macroeconomic fundamentals of individual countries were significant determinants.

Bantwal and Kunreuther (2000) observed that catastrophe bond spreads were higher than those of equivalent-rated corporate bonds and tried to explain this puzzle by behavior economics (e.g., reluctance of investment managers to invest in cat bonds). Zanjani (2002) suggested that the “extra” risk premiums might be due to the threats of catastrophes on risk bearers’ solvencies. Dieckmann (2010) proposed a habit process to explain the extra premiums in which catastrophes were rare economic shocks that could bring investors closer to their subsistence level.

The determinants of the rate spreads that can be expected from investing in life settlements have not yet been examined, albeit the importance of this product. Life settlements are life insurance policies sold in a secondary market. The policyholder involved in a life settlement transaction receives a payment exceeding the surrender value but less than the death benefit. The investor in such a transaction assumes the role of paying premiums, and the investment return depends on the quality of the life expectancy estimates provided by medical underwriters. Life settlements are an increasingly popular asset class because they seemed to render good returns and/or diversification benefits to widely held assets (Gatzert, 2010; Braun, Gatzert, and Schmeiser, 20121). Life settlements may develop to be a strong market with the potential exceeding $140 billion by 2016 (Conning & Company, 2011).

This study intends to extend the scope of the literature on the spread determinants of risky assets to life settlements.[1] Identifying the determinants and understanding their relative significance will help market participants assess the value and risk of life settlements. Our results is therefore of interest to the buyers, sellers, originators, and other stakeholders of life settlements, in addition to scholars.

In estimating the rate spreads of life settlements, we first regard the life expectancy of an insured underlying the corresponding life settlement as the expected maturity of a corporate bond. The former is subject to mortality risk while the latter default risk. Thus we may calculate the internal rate of return (IRR) of a life settlement given the insured's life expectancy in the similar way as we calculate the yield to maturity (YTM) of a corporate bond.

The assumption behind such calculations is that the underlying insureds live out the life expectancies, which implies certain death time. We propose two methods to incorporate the uncertainty about death time. Within the first method, we insert into the above IRR calculation the probabilities of paying premiums if the insureds are alive at the beginning of a period and the probabilities of receiving death benefits if the insureds decease at the end of

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[1] This study extends the boundary of the researches on life settlements as well. Many studies investigated the economic impacts of life settlements on life insurance markets (e.g., Giacalone, 2001; Ingraham and Salani, 2004; Ziser, 2006 and 2007; Smith and Washington, 2006; Seitel, 2006 and 2007; Sherman, 2007; Leimberg et al., 2008). Some focused on the actuarial modeling and valuation of life settlements (e.g., Russ, 2005; Perera and Reeves, 2006; Milliman, 2008). Others cover issues such as securitization (Stone and Zissu, 2006; Ortiz, Stone, and Zissu, 2008), life expectancy estimation risk (Perera and Reeves, 2006; Stone and Zissu, 2007), and hedging benefits (Wang, Hsieh, and Tsai, 2011) of life settlements.

the period. By the second method, we calculate the IRRs of dying at different ages and then obtain a mortality-weighted IRR. Both types of IRRs reflect the spreads for bearing non-systematic mortality risk. We further estimate the spreads that consider potential mortality improvements in the future.

Our data on life settlements are from Coventry, a leading market maker. The given data contain the life expectancies estimated by one of Coventry's major medical underwriters. Note that the rate spreads investigated in this paper is *ex ante*, i.e., the expected spreads.[2]

The differences between the calculated IRRs and the risk-free rates at the inception of life settlements represent the spreads expected by the investors who bear all sorts of the risks associated with life settlements and/or want to seize the profitable opportunities associated with the surrender behaviors of the underlying insureds. We use the spot rates of US government bonds that have the maturities matching with the life expectancies of life settlements as the risk-free rates. The tested independent variables include: mortality risk premium, the premium for bearing the default risk of the underlying insurer, and some variables determining the insureds' motivations/tendencies to surrender their policies since life settlements are substitutes for surrenders. Then we conduct three sets of regression analyses that correspond to the three ways in estimating the expected IRRs to investigate the determinants of the rate spreads on life settlements.

Observing that the IRRs considering the uncertainty about death time have a higher mean than those obtained under the assumption of certain death time, we infer that there are positive premiums for bearing non-systematic mortality risk. We also observe that introducing uncertain mortality improvements increases expected IRRs. This implies positive premiums for bearing systematic mortality risk.[3]

The rate spreads of life settlements contain risk premiums associated with the default

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[2] CAPM and APT proposed *ex ante* relations between risk and return. Most papers on fixed-income products investigated the *ex-ante* relations as well.
[3] We also observe that mortality improvements reduce the expected IRRs with limited extents only.

risks of the underlying insurers, which is supported by the negative and significant coefficient of the rating-ranked variable. The rest portions of the spreads mainly related to the surrender tendencies of the underlying policies. The holders of older policies have less motivation to surrender their policies, which in turn leads to worse terms for the investors when contracting life settlements. Our regression results show negative and significant coefficient of policy year with respect to IRR.

Our results also present positive coefficients for the insureds' age and gender. This is as expected since empirical statistics show that healthier people have less motivation to surrender their policies. We further decompose the information conveyed by the life expectancy estimated by medical underwriter into the information reflected by age and gender and the information captured by medical underwriting. The regression results are consistent with our expectations. Older people and males are unhealthier, ceteris paribus, and thus have positive and significant coefficients. The underwriting information variable reflecting "extra" healthiness also has negative and significant coefficients.

The remainder of this article is organized as follows. We delineate the life settlements obtained from Coventry in Section 2. In Section 3, we explain how to calculate IRRs in different ways and compare the resulted IRR distributions. In Section 4 we speculate possible sources of life settlement' rate spreads and specify corresponding variables. Section 5 contains regression results and analyses. We draw conclusions and outline possible extensions in Section 6.

2. Life Settlement Samples

The data used in this study are from Coventry, one of the major originators and market makers in the US life settlement market. The samples that Coventry provided for us were a subset of the policies originated during the period from July 2009 to April 2011. They are 346 universal life insurance policies with descriptive statistics presented in Table 1.

[Insert Table 1 about Here]

The insureds of the policies underlying the sampled life settlements were seniors, with a mean age of 76 and the range from 63 to 87 at the times when the policies were acquired by Coventry. At those times their life expectancies estimated by one of Coventry's major medical underwriters ranged from 6 years to 20 years with a mean of 13 years. Three quarters of the insureds were male. The insurance policies were acquired by Coventry at early stages. The average policy year when the policies were bought was about 3; the youngest one was just one month old while the oldest was bought in its 24th policy year. Most policies had large amounts of death benefits: the average is 4 million dollars and the largest one reaches $20 million. Their acquisition costs had a mean of $0.44 million and a range of $20,000 - $6.8 million. Standard & Poor's credit rating of insurance policies' original carrier is also provided. The rating ranges from BBB- to AAA, with almost all carriers above investment grade. We convert the credit ratings to a numerical scale 1 to 5 with AA- and below equals to 1, AA equals to 2 and so forth.

3. Calculating Expected Returns and Risk Premiums

The first set of IRRs, $IRR^{(1)}$, is obtained by solving the following equation:

$$AC = -\sum_{t=1}^{LE} \frac{Premium_t}{\left(1+IRR^{(1)}\right)^t} + \frac{NDB}{\left(1+IRR^{(1)}\right)^{LE}}, \qquad (1)$$

in which *AC* stands for the acquisition cost of a life settlement, *PV* denotes the operator of calculating present value, $Premium_t$ indicates the premium expected to be paid in month *t*, NDB_{LE} is the net death benefit to be paid at the expected death time. The unit of time used in the present value operator is month since the premium schedules of life settlements are in terms of months. The monthly premium projection were provided by Coventry and used as a basis of pricing calculation. The obtained monthly IRRs are then multiplied by 12 to become annual rates.

All IRRs were computed in two ways to make sure they are numerically stable. The first algorithm is a built-in function called "irr" in MATLAB R2011b. To ensure of the regularity/reasonableness of the obtained IRRs,[4] we also use the grid-search method on the present value of each life settlement by varying the discount rate between -100% and 100%. None of them have multiple rates. We detect 2 life settlements having negative IRRs. After inspecting the projected premium schedule, we found that it is the sums of expected premium payments exceed the nominal death benefits caused negative IRRs. We drop all samples with negative IRR for robustness. The distribution and the associated summary statistics are presented in Figure 1.

[Insert Figure 1 about Here]

The mean IRR of the sampled life settlements is 8.62%. The standard deviation of the IRR distribution is small: 1.21%. The distribution is right-skewed: the skewness is about -0.91. The distribution is leptokurtic with kurtosis equals to 4.22.

We further calculated the expected IRRs of life settlements under uncertain death time. To take into account the differences between the policyholders underlying life settlements and the general public and to facilitate the comparisons between certain and uncertain death time cases, we scaled the death probabilities (q_x; the probability of the insured, age of x, dying within one year) of 2008 Valuation Basic Table (VBT 2008) from the Society of Actuaries (SOA) so that the life expectancy for each life settlement would be equal to that in Coventry's dataset. That is, let ex be a function that maps death probabilities q_x to life expectancy. We find the scaling parameter α so that

$$ex(\alpha q_x) = \mathrm{LE}_{\mathrm{coventry}},$$

where $\mathrm{LE}_{\mathrm{coventry}}$ is the life expectancy estimate provided by the medical underwriter. And we use the adjusted probability of death $q_x^{adj} = \alpha q_x$ to compute the expected cash flow and solve for IRR.

[4] Brealey, Myers, and Allen (2011) pointed out that an investment may have no or multiple IRRs.

We solve the following equation to obtain the ex-ante $IRR^{(2)}$:

$$AC = -\sum_{t=1}^{\omega} {}_{t}p_x^{adj} \frac{Premium_t}{1+IRR^{(2)}} + {}_{t-1}p_x^{adj} q_{x+t-1}^{adj} \frac{NDB}{1+IRR^{(2)}}, \qquad (2)$$

where ω indicates the ultimate age of the mortality table and ${}_{t-1}p_x^{adj}$ stands for the probability that an insured with age of x survives to time t with the adjustment to life expectancy provided by Coventry. The ultimate age ω is 110 years old for both US male and female mortality tables in VBT 2008. We assumed uniform distribution of death (UDD) for fractional years. The distribution of the IRRs obtained by Equation (2) is presented as Figure 2.

[Insert Figure 2 about Here]

The IRRs calculated under uncertain death time has a higher mean of 11.25% than that of the IRRs assuming the death of the underlying insured happens at exactly the moment life expectancy predicts. Higher IRR on average may be justified by two observations/speculations. Firstly, the changes in IRRs are asymmetric between death is happening earlier and later than expected. Because IRR is a convex function in death time, the increases in IRRs due to earlier deaths are larger on average than the decreases in IRRs because of later deaths. Take the policy that has 11.167 years (i.e., 134 months) of life expectancy as an example. The IRR when life expectancies are 50 months, 134 months, and 218 months from the inception of the life settlement is 39.29%, 6.87%, and 6.71%, respectively.[5]

Secondly, $IRR^{(2)}$ has a higher mean than the first set because it incorporates the uncertainty of death time. The difference between these two sets of IRR may be regarded as the risk premium for the non-systematic mortality risk. The risk premium is high probably because the IRR of a life settlement is quite sensitive to the death time of the underlying insured and the diversification of non-systematic mortality risk is difficult for life settlement

[5] The associated probability is 0.37%, 0.46%, and 0.29% respectively.

buyers to implement (possibly due to the underlying policy is usually large and the limited availability of life settlements in each trade).

The IRRs calculated under uncertain death time has a higher standard deviation of 1.76% than that of the IRRs assuming the death of the underlying insured happens as life expectancy predicts. This is reasonable due to the introduction of non-systematic mortality risk. The IRR distribution becomes positively skewed under uncertain death time can be explained by the asymmetry effect of changes in death time on IRRs.

We also consider the effect of tax to life settlement. Internal Revenue Service (2009) addresses the tax treatment to life settlement. The gain of life settlement investor should be treated as ordinary income instead of capital gain. Since we do not have income data of individual investors, we simply assume the income tax rate is 15% in the calculation henceforth. Adding tax in fact lowered IRR for a considerable amount. In our calculation of certain lifetime IRR, 15% capital tax lowered the rate to return by about 1%, to 7.73%. For uncertain lifetime IRR, the tax effect lowered the rate of return by 2%. This is hardly surprising because in the latter the expected tax payment is distributed throughout the whole schedule instead of at expected death time.

The second way of incorporating uncertain death time is by estimating the distribution of IRR. That is, we compute IRR for every possible death time and use the result as an empirical distribution. These IRRs represent the actual rate of return the policyholder in every state of the world. Then we weigh them via the probability distribution of death time K_x to get the expected IRR.

In this case the expected IRR of the policy is computed as follows:

1. Given that the insured is age x, for each possible death time $t = x + 1m, x + 2m, \ldots, \omega - 1m, \omega$, we compute IRR_t assumes that insured dies at time t.
2. If time of death is within first 12 months of funding date, then instead receive the death benefit immediately, the insured will receive the death benefit at the end of

that year.

3. The expected IRR is $IRR^{(3)} = \sum_{t=1m}^{\omega} P(K_x = x + t) IRR_t$.

Figure 3 presents the distribution of expected IRR. The average expected IRR is 17.56% with standard deviation 3.18%. It can be shown this approach will always yield higher expected IRR than the previous approach from Jensen's inequality.

4. Possible Determinants of the Spreads and Variable Specifications

The rate spreads of investing on life settlements have several sources including the mortality risk premium, the premium for bearing the default risk of the underlying insurer, the illiquidity premium, tax benefits of life insurance policies, and the profitability resulting from insureds' surrender behaviors. Life settlements can be regarded as substitutes for surrenders. Stronger motivations to surrender insurance policies imply more willingness to enter life settlement transactions. The terms of such transactions will be worse as a result, which implies higher rate spreads of life settlements. The determinants of surrender rates therefore will also be those of life settlements' spreads.

There were few papers studying the determinants of surrender rates. Some of them identified macroeconomic variables affecting the surrender rates such as Tsai, Kuo, and Chen (2002), Kuo, Tsai, and Chen (2003), and Kim (2005); some (e.g., Transactions of Society of Actuaries Reports; Taiwan Standard Ordinary Experience Mortality and Lapse Rate Reports; Fier and Liebenberg, 2012) investigated individual/family variables affecting surrender behaviors. These reports/studies found that policy year and healthiness are negatively correlated with surrender rate.

Therefore, the independent variables used in this study include healthiness proxy variables, policy year, and rating. We first use the life expectancy estimated from the VBT 2008 as the proxy for the health indicator of the underlying insured. Since Transactions of Society of Actuaries Reports indicates that healthier people have smaller tendency to surrender their policies and thus are less willing to enter life settlement transaction, we expect life expectancy to be negatively correlated with the rate spread.

For comparison's sake we also decompose the healthiness variable to age, gender, and a private information indicator. We expect age is negatively correlated with healthiness of the insured. We also expect male is more risky than female. Other than that, we expect the life expectancy estimated by medical underwriter contains proprietary information that may tell us more than age and gender. We use the difference between life expectancy from Coventry and VBT 2008 to represent he private information indicator.

We retrieve the private information contained in the life expectancy estimated by Coventry as follows:

$$Underwriting\ information = LE_{Coventry} - LE_{VBT},$$

in which LE_{VBT} indicates the life expectancy estimated from the 2008 SOA VBT mortality tables. The difference between two life expectancy estimates may be regarded as proprietary information provided by the medical underwriter. We expect that the person identified by the Coventry's medical underwriter to be healthier than usual (given the same age and gender) will result in a lower spread of the life settlement.

Tsai, Kuo, and Chiang (2009) showed that policy year is negatively correlated with surrender rate. We therefore expect policy year to be negatively correlated with the spread.

The rating of the underlying insurer reflects the default risk undertook by the life settlement investor. The investor will require risk premiums. Therefore, we expect that the life settlement originated from a policy issued by a better-rated insurer will render a lower spread.

With regard to the dependent variable, we calculate the ex-ante rate spread as the difference between the expected IRR of a life settlement and a matching risk free rate close to the funding date of that life settlement. We first collect the term structure of the zero rates of US government bonds in the first trading day of the month in which the life settlement is funded and then specify the matching risk-free rate to be the spot rate with the same time to maturity to the life expectancy of the life settlement. Since the life expectancy of a life settlement is expressed in terms of months, we interpolate linearly to match the maturity with life expectancy. We have three sets of spreads corresponding to the IRRs calculated by Equations (1) to (3) respectively.

5. Empirical Results

We conduct regression analyses to investigate the significant determinants of the spreads. Since we have three sets of IRRs calculated under certain death time, uncertain death time, and uncertain death time with mortality improvements, we report three sets of regression results. The results are presented in Table 2.

[Insert Table 2 about Here]

The first set of regressions has high R^2, and the signs of all coefficients are consistent with our expectation. Both age and gender have a positive and significant coefficient, which is consistent with the conjecture that healthier persons have less motivation to surrender their policies and thus lead to less favorable terms to life settlements' investors.
Underwriting information reflect the evaluation from the third party medical underwriter, indicating policyholder's healthiness (or unhealthiness) beyond standard mortality tables. It has a negative coefficient and is significant.

Policy year is expected to have a negative coefficient since older policies exhibit lower surrender rates and lead to lower IRRs of life settlements eventually. Our regression results are consistent with this expectation. This expectation is valid for alternative healthiness proxies.

The rating of the insurer that issues life insurance policies underlying life settlements should be negatively correlated with the expected IRR of life settlements since higher rating means lower default risk (premium). However, the coefficient result is not significant, though negative, in the regression.

We add one more control variables to the regression, which is the size of the policy at the funding date. The policy size can be an indicator to policyholder's net worth, from the perspective of insurable interest. Rich people tend to be healthier because of the better living condition, or, if sick, can afford better medical care. We use the logarithm of net death benefit to control for policy size.

The results of the second set of regressions are similar to those of the first set. All coefficients have the same signs. The major differences are in the adjusted R^2 and the significance levels of policy year and rating. The R^2 is lower (55%) when *age and gender* are used as the healthiness proxy. The underwriting information is positively significant. Policy year and rating are both significant in this case. The results of the third set of regressions are rather similar to those of the second set. All coefficients have the same signs, and most have the same significance levels.

6. Conclusions and Future Work

Scholars as well as practitioners have paid significant attention to the determinants of rate spreads on various investment products including common stocks, corporate bonds, sovereign bonds, corporate loans, and catastrophe bonds. The determinants of the rate spreads on life settlements have not yet been examined, even though life settlements are an increasingly popular asset class with growth potentials. This study extends the scope of the literature on the spread determinants of risky assets to life settlements, in addition to extending the boundary of the literature on life settlements.

We first estimate the IRR of a life settlement given the insured's life expectancy as people calculate the YTM of a corporate bond. Then we calculate the IRRs under uncertain death time with and without considering mortality improvements. Our data on life settlements are from Coventry, and the data contain attributes of the underlying policies and insured as well as the life expectancies estimated by a major medical underwriter. The differences between the calculated IRRs and the spot rates of US government bonds that have the maturities matching with the expected death time represent the rate spreads expected by the investors. The tested independent variables include: mortality risk premium, the premium for bearing the default risk of the underlying insurer, and some variables associated with the insureds' motivations to surrender their policies.

The regression results show that the rate spreads of life settlements contain the risk premiums associated with the default risks of the underlying insurers. The rate spreads also relate to the factors affecting the surrender tendencies of the underlying policies including policy year, the policy value normalized by death benefit, and the proxies for the healthiness of the insureds. All regression models have adequate adjusted R^2 and consistent results. We further infer that there are significant, positive premiums for bearing non-systematic mortality risk from the differences between the IRRs obtained under uncertain death time and those under certain death time.

We plan to extend the current study in the near future in several aspects. Firstly, we would like to make our inference about the non-systematic mortality risk be more rigorous by introducing risk-neutral valuations. Such valuations can also be applied to estimate the premiums for bearing systematic mortality risk resulting from uncertain mortality improvements. We plan to build an empirical CBD model (Cairns, Blake and Dowd, 2006), in addition to the current Lee-Carter model, to simulate future mortality rates since the CBD model is better suited for senior people. We also plan to estimate the spreads resulting from the tax benefits of life insurance.

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Table 1: Summary Statistics on the Samples of Life Settlements

	Mean	Median	Standard Deviation	Minimum	Maximum
Insured's Age	75.63	75.38	4.60	63.42	86.75
Life Expectancy	12.98	13.13	2.78	5.92	19.67
Gender (Male = 1)	0.75	1.00	0.43	0.00	1.00
Policy Year	2.82	2.33	3.22	0.08	23.67
Rating	2.57	3.00	0.99	1.00	5.00
Acquisition Cost ($million)	0.44	0.23	0.66	0.02	6.80
Death Benefit ($million)	4.09	3.00	3.77	0.23	20.00

Table 2: Regression Results: Certain Death Time vs. Uncertain Death Time (with or without Mortality Improvements)

	(1)		(2)		(3)	
	sprd_LE	**sprd_LE**	**sprd_q1**	**sprd_q1**	**sprd_q2**	**sprd_q2**
(Intercept)	10.36***	-3.06	15.13***	-0.93	35.64***	-27.55***
	(1.14)	(1.96)	(1.49)	(2.95)	(1.88)	(3.84)
LE_VBT	-0.02***		-0.02***		-0.08***	
	(0.00)		(0.00)		(0.01)	
Underwriting Information	-0.03***	-0.03***	-0.02***	-0.02***	-0.07***	-0.07***
	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Policy Year	-0.12***	-0.11***	-0.14***	-0.14***	-0.13***	-0.11**
	(0.02)	(0.03)	(0.04)	(0.04)	(0.04)	(0.05)
Rating	-0.05	-0.05	0.01	-0.01	0.26	0.25
	(0.08)	(0.08)	(0.11)	(0.11)	(0.16)	(0.16)
logAcqCost	-0.21**	-0.19**	-0.34***	-0.33***	-0.80***	-0.73***
	(0.09)	(0.09)	(0.11)	(0.11)	(0.14)	(0.15)
Insured Age		0.13***		0.16***		0.63***
		(0.02)		(0.03)		(0.05)
Insured Gender		0.38**		0.04		1.74***
		(0.18)		(0.26)		(0.32)
R^2	0.21	0.20	0.15	0.15	0.54	0.53
Adj. R^2	0.20	0.18	0.14	0.14	0.54	0.52
Num. obs.	343	343	341	341	343	343

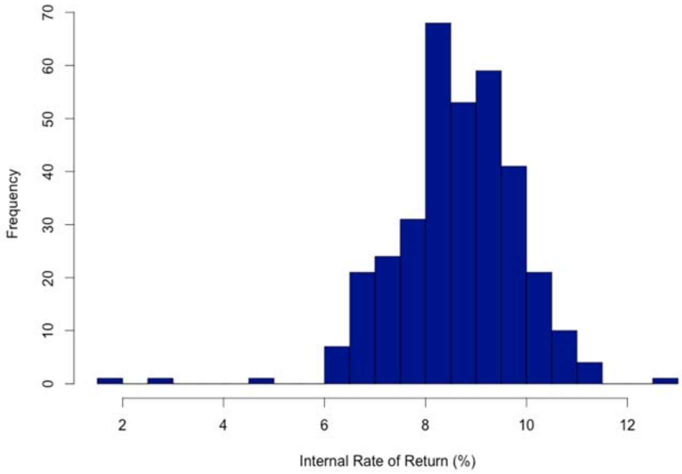
	Life settlement spread (certain death time)	Life settlement spread (uncertain death time)	Life settlement spread (uncertain death time, certain mortality improvement)

Note: Heteroskedasticity-consistent standard errors are reported in parentheses. The asterisks ***, **, and * indicate significance level of 1%, 5%, and 10% respectively.

Table 3: Correlation

	IRRLE spread	Age	Gender	Policy Year	Underwriting information	Rating
IRRLE spread						
Age	0.17***					
Gender	0.11**	-0.28***				
Policy Year	-0.16***	0.03	0			
Underwriting information	-0.17***	0.48***	-0.24***	-0.17***		
Rating	-0.03	0.14**	-0.09	0	0.06	
Log policy size	-0.05	0.18***	-0.06	-0.12**	-0.01	0.09*

Note: The asterisks ***, **, and * indicate significance level of 1%, 5%, and 10% respectively.



Mean: 8.62%

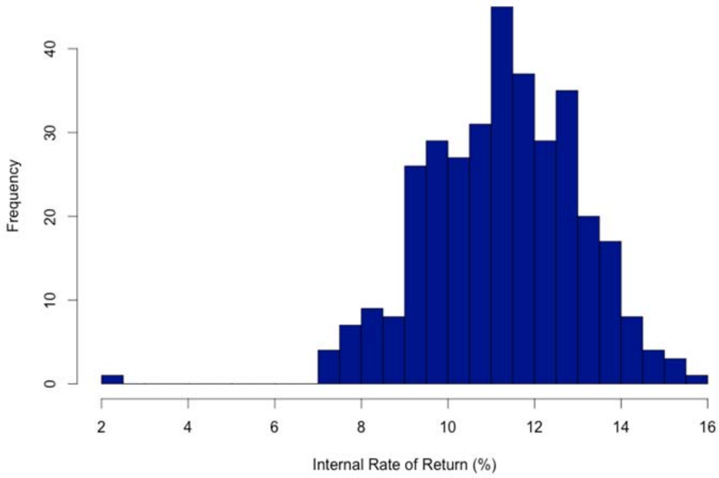
Median: 8.70%

Std. Dev.: 1.21%

Skewness: -0.91

Kurtosis: 4.22

Figure 1: Histogram of IRRs under certain death at life expectancy



Mean: 11.25%

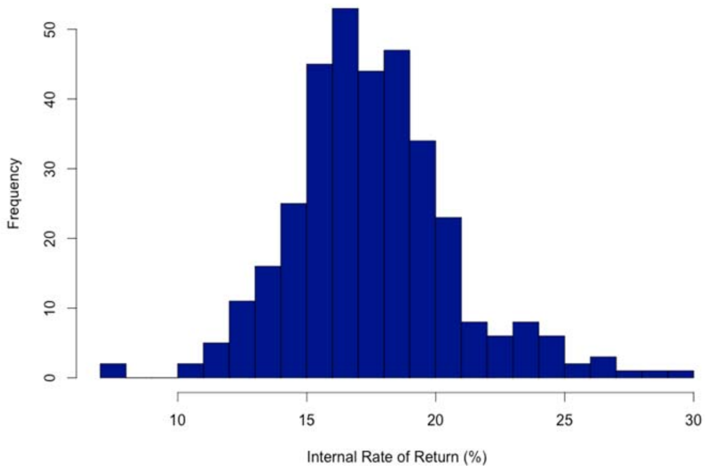
Median: 11.33%

Std. Dev.: 1.76%

Skewness: -0.43

Kurtosis: 1.33

Figure 2: Histogram of IRRs under uncertain death time



Mean: 17.56%

Median: 17.35%

Std. Dev.: 3.18%

Skewness: -0.58

Kurtosis: 1.56

Figure 3: Histogram of IRRs under probabilistic IRR

Empirical Tests on a Relational Model of Mortality Rates with Applications to Internal Hedging

Linus Fang-Shu Chan
Assistant Professor, Department of Financial Engineering and Actuarial Mathematics
Soochow University
No. 56, Section 1, Kuei-Yang Street, Taipei City, Taiwan 10048
(Tel) +886-2-2311-1531#3628; (Fax) +886-2-2381-2510; E-mail: linuschan@scu.edu.tw

Cary Chi-Liang Tsai (Corresponding Author)
Associate Professor, Department of Statistics and Actuarial Science
Simon Fraser University
8888 University Drive, Burnaby, BC V5A1S6, Canada
(Tel) +778-7827044; (Fax) +778-7824368; E-mail: cltsai@sfu.ca

Chenghsien Tsai*
Professor, Department of Risk Management and Insurance
Director, Risk and Insurance Research Center
National Chengchi University
No. 64, Section 2, ZhiNan Road, Taipei City, Taiwan 11605
(Tel) +886-2-2936-9647; (Fax) +886-2-2939-3864; E-mail: ctsai@nccu.edu.tw

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Empirical Tests on a Relational Model of Mortality Rates
with Applications to Internal Hedging

ABSTRACT

Modeling and forecasting mortality rates are crucial to life insurers, social benefits programs, and the society. There is a vast mass of literature on the methods to model and/or forecast mortality rates. Relational modeling (Brass, 1971; Tsai and Jiang, 2010) has merits, but its performance had not yet been compared with other types of models. To fill this gap, we use empirical data to test how a linear hazard transform (LHT) model compares with the renowned Lee-Carter and CBD models in terms of in-sample fitting and out-of-sample forecasting. We find that the LHT model produces the smallest errors on the data of US and UK that cover both genders from 1951-2007 for the people 25 years old or more.

Moreover, the proposed LHT model provides better ways of establishing mortality immunization strategies than Wang et al. (2010). It is more general and may give explicit formulas for mortality durations. The generality of our model further reveals the deficiency of internal hedging that has not yet been identified in the literature. This finding provides support for the development of mortality-linked assets.

Keywords: mortality rates, fitting, forecasting, hedging, duration

INTRODUCTION

Modeling the changes/dynamics of mortality rates is critical to the solvency of life insurers, social benefits programs, and the society as a whole. Mortality rates are one of the key factors in determining the premiums and reserves of life insurance and annuity products. Ignoring possible mortality rate changes can lead to significant under-pricing and under-reserving that may impair the profitability and solvency of a life insurer. Retirement programs and long-term care systems need to consider the dynamics of mortality rates as well since incomes and benefit outgoes depend on mortality rates. Under-estimating improvements in mortality rates can jeopardize the programs' solvency and continuity. Mortality rates are also a significant factor in shaping the population structure of a country that in turn affects the growth prospects of many industries. Therefore, modeling the dynamics of mortality rates is important.

Many scholars recognized the importance of mortality rate dynamics and developed various models to understand and/or forecast mortality rates. Demographers and sociologists developed explanatory models to understand which factors affected the mortality rates of certain populations with respect to age, gender, region, race, period, etc. (see Stallard (2006) and the references therein). Lee and Carter (1992) established a fitting and forecasting model in which one common factor drove mortality rate dynamics of all ages and a pair of age-indexed parameters differentiated the changes of age-specific mortality rates. Later extensions such as Renshaw and Haberman (2003) and Hyndman and Ullah (2007) identified more common factors. Another stream of literature has assumed specific functional forms for the mortality rate curve and established time-series models for the function parameters to project mortality rates (e.g., McNown and Rogers, 1989; Cairns, Blake, and Dowd, 2006a; Plat, 2009; Blackburn and Sherris, 2011). The aforementioned common-factor models and curve-fitting models could take the cohort effect into account (e.g., Renshaw and Haberman, 2006). A more recently developed stream of literature is continuous-time models. Some applied the idea of the term structure modeling on interest rates to the age structure of mortality rates (Dahl, 2004; Dahl and Møller, 2005; Biffis, 2005; Cairns, Blake, and Dowd, 2006b). Others, such as Biffis (2005) and Luciano and Vigna (2005), borrowed from credit risk modeling.

Another distinct method is called relational modeling. It hypothesizes the existence of a relation between two mortality rate tables/curves.[1] The relations might emerge across genders, sub-groups of populations, geographic areas, and time. For instance, Brass (1971)

[1] The term "mortality rates" in this paper is used in a broad way to convey the general concept of mortality and survival. Similarly, the term "mortality rate curves" encompasses survival probability curves.

assumed that two mortality rate tables could be related to each other linearly in terms of their survival probabilities. Later extensions added more parameters to allow for the bends in survival functions (e.g., Zaba, 1979; Ewbank, De Leon and Stoto, 1983) or inserted age-specific terms to capture deviations from linearity (Murray et al., 2003). The above papers analyzed mainly the relations of mortality tables across regions. Tsai and Jiang (2010) focused on the relations across time, on the other hand. They assumed that the forces of mortality of two mortality sequences could be modeled by a linear relation. Then they tried fitting and forecasting on the 1980 and 2001 CSO tables.

Relational modeling has its merits. It starts from a mortality rate curve that contains information on how the mortality rates of different ages relate to each other. These relations may result from biological reasons (e.g., new-born babies have higher mortality rates; mortality rates increase with ages for adults) or social reasons (e.g., the speed driving by young adults leads to higher mortality rates). Common-factor models did not fully capture such information. The second step of relational modeling specifies a relation between two curves. For instance, mortality rates on the curve of a later year can be regarded as a transformation of those on the curve of an earlier year. The justification for such specifications is that mortality rate curves move slowly with small changes and shift stably in terms of shape. Small and stable changes might result from biological constraints and/or the rigidity of social changes (e.g., health care systems, living habits, medical technology improvements, and medicine inventions).

We are the first to empirically assess the fitting and forecasting capabilities of a relational model relative to other types of models. More specifically, we assume that there exists a linear relation, called linear hazard transform (LHT), between the forces of mortality (hazard rates) of two curves. Then we estimate the parameters of the LHT using the empirical data of US and UK that cover both genders from 1950 to 2007. To evaluate the performance of LHT model, we choose the well-known Lee-Carter model (Lee and Carter, 1992) and CBD model (Cairns, Blake and Dowd, 2006a) as benchmarks and conduct in-sample fitting and out-of-sample forecasting comparisons. Both models are good representatives of their types and have been compared extensively in extant research (e.g., Lee and Miller, 2001; Brouhns, Denuit and Vermunt, 2002; Czado, Delwarde and Denuit, 2005; Booth, Tickle and Smith, 2005; Cairns et al., 2009). We find no papers that reported empirical comparison of relational modeling with other types of models.

The second contribution of this paper is applying our LHT model to constructing the insurance portfolios that are to be immunized from mortality rate risk. Our LHT model requires merely two parameters to depict the dynamics of the entire mortality rate curve.

This feature enables us to use the durations with respect to these two parameters to construct a portfolio immunized from sophisticated changes of mortality rate curves. Our LHT model is more general than that of Wang et al. (2010) which assumed that the force of mortality is constant within each age interval and changes proportionally. Their method is indeed a special case of ours: setting one of the two parameters of our LHT model to zero. Another advantage of our method is that it may render explicit formulas for mortality durations, which makes risk management easier and more accurate.

Statistical comparisons show that LHT produces the smallest fitting and forecasting errors. In fitting tests, it yields 71.79% and 70.49% lower RMSEs (root of mean square error) and MAEs (mean absolute error) on average than the Lee-Carter model on the US data and 55.42% and 58.38% lower errors on the UK data. Our mean fitting errors are 87.22% and 86.65% lower than those of CBD in US and 85.54% and 84.95% lower in UK. The median, standard deviation, minimum and maximum of our fitting errors across the sampling periods are smaller than those of the two benchmark models (ranging from 42.22% to 94.69%) as well.

In forecasting tests, the average errors of LHT during the forecasting period are 12.80% (RMSE) and 14.91% (MAE) of Lee-Carter's on the US data and 33.15% (RMSE) and 31.36% (MAE) in UK. LHT's mean forecasting errors are lower than those of CBD by 93.00% (RMSE) and 92.48% (MAE) in US and 84.64% (RMSE) and 83.83% (MAE) in UK. All other statistics of LHT's forecasting errors are also smaller than those of the benchmark models. The good performance of LHT is robust across genders, periods, and the sub-groups of ages larger than 45.

We then derive and calculate the dollar durations of the reserves at inception with respect to the two parameters of the LHT model for ten life insurance and annuity products to quantify their mortality rate risks. We find that the reserves of these products are more sensitive to the shocks from the parallel shifts of force-of-mortality curves than to the proportional changes. This fact highlights the importance of our extension to Wang et al. (2010) since their immunization strategies were based on the assumptions that the force of mortality is constant within each age interval and shifts proportionally. We further construct some portfolios that are immunized from the risks of both types of shifts in mortality rate curves. We find that natural/internal hedging might be infeasible due to the close relations among premiums/reserves of insurance and annuity products. This finding is new to the literature and appeals for the development of mortality-linked assets.

The remainder of this paper is organized as follows. Section 2 specifies the relations between two mortality rate curves. Section 3 delineates how we conduct statistical tests to compare the LHT model with the two benchmarks. Section 4 illustrates how our model can generate mortality immunization strategies and reveal the deficiency of internal hedging. Section 5 summarizes and concludes the paper.

RELATIONS BETWEEN TWO MORTALITY RATE CURVES

We regard the changes of mortality rates across time as transformations from one curve to another. More specifically, we assume that there is a linear relation (plus an error term) between the forces of mortality (i.e., hazard rates) of two mortality rate curves for years A and $B = A + a$, where $a \in N$. The mathematical form is:

$$\mu_{x,n}^{B}(t) = (1+\alpha_{x,n}^{A,B}) \times \mu_{x,n}^{A}(t) + \beta_{x,n}^{A,B} + \varepsilon_{x,n}^{A,B}(t), \quad t \in [0, n], \tag{1}$$

where $\mu_{x,n}^{A}$ and $\mu_{x,n}^{B}$ denote the forces of mortality for years A and B respectively, x indicates the starting age of the mortality rate curve to be studied, $n = \omega - x$, ω represents the end age of the studied rate curve, $\alpha_{x,n}^{A,B}$ and $\beta_{x,n}^{A,B}$ are constants to be estimated, and $\varepsilon_{x,n}^{A,B}$ is the error term. Year A is called the base year in the following while year B is called the target year.

Casual observations seem to support the linear assumption. Figure 1 plots the relations between $\mu_{45,64}^{A}(t)$ and $\mu_{45,64}^{A+1}(t)$ with A = 1950 and 2000 for the females of UK and US, respectively.[2] It renders preliminary support for the reasonableness of the linear assumption.

[Insert Figure 1 Here]

Since the k-year survival probability ${}_k p_x = e^{-\int_0^k \mu_x(s)ds}$, Equation (1) implies the following relation between ${}_k p_x^A$ and ${}_k p_x^B$:

$${}_k p_x^B = e^{-\int_0^k \mu_{x,n}^{B}(t)dt} = e^{-(1+\alpha_{x,n}^{A,B})\times\int_0^k \mu_{x,n}^{A}(t)dt - \int_0^k \beta_{x,n}^{A,B}dt - \int_0^k \varepsilon_{x,n}^{A,B}(t)dt} = [{}_k p_x^A]^{1+\alpha_{x,n}^{A,B}} \times e^{-\beta_{x,n}^{A,B}\times k} \times e^{-\int_0^k \varepsilon_{x,n}^{A,B}(t)dt}. \tag{2}$$

Taking the natural logarithm on both sides yields:

$$(-\ln {}_k p_x^B) = (1+\alpha_{x,n}^{A,B}) \times (-\ln {}_k p_x^A) + \beta_{x,n}^{A,B} \times k + \int_0^k \varepsilon_{x,n}^{A,B}(t)dt. \tag{3}$$

[2] $\mu_{45,64}^{A}(t)$ is derived by $\mu_{45,64}(t) = -\ln(1-q_{45+t})$, $t \in [0, 64]$.

Then we may estimate the parameter pair ($\alpha_{x,n}^{A,B}, \beta_{x,n}^{A,B}$) using regular regression analysis that

minimizes the sum of squares of integrated errors $\sum_{k=1}^{n}\left[\int_0^k \varepsilon_{x,n}^{A,B}(t)dt\right]^2$ on the data set

$\{(-\ln {}_k p_x^A, -\ln {}_k p_x^B)$: k=1, 2,…, n}.

We can grasp the economic meaning of the transformation as well as the meanings of α and β from Equation (1).[3] The transformation decomposes the changes in the forces of mortality from the curve of an earlier year to the curve of a later year into two components: a proportional change reflected by α and a parallel shift determined by β. Assuming $\beta = 0$ implies that the curve shifts proportionally. In this case, higher forces of mortality have larger improvements or deteriorations (depending on whether α is negative or positive). This type of curve changing behavior is also called the proportional hazard transform. Assuming $\alpha = 0$ corresponds to the case of a parallel shift of the force-of-mortality curve.

STATISTICAL TESTS

Data, Benchmarks, and Measures

We draw historical mortality rates q_x from the Human Mortality Database (HMD).[4] The drawn data cover both genders of US and UK, the countries that have probably been studied the most. The sampling period is from 1950 to 2007, a few years after the World War II to the most recent year for which data are available.[5]

Since the majority of the persons purchasing life insurance and annuities are young adults and older people, we focus on the mortality rates of ages 25 and above. More specifically, we test the LHT model on the 25+ section of mortality rate curves.[6] We further set $\omega = 109$ to avoid abnormal disruptions on the curves between ages 109 and 110 caused by setting p_{110} to be 0 in the original US and UK mortality tables.

We choose two well-known models as the benchmarks to be compared with the LHT

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[3] We omit the superscripts and/or subscripts of the parameters whenever the omission causes no confusions.
[4] Many researches such as Steinsaltz and Wachter (2006), Bhaskaran et al. (2008), Wang et al. (2010), and Vaupel (2010) obtained morality rate data from the HMD as well.
[5] As of March of 2012, the most recent mortality rates available for US are those of 2007 while the data of UK are updated to 2009. We preferred the same length of history for both countries.
[6] We also test two other sections, 35+ and 45+. The results are consistent with those from the 25+ section, which may also be inferred from Tables 4 and 8. We do not present these results for the sake of brevity.

model: the Lee-Carter model and the CBD model. The Lee-Carter model is a one-factor, linear model assuming that:

$$\log q_{x,A} = a_x + b_x K_A + \varepsilon_{x,A}, \tag{4}$$

where $q_{x,A}$ denotes the one-year death rate of age x in year A, a_x and b_x are age-specific parameters, K_A represents the time-varying factor of mortality rates, and $\varepsilon_{x,A}$ indicates the fitting error associated with age x in year A.[7] We follow Lee and Carter (1992) to estimate and forecast the parameters.

The other benchmark that we choose is the CBD model that became popular recently. The model specification is:

$$\text{logit}\, q_{x,A} = K_A^{(1)} + K_A^{(2)}(x - \bar{x}) + \varepsilon_{x,A}, \tag{5}$$

where $\text{logit}\, q_{x,A} = q_{x,A} / (1 - q_{x,A})$, and both $K_A^{(1)}$ and $K_A^{(2)}$ are modeled as random walks with drifts (Cairns et al., 2009).

We adopt two accuracy measures, RMSE and MAE. Their definitions are:

$$RMSE = \frac{1}{T} \sum_{A=A_1+1}^{A_1+T} \sqrt{\frac{1}{2(\omega - x_I)} \sum_{s \in \{m,f\}} \sum_{x=x_I}^{\omega-1} (q_{s,x,A} - \hat{q}_{s,x,A})^2} \quad \text{and}$$

$$MAE = \frac{1}{2T(\omega - x_I)} \sum_{A=A_1+1}^{A_1+T} \sum_{s \in \{m,f\}} \sum_{x=x_I}^{\omega-1} \left| q_{s,x,A} - \hat{q}_{s,x,A} \right|,$$

where $q_{s,x,A}$ indicates the observed one-year death rate for gender s (*m* and *f* denote male and female, respectively) and age x in year A, $\hat{q}_{s,x,A}$ represents the fitted/forecasted value, and x_I is the starting age of the section on mortality rate curves. These two measures have been used in the literature (e.g., Carter, 1996; Gakidou and King, 2006).

In-Sample Fitting

[7] The original model in Lee and Carter (1992) is $\log m_{x,A} = a_x + b_x K_A$, where $m_{x,A}$ is the central death rate of age x in year A. We substitute the one-year death rate for the central death rate to ensure that the LHT and Lee-Carter models use the same raw data and avoid the disturbances from the differences between m_x and q_x.

In-sample fitting is done by fitting Equation (3) using two mortality rate curves. We first draw the q_x of two different years from our dataset, and calculate two sets of corresponding ${}_k p_x$: ${}_k p_{x,A}$ and ${}_k p_{x,B}$. Taking the natural log of these ${}_k p_x$ and then running the regular regression analysis on Equation (3) yields $\hat{\alpha}$ and $\hat{\beta}$. Combining the estimated $\hat{\alpha}$ and $\hat{\beta}$ with the input ${}_k p_{x,A}$ gives us ${}_k \hat{p}_{x,B} = [{}_k p_{x,A}]^{1+\hat{\alpha}} \times e^{-\hat{\beta}\times k}$. We compute the corresponding $\hat{q}_{x,B} = 1 - \hat{p}_{x,B}$ and then RMSE and MAE to measure the fitting errors. Repeating the steps for A = 1950-2006 with $a = 1$, we obtain the following table.[8]

[Insert Table 1 here]

Table 1 shows that LHT produces the smallest errors in in-sample tests. In both US and UK, LHT generates smaller mean, median, standard deviation, minimum, and maximum of fitting errors than the two benchmark models do. For instance, Table 1a exhibits that LHT's RMSE averaged across ages, sampled years, and genders is 28.21% of Lee-Carter's and 12.78% of CBD's in US and 44.58% and 14.46% of theirs in UK. The improvements of LHT upon the benchmark models with respect to the median RMSE of fitting errors are similar to those in the mean errors. In addition, the standard deviations of LHT's RMSEs are 22.38% and 22.39% of Lee-Carter's and CBD's in US and 30.95% and 17.61% in UK. LHT also performs well on both ends of error distributions. For example, the minimum RMSEs of LHT are 52.11% and 5.31% of Lee-Carter's and CBD's in US and 103.89% and 9.20% in UK. LHT's maximum RMSEs are 29.40% and 35.43% of Lee-Carter's and 19.52% and 21.48% of CBD's for US and UK data, respectively. Table 1b further demonstrates that LHT outperforms the benchmarks to a similar extent when MAE is used as the fitness measure.

Table 2 indicates that LHT renders better fitting than the two benchmark models for both genders among US and UK adults. The LHT model produces mean RMSEs that are 30.11% and 25.89% of those by the Lee-Carter model for US males and females, respectively. The mean fitting errors in UK are also smaller: 42.41% for males and 48.52% for females. Compared with the CBD model, LHT's RMSEs are 15.01%, 10.55%, 14.41%, and 14.53% of CBD's for US males, US females, UK males and UK females respectively. Table 2b displays that the LHT model outperforms the benchmark models to a similar extent when we

[8] The sample size T in estimating the LHT model is thus equal to 2006-1950+1 = 57. On the other hand, the sample size used to estimate the Lee-Carter and CBD models is 58 since the estimation is done upon single-year data rather than the data of two-year differences.

switch the fitness measure to MAE.

[Insert Table 2 Here]

Table 3 shows that the LHT model renders better fitting than both benchmark models in all sub-periods of the sampling period in US and UK with respect to the two accuracy measures. The fitting errors averaged over genders and ages during each decade of the sampling period produced by LTH are smaller than those by Lee-Carter and CBD. For instance, its mean RMSEs for US males during the decades of 50s, 60s, 70s, 80s, 90s, and 00s are 57.88%, 50.60%, 31.04%, 18.53%, 24.40%, and 15.42% respectively of the Lee-Carter's. The ratios are 40.23%, 38.92%, 37.64%, 40.21%, 39.37%, and 34.47% for UK females in terms of MAE when compared with the CBD model.

[Insert Table 3 Here]

The better performance of LHT than the benchmark models can be attributed to the better fitting upon the populations of 45 years old or more. Table 4 demonstrates that LHT's fitting errors are smaller than Lee-Carter's and CBD's for the age groups of 45-64, 65-74, 75-84, and 85-109 with respect to genders, countries, and accuracy measures.[9] For instance, the ratios of LHT's fitting errors to Lee-Carter's for US males in terms of RMSE for these age groups are 88.70%, 73.17%, 81.13%, and 25.88% respectively. The ratios to CBD are 61.82%, 55.57%, 62.15%, and 32.98% for UK females in terms of MAE. On the other hand, both Lee-Carter and CBD models outperform LHT for the age groups of 24-34 and 35-44. Since LHT provide better fitting for most age groups (45-109) than the benchmark models do, its overall performance is better as shown by previous tables.

[Insert Table 4 Here]

Table 4 implies that LHT will perform better than Lee-Carter and CBD in the sub-sections of mortality rate curves with ages greater than 25. This is confirmed by replicating Tables 1-3 using the data of age groups 35+ and 45+.[10] The superiority of LHT in the 35+ and 45+ sections of the mortality rate curve is as good as, if not better than, that in the 25+ section.

Out-of-Sample Forecasting

For simplicity as well as following Lee-Carter (1992), Nelson and Siegel (1987), and Cairns et al. (2009), we assume that the dynamics of the two parameters in Equation (3) follow the random walk with a drift individually. More specifically, we assume that:

[9] There is one exception: the 45-64 age group of UK males when measured by MAE. LHT's MAE is 103.1% of the Lee-Carter's.

[10] We do not present the replicated tables for the sake of brevity.

$$\gamma_{x,n}^{A,A+1} - \gamma_{x,n}^{A-1,A} = \Delta\gamma_{x,n}^{A} = \overline{\Delta\gamma_{x,n}} + \varepsilon_{x,n}^{\gamma},$$

where $\overline{\Delta\gamma_{x,n}}$ denote the long-term mean change (i.e., drift) of $\gamma_{x,n}^{A,A+1}$, $\gamma = \alpha$ or β, and

$\varepsilon_{x,n}^{\gamma} \sim N(0, \sigma_{x,n}^{\gamma})$. This set of assumptions are validated by the Dickey-Fuller tests in which

we found no unit roots in the time series of $\hat{\alpha}$ and $\hat{\beta}$.

We estimate the drifts using the *F*-year periods of data prior to the "current" year *A* upon which the projection would be made. For instance, when we have mortality data up to 1989 (i.e., $A = 1989$) and want to make projections for 1990, we will use the period from $(1989 - F)$ to 1989 to estimate the drifts. Our estimators for the drifts are simply the averages of changes in parameter values over the corresponding *F*-year period:

$$\overline{\Delta\hat{\gamma}}_{x,n}^{A-1,F} = \frac{1}{F-1} \sum_{i=A-F+1}^{A-1} \Delta\hat{\gamma}_{x,n}^{i} \quad , \tag{6}$$

where $\Delta\hat{\gamma}_{x,n}^{i} = \hat{\gamma}_{x,n}^{i,i+1} - \hat{\gamma}_{x,n}^{i-1,i}$ and the pairs $(\hat{\gamma}_{x,n}^{i-1,i}, \gamma_{x,n}^{i,i+1})$ are estimated in in-sample fitting.

We set $F = 40$ for out-of-sample forecasting tests after taking into account the tradeoff between sufficiency of the in-sample size and the number of out-of-sample tests.

The projected parameter $\tilde{\gamma}_{x,n}^{A,A+1}$ is assumed to satisfy:

$$\tilde{\gamma}_{x,n}^{A,A+1} = \hat{\gamma}_{x,n}^{A-1,A} + \overline{\Delta\hat{\gamma}}_{x,n}^{A-1,F} \quad . \tag{7}$$ [11]

Plugging the projected parameters into Equation (3) to the mortality rates of year A would yield the projected mortality rates of the person aged *x* in year *A*+1 (i.e., ${}_k\tilde{p}_x^{A+1}$). Then we calculate RMSE and MAE in the same way as in the in-sample fitting section to measure the forecasting errors. Repeating the above procedures for $A = 1990 - 2007$ produces the following table.

[Insert Tables 5 Here]

Table 5 displays that the out-of-sample forecasting errors of LHT are smaller than those of the Lee-Carter and CBD models in both cases of US and UK. For instance, the mean RMSEs of LHT as shown in Table 5a are 12.80% (US) and 33.15% (UK) of Lee-Carter's and 7.00% (US) and 15.36% (UK) of the CBD's. LHT has lower error variations as well. For example, Table 5b indicates that the standard deviations of MAEs

[11] We use the top script $\tilde{}$ to indicate a projected value, $\hat{}$ to denote an estimated value, and $\bar{}$ for an averaged value.

associated with LHT are 0.000239 and 0.000385 for US and UK respectively. They are 11.11% and 20.57% of Lee-Carter's and 15.88% and 23.96% of CBD's. Other statistics in Table 5 also support the superiority of LHT to the benchmark models. One example is the smaller maximum RMSE produced by LHT, 0.001872, compared to 0.015735 by Lee-Carter and 0.020640 by CBD for US.

We further find that the superiority of LHT is more significant in forecasting than in in-sample fitting. The error ratios of LHT to Lee-Carter with respect to almost all statistics in terms of both accuracy measures are smaller in forecasting tests. This can be seen by comparing 12.80%, 33.15%, 7.00% and 15.36% (the ratios of LHT's RMSEs to benchmark models' as presented in the previous paragraph) with the corresponding 28.21%, 44.58%, 12.78% and 14.46% mentioned in the paragraph after Table 1. That the advantage of LHT over the benchmark models is more significant in forecasting has material practical implications.

For each gender in US and UK, LHT's forecasting errors are the smallest. Table 6a exhibits that the ratios of LHT's mean RMSEs to Lee-Carter's are 13.97%, 11.26%, 31.60% and 35.99% for US males, US females, UK males, and UK females, respectively. The corresponding ratios with respect to CBD are 9.01%, 5.14%, 17.43% and 12.90%. The better performance of LHT than Lee-Carter and CBD models remains intact when the accuracy measure changes to MAE. The mean MAEs generated by LHT are lower than those by Lee-Carter and CBD as Table 6b displays.

[Insert Tables 6 Here]

Table 7 shows that the good forecasting performance of LHT is robust across the decades of 1990s and 2000s. For instance, the ratios of LHT's mean forecasting RMSEs to Lee-Carter's for females are 11.00% (US) and 24.54% (UK) in 1990s and 8.81% and 48.84% in 2000s. When compared with CBD in terms of mean MAE, the ratios for males are 10.62% (US) and 17.18% (UK) in 1990s and 8.25% and 20.04% in 2000s.

[Insert Table 7 Here]

Table 8 tells a story similar to Table 4. LHT performs well for ages 45+ while Lee-Carter and CBD perform better for ages 25-44. For instance, the ratios of LHT's mean RMSEs to Lee-Carter's for the age group of 85-109 are 11.69% (US males), 8.84% (US females), 24.41% (UK males), and 26.81% (UK females). On the other hand, the corresponding ratios for the age group of 25-34 are 128.57%, 347.92%, 367.21%, and 560.42% respectively. The mean MAEs of LHT are smaller than those of CBD for US and UK adults aged 45 years old or more, but the relative performance reverses for the age groups of 25-34 and 35-44. LHT's overall forecasting performance is better than the benchmark models, as we have seen from Table 5, due to its better performance for more age groups. The relative advantage of LHT in the age groups of 45 and above has practical implications since the people in these age groups are the major customers of life insurers.

[Insert Table 8 Here]

RISK MANAGEMENT

One of the major usages of mortality modeling/projection for life insurers is to manage mortality rate risk. Such management might involve constructing internally/naturally hedged portfolios of life insurance and annuity products so that reserves will not deviate from the expected to a significant extent.[12] We illustrate in this section how our LHT modeling has advantages over the models proposed in the existing literature to establish hedged portfolios.

Mortality Durations

We may regard the LHT model as a two-parameter model of mortality improvement or deterioration. Mortality risks can thus be measured and managed by the "durations" with respect to parameters α and β.[13] More specifically, the sensitivity of a policy's reserves R_* based on the adjusted force of mortality $\mu_{x^*}(t) = (1+\alpha)\times\mu_x(t)+\beta$ (see Tsai and Jiang, 2011) to a change in α or β can be defined as

$$DD_\gamma(R_*) = -\frac{\partial R_*}{\partial \gamma}, \tag{8}$$

where *DD* denotes dollar duration and $\gamma=\alpha$ or β.[14] DD measures the change of the reserves caused by the change of a mortality parameter. It can also be deemed as the slope of the reserve-to-mortality-parameter curve with the opposite sign. Most models presented in the literature have many more parameters. Life insurers hence cannot use these models for mortality duration management since there will be many durations to match, which

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12 The conventional way to manage mortality rate risk is reinsurance. Alternative ways involve asset-side products/derivatives linked to mortalities. Few products are currently available though.

13 The idea is the same as the duration management for interest rate risk. Many financial institutions, especially banks and life insurers, calculate the interest rate durations for individual assets and liabilities to measure their exposures to interest rate risk. The use of interest rate duration in financial markets is extensive (Bierwag and Fooladi, 2006). We borrow the idea of duration management but substitute interest rate for mortality rate as the underlying risk factor.

14 Another popular risk measure is modified duration (MD) defined as: $MD_\gamma(R_*) = -\frac{\partial R_*}{\partial \gamma}\times\frac{1}{R_*} = \frac{DD_\gamma(R_*)}{R_*}$.

The economic meaning of MD is the percent change of reserves caused by the change of a mortality parameter. DD is more suitable for life insurance since it avoids irregularities caused by small or zero reserves (Tsai, 2009).

demonstrates one advantage of our LHT modeling.

Under the LHT model, mortality durations may have explicit formulas that facilitate risk management. Mortality rates of a future year are a function of the current-year mortality rates with parameters of α and β. Since appropriate reserves determined today should take into account the expected changes in mortalities, reserves are functions of α and β as well. The partial derivatives of reserves with respect to α and β may have explicit formulas so that we can derive closed-form solutions for mortality durations.

In the following we define and derive mortality durations of the premiums and reserves at inception for several products including n-year temporary life annuities-due, endowment, and term life insurance as illustrations. Denote the net single premium of an m-year deferred, n-year temporary life annuity-due issued to an individual aged x by:

$$ {}_{m|}\ddot{a}_{x:\overline{n}|} = {}_{m|}\ddot{a}_{x:\overline{n}|(\mu_x+\delta)} = \sum_{k=m}^{m+n-1} {}_k p_x v^k = \sum_{k=m}^{m+n-1} e^{-\int_0^k [\mu_x(t)+\delta]dt}, \tag{9} $$

where $v = 1/(1+i)$ and $\delta = \ln(1+i)$. Note that $m = 0$ (the case of n-year temporary life annuity-due), $n = 1$ (the case of m-year pure endowment) and $n = \omega - x - m$ (the case of m-year deferred whole life annuity-due) are three commonly seen products. When the force of mortality μ_x changes proportionally to $(1+\alpha)\mu_x$, the underlying curve becomes

$(1+\alpha)\times\mu_x+\delta$ and the associated net single premium is:

$$ {}_{m|}\ddot{a}_{x:\overline{n}|((1+\alpha)\times\mu_x+\delta)} = \sum_{k=m}^{m+n-1} e^{-\int_0^k [(1+\alpha)\times\mu_x(t)+\delta]dt}. \tag{10} $$

We can then define the dollar duration of the net single premium for an m-year deferred, n-year temporary life annuity-due with respect to α by:

$$ DD_\alpha({}_{m|}\ddot{a}_{x:\overline{n}|}) = -\lim_{\alpha\to 0}\frac{{}_{m|}\ddot{a}_{x:\overline{n}|((1+\alpha)\times\mu_x+\delta)} - {}_{m|}\ddot{a}_{x:\overline{n}|(\mu_x+\delta)}}{\alpha} = \sum_{k=m}^{m+n-1}(-\ln {}_k p_x)\times {}_k p_x \times v^k. \tag{11} $$

Similarly, we define the dollar duration with respect to β by:

$$ DD_\beta({}_{m|}\ddot{a}_{x:\overline{n}|}) = -\lim_{\beta\to 0}\frac{{}_{m|}\ddot{a}_{x:\overline{n}|(\mu_x+\beta+\delta)} - {}_{m|}\ddot{a}_{x:\overline{n}|(\mu_x+\delta)}}{\beta} = \sum_{k=m}^{m+n-1} k\times {}_k p_x \times v^k. \tag{12} $$

Assume that the death benefit is payable at the end of the death year. Denote the net single premium of an n-year endowment policy as:

$$ A_{x:\overline{n}|} = \sum_{k=0}^{n-1}({}_k p_x - {}_{k+1}p_x)\times v^{k+1} + {}_n p_x \times v^n = 1 - d\times \ddot{a}_{x:\overline{n}|}, \tag{13} $$

where $d = 1 - v$. This implies $DD_\gamma(A_{x:\overline{n}|}) = -d\times DD_\gamma(\ddot{a}_{x:\overline{n}|})$, $\gamma = \alpha$ or β. Setting

$n=\omega-x$ yields $DD_\gamma(A_x)=-d\times DD_\gamma(\ddot{a}_x)$ that represents the relation between the dollar durations of whole life insurance and whole life annuity-due. For the n-year term life insurance, we have $DD_\gamma(A^1_{x:\overline{n}|})=-d\times DD_\gamma(\ddot{a}_{x:\overline{n}|})-DD_\gamma({}_n|\ddot{a}_{x:\overline{1}|})$ since its net single premium is $A^1_{x:\overline{n}|}=A_{x:\overline{n}|}-{}_n|\ddot{a}_{x:\overline{1}|}$. The above durations of premiums can help life insurers to assess the sensitivity of premiums to mortality rate variations.

For the level h-payment, m-year deferred $(m\geq h\geq 1)$, and n-year temporary life annuity-due, the dollar duration of its reserves at time 0 is

$$DD_\gamma({}_0R({}_hP({}_m|\ddot{a}_{x:\overline{n}|})))=DD_\gamma({}_m|\ddot{a}_{x:\overline{n}|})-{}_hP({}_m|\ddot{a}_{x:\overline{n}|})\times DD_\gamma(\ddot{a}_{x:\overline{h}|}), \tag{14}$$

where ${}_hP({}_m|\ddot{a}_{x:\overline{n}|})={}_m|\ddot{a}_{x:\overline{n}|}/\ddot{a}_{x:\overline{h}|}$. Note that for the case of $h=1$ (single premium), $DD_\gamma(\ddot{a}_{x:\overline{1}|})=0$ and Equation (14) reduces to (11) and (12). Similarly, the dollar durations of the reserves at time 0 for the level h-payment, n-year endowment and term life insurance $(n\geq h\geq 1)$ can be derived respectively as:

$$DD_\gamma({}_0R({}_hP(A_{x:\overline{n}|})))=DD_\gamma(A_{x:\overline{n}|})-{}_hP(A_{x:\overline{n}|})\times DD_\gamma(\ddot{a}_{x:\overline{h}|}), \tag{15}$$

and

$$DD_\gamma({}_0R({}_hP(A^1_{x:\overline{n}|})))=DD_\gamma(A^1_{x:\overline{n}|})-{}_hP(A^1_{x:\overline{n}|})\times DD_\gamma(\ddot{a}_{x:\overline{h}|}), \tag{16}$$

where ${}_hP(A_{x:\overline{n}|})=A_{x:\overline{n}|}/\ddot{a}_{x:\overline{h}|}$ and ${}_hP(A^1_{x:\overline{n}|})=A^1_{x:\overline{n}|}/\ddot{a}_{x:\overline{h}|}$.

We give some numerical illustrations in the following. Assume that $i=3\%$, $x=45$, and ${}_kp_x$ $(k=1,2,\cdots,n=20)$ represent the out-of-sample forecasting mortality rates when using the LHT model with $A=2007$ and $F=40$. We calculate DD_α and DD_β of the reserves at time 0 for 10 products. The results are as shown in Table 9.

[Insert Table 9 Here]

We find that annuity products and pure endowment have positive mortality durations, implying exposures to longevity risk. For instance, the (DD_α, DD_β) of the reserves at time

0 for the single-payment, 20-year deferred whole life annuity-due and the 20-payment, 20-year pure endowment are (3.16, 183.86) and (0.06, 5.48) respectively. The whole life insurance, on the other hand, has negative mortality durations that implies exposures to mortality deterioration risk.[15] For example, the DD_α and DD_β of the reserves at time 0 for the 20-payment whole life insurance are -0.13 and -12.11 respectively.

The DD_β s of the reserves for the life insurance and annuity products have much larger magnitudes than DD_α s, which means that the reserves are more sensitive to shocks/changes of β (parallel shift of force-of-mortality curves) than to those of α (proportional shift). This fact highlights the importance of our extension to Wang et al. (2010) since their immunization strategies are based on the assumption that the force of mortality is constant within each age interval and moves proportionally and thus utilize DD_α only.

The reserves of the whole-life annuities have the largest mortality duration figures, (3.16, 183.86) for single-payment and (2.84, 131.43) for 20-payment. This implies that whole-life annuities have the largest mortality rate risk. The term life insurance has the second largest sensitivity to the parallel shifts of the forces of mortality ($DD_\beta =$ -12.86 and -13.76 for single-payment and 20-payment, respectively), while the whole life insurance has the second largest sensitivity to the proportional shifts ($DD_\alpha =$ -0.11 and -0.13). The endowment is the least sensitive to the proportional shifts ($DD_\alpha =$ -0.02 and -0.05).

Internal Hedging

After calculating the mortality durations of reserves for several life insurance and annuity products, we can now measure the mortality rate risk of the portfolios consisting of these products and construct the portfolios with minimal mortality rate risks. The reserve duration of a portfolio with respect to a model parameter is simply the weighted average of

[15] Following Wang et al. (2010), we define that a product is subject to longevity risk if mortality improvements would increase the reserves of that product. A product is subject to mortality deterioration risk if increases in mortality rates would raise the product's reserves. The term "mortality rate risk" is used as a general term referring to the risks of reserves subject to changes in mortality rates.

the mortality durations of individual products' reserves. By combining products in different ways, we may find a portfolio with zero mortality durations. This means that the portfolio is "immunized" from mortality rate risk. Managing mortality rate risk by carefully constructing product portfolios is called the natural hedging strategy in the literature (e.g., Cox and Lin, 2007; Wang et al., 2010).

For the purpose of demonstration, we construct some portfolios that have minimal exposure to mortality rate risk with regard to the reserves at time 0, i.e., the portfolios having zero DD_α and DD_β. Such portfolios are composed of at least three products since we have three equations as constraints to be solved:

$$\begin{aligned} &\sum_{i=1}^{3} w_i \times DD_\alpha{}^i = 0, \\ &\sum_{i=1}^{3} w_i \times DD_\beta{}^i = 0, \\ &\sum_{i=1}^{3} w_i = 1, \text{ and} \\ &w_i > 0,\ i = 1,\ 2,\ 3. \end{aligned} \tag{17}$$

The solutions to equation system (17) are:

$$w_1 = \begin{vmatrix} DD_\alpha{}^2 & DD_\alpha{}^3 \\ DD_\beta{}^2 & DD_\beta{}^3 \end{vmatrix} / D, w_2 = \begin{vmatrix} DD_\alpha{}^3 & DD_\alpha{}^1 \\ DD_\beta{}^3 & DD_\beta{}^1 \end{vmatrix} / D, w_3 = \begin{vmatrix} DD_\alpha{}^1 & DD_\alpha{}^2 \\ DD_\beta{}^1 & DD_\beta{}^2 \end{vmatrix} / D\,, \tag{18}$$

where $D = \begin{vmatrix} DD_\alpha{}^2 & DD_\alpha{}^3 \\ DD_\beta{}^2 & DD_\beta{}^3 \end{vmatrix} + \begin{vmatrix} DD_\alpha{}^3 & DD_\alpha{}^1 \\ DD_\beta{}^3 & DD_\beta{}^1 \end{vmatrix} + \begin{vmatrix} DD_\alpha{}^1 & DD_\alpha{}^2 \\ DD_\beta{}^1 & DD_\beta{}^2 \end{vmatrix}$.

When generating an optimal portfolio, the key difference between the LHT model and Wang et al. (2010) lies in the calculations of mortality durations. Our method involves calculating two mortality durations while Wang et al. (2010) compute only one with an implicit assumption of how the mortality rate curve changes. They assumed that the force of mortality within each age interval $(x, x+1)$ is constant and is changed by a certain percentage (e.g., -10%). The resulting changes in reserves are then used to calculate the mortality duration (see Equation (11) with $\lim\limits_{\alpha \to 0}$ removed). Therefore, the numerical value of their mortality duration depends on the size and sign of the designated percentage change. Furthermore, the assumption of constant force of mortality is inconsistent with the empirical mortality data. Neither is the equal percentage change for all forces of mortality consistent with observed behaviors of mortality rate curves. Their assumption of a uniform -10%

change is indeed equivalent to $\alpha = -0.1$ (if DD_α would have been calculated with the method of finite differences) and ignores the parallel shifts of mortality rate curves under the LHT model. Therefore, our method is more general than theirs with the extra benefit of having explicit formulas for durations.

Immunization Illustrations

We formulate three portfolios with both DD_α and DD_β equal to 0 as displayed in Table 10. The weights are calculated using Equation (18). Portfolio 1 consists of life insurance products: whole life, term life, and pure endowment (all 20-payment). The pure endowment accounts for 69.51% while whole life insurance makes up 23.25%. The term life insurance accounts for only 7.24% of the portfolio. Portfolios 2 (all single-premium) and 3 (a mix of 20-payment and single-premium) are examples of natural hedging between life insurance and annuity products. The mortality deterioration risk of whole life insurance is hedged by the longevity risks of annuities and pure endowment. The whole life insurance accounts for about two thirds of Portfolios 2 and 3 (66.61% and 60.32% respectively), and pure endowment makes up thirty some percent (31.79% and 38.34%). The weights of annuities are low due to their large mortality durations. These compositions show the substitution effect between annuities and pure endowment in hedging mortality rate risk.

[Insert Table 10 Here]

Equation (17) might have no solutions, however. It can be proved that w_1, w_2, and w_3 fall within the interval (0,1) if and only if the three determinants in D are either all positive or all negative. The determinants might not have the same signs because of the close relations among the net single premiums of insurance and annuity products (e.g., Equation (13)) and the resulting linkages among mortality durations. We present two portfolios that have negative weights in Table 11. Portfolio 4 is formed by replacing the 20-payment, 20-year pure endowment of Portfolio 1 with the 20-payment, 20-year deferred whole life annuity-due. Portfolio 5 is constructed by substituting the single-premium, 20-year term life insurance for the single-premium whole life insurance in Portfolio 2. One would probably expect natural hedging to be feasible. However, the close relation $A_x = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\,1} - d \times {}_n|\ddot{a}_x$ (where $A_{x:\overline{n}|}^{\;\,1}$ is the net single premium of the *n*-year pure endowment) among the net single premiums of these three underlying products prevents all weights from being positive. The insurer has to "buy" rather than "sell" the product that has a negative weight in order to hedge mortality rate risk.

These examples did not show up in Wang et al. (2010) since they calculated one mortality duration only. In their settings, two products are enough to hedge the sole duration, and the weights of the products will be positive as long as the reserve durations of these two products have different signs. When the mortality rate curve changes in more complex ways and demands more than one parameter to model its dynamics, it takes at least three products for mortality immunization. All weights will be positive only when mortality durations meet the aforementioned necessary and sufficient conditions. Therefore, life insurers may not be able to internally hedge mortality rate risk to the full extent.

[Insert Table 11 Here]

External hedging arrangements are needed therefore. The negative weights shown up in the above examples mean that a life insurer may have to buy life insurance products from other issuers to reduce its exposure to mortality rate risk to a desired level. Life settlements seem to fit this demand. Other mortality securities like mortality bonds and derivatives may also help life insurers hedge mortality rate risk externally and complement internal hedging. The incompleteness of internal hedging found in this paper is new to the literature.

CONCLUSIONS AND REMARKS

Modeling and projecting mortality rates are vital to life insurers, social benefits programs, and the society. Extant literature contains extensive studies on mortality rates. Demographers and sociologists developed cross-sectional, explanatory models. Lee and Carter (1992) developed a one-factor model and stimulated later papers on factor models. Cairns, Blake and Dowd (2006a) represented another type of modeling that presumed a function for age-specific mortality rates. Some scholars applied interest rate and credit risk modeling methods developed in the finance field to mortality rates.

Relational modeling distinguishes itself from the above models in that it is based on an existing mortality rate curve with assumptions on how another curve is related to the existing one. Brass (1971) and the extensions employed this method to analyze the curves across regions while Tsai and Jiang (2010) applied it to the curves across time. One merit of relational modeling is that it takes full account of the information on how the mortality rates for different ages relate to each other by taking an existing curve as given. When applied to a sequence of curves, this methodology might be suitable because mortality rate curves change in small and stable ways due to biological factors and/or the rigidity of changes in social systems.

We do not come across any empirical assessment in the literature of how relational modeling performs relative to other types of models, despite its reasonableness and potential. To fill this gap, we assume that the force of mortality on the curve for a later year is a linear transformation of that on the curve for an earlier year and employ empirical data to compare the performance of our model with those of the well-known Lee-Carter and CBD models. We conduct both in-sample fitting and out-of-sample forecasting tests using the data of US and UK that covered both genders from 1950 to 2007. The test results show that LHT produces the smallest errors in both types of tests.

We then apply our model to construct the portfolios immunized from mortality rate risk. Since our model is parsimonious with parameters, we need to calculate only two durations (with respect to these two parameters) to construct an immunized portfolio consisting of three life insurance and/or annuity products. Our model is more general than Wang et al. (2010) that assumed the force of mortality within each age interval is constant and shifts proportionally. Furthermore, our model exposes the deficiency of internal hedging in achieving immunization. This finding is new to the literature and has significant implications to the mortality rate risk management of life insurers. It also supports the development of mortality-linked assets for life insurers to manage mortality rate risk.

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Table 1: Summary Statistics of In-Sample Fitting Errors

	1a: RMSE														
	Mean			Median			Std. Deviation			Min			Max		
Country	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US	**0.001510**	0.005351	0.011811	**0.001354**	0.004765	0.010690	**0.000677**	0.003026	0.003025	**0.000402**	0.000772	0.007571	**0.003777**	0.012847	0.019355
		(28.21%)	(12.78%)		(28.41%)	(12.66%)		(22.38%)	(22.39%)		(52.11%)	(5.31%)		(29.40%)	(19.52%)
UK	**0.002419**	0.005426	0.016732	**0.002218**	0.005061	0.014347	**0.000946**	0.003056	0.005370	0.000830	**0.000798**	0.009021	**0.005739**	0.016198	0.026723
		(44.58%)	(14.46%)		(43.82%)	(15.46%)		(30.95%)	(17.61%)		(103.89%)	(9.20%)		(35.43%)	(21.48%)
	1b: MAE														
	Mean			Median			Std. Deviation			Min			Max		
Country	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US	**0.000869**	0.002946	0.006509	**0.000824**	0.002742	0.005876	**0.000349**	0.001531	0.001791	**0.000285**	0.000493	0.004313	**0.001985**	0.006744	0.011020
		(29.51%)	(13.35%)		(30.06%)	(14.03%)		(22.80%)	(19.49%)		(57.78%)	(6.60%)		(29.44%)	(18.02%)
UK	**0.001301**	0.003125	0.008642	**0.001212**	0.002966	0.007844	**0.000505**	0.001698	0.002614	0.000499	**0.000478**	0.004404	**0.003064**	0.009558	0.013632
		(41.62%)	(15.05%)		(40.85%)	(15.45%)		(29.76%)	(19.33%)		(104.36%)	(11.33%)		(32.05%)	(22.47%)

Notes:

1. The mean, median, standard deviation, min, and max of RMSE and MAE are the statistics of $\left\{ \sqrt{\frac{1}{2\times(109-25+1)}\sum_{s\in\{m,f\}}\sum_{x=25}^{109}(q_{s,x,A}-\hat{q}_{s,x,A})^2} \,\middle|\, A=1951,\cdots,2007 \right\}$

and $\left\{ \sum_{s\in\{m,f\}}\sum_{x=25}^{109}\left|q_{s,x,A}-\hat{q}_{s,x,A}\right| \,\middle|\, A=1951,\cdots,2007 \right\}$ respectively.

2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 2: Fitting Errors by Genders

	2a: RMSE						2b: MAE					
	Male			Female			Male			Female		
Country	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD
US	**0.001776**	0.005897	0.011834	**0.001244**	0.004806	0.011788	**0.001037**	0.003310	0.006322	**0.000702**	0.002581	0.006697
		(30.11%)	(15.01%)		(25.89%)	(10.55%)		(31.32%)	(16.40%)		(27.19%)	(10.48%)
UK	**0.002968**	0.006997	0.020591	**0.001870**	0.003855	0.012872	**0.001594**	0.004029	0.010523	**0.001008**	0.002221	0.006761
		(42.41%)	(14.41%)		(48.52%)	(14.53%)		(39.55%)	(15.14%)		(45.37%)	(14.90%)

Note:

1. The cells in Tables 2a and 2b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD for a specific gender. For instance, the mean RMSEs and MAEs for males are obtained by $\frac{1}{57}\sum_{A=1951}^{2007}\sqrt{\frac{1}{(109-25+1)}\sum_{x=25}^{109}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{57\times(109-25+1)}\sum_{A=1951}^{2007}\sum_{x=25}^{109}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 3: Fitting Errors by Periods

3a: RMSE

	1950s			1960s			1970s			1980s			1990s			2000s		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	**0.002688**	0.004643	0.011780	**0.005156**	0.010190	0.016223	**0.002256**	0.007267	0.009761	**0.001384**	0.007471	0.008925	**0.001343**	0.005502	0.012576	**0.001322**	0.008573	0.017859
		(57.88%)	(22.81%)		(50.60%)	(31.78%)		(31.04%)	(23.11%)		(18.53%)	(15.51%)		(24.40%)	(10.68%)		(15.42%)	(7.40%)
US-Female	**0.002120**	0.004203	0.009872	**0.004432**	0.009610	0.015228	**0.001430**	0.004908	0.009759	**0.000947**	0.005557	0.009717	**0.000814**	0.005003	0.013761	**0.000843**	0.007688	0.018449
		(50.43%)	(21.47%)		(46.12%)	(29.11%)		(29.14%)	(14.66%)		(17.04%)	(9.74%)		(16.27%)	(5.91%)		(10.97%)	(4.57%)
UK-Male	**0.004430**	0.010749	0.021331	**0.006795**	0.011092	0.028912	**0.003065**	0.007309	0.025373	**0.002404**	0.007732	0.023962	**0.002400**	0.006736	0.015202	**0.002779**	0.005454	0.012663
		(41.21%)	(20.77%)		(61.26%)	(23.50%)		(41.93%)	(12.08%)		(31.10%)	(10.03%)		(35.63%)	(15.79%)		(50.96%)	(21.95%)
UK-Female	**0.002888**	0.005136	0.015381	**0.005206**	0.009787	0.019422	**0.001796**	0.003347	0.013053	**0.001673**	0.003286	0.011287	**0.001365**	0.004282	0.010235	**0.001796**	0.003408	0.013250
		(56.23%)	(18.78%)		(53.19%)	(26.80%)		(53.66%)	(13.76%)		(50.92%)	(14.82%)		(31.88%)	(13.34%)		(52.70%)	(13.55%)

3b: MAE

	1950s			1960s			1970s			1980s			1990s			2000s		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	**0.001414**	0.002489	0.005563	**0.001126**	0.003165	0.005507	**0.001156**	0.003499	0.004875	**0.000881**	0.003737	0.004957	**0.000855**	0.002767	0.007385	**0.000773**	0.004325	0.010381
		(56.81%)	(25.42%)		(35.57%)	(20.44%)		(33.05%)	(23.72%)		(23.57%)	(17.77%)		(30.90%)	(11.58%)		(17.87%)	(7.45%)
US-Female	**0.001000**	0.002325	0.005628	**0.000824**	0.002989	0.006317	**0.000861**	0.002189	0.005714	**0.000541**	0.002440	0.005417	**0.000483**	0.002211	0.007577	**0.000490**	0.003488	0.010101
		(43.01%)	(17.77%)		(27.57%)	(13.04%)		(39.35%)	(15.07%)		(22.17%)	(9.99%)		(21.83%)	(6.37%)		(14.05%)	(4.85%)
UK-Male	**0.047245**	0.075536	0.103539	**0.041173**	0.060915	0.108856	**0.038435**	0.062413	0.112838	**0.037068**	0.064822	0.109658	**0.038381**	0.059598	0.089734	**0.036308**	0.055589	0.084210
		(62.55%)	(45.63%)		(67.59%)	(37.82%)		(61.58%)	(34.06%)		(57.18%)	(33.80%)		(64.40%)	(42.77%)		(65.31%)	(43.12%)
UK-Female	**0.037712**	0.054652	0.093737	**0.033973**	0.055565	0.087297	**0.030650**	0.040871	0.081432	**0.029702**	0.041873	0.073875	**0.028565**	0.045652	0.072561	**0.028985**	0.041497	0.084083
		(69.00%)	(40.23%)		(61.14%)	(38.92%)		(74.99%)	(37.64%)		(70.93%)	(40.21%)		(62.57%)	(39.37%)		(69.85%)	(34.47%)

Note:

1. The cells in Tables 3a and 3b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD during different periods. For instance, the mean RMSEs and MAEs for males during 1950s are obtained by $\frac{1}{9}\sum_{A=1951}^{1959}\sqrt{\frac{1}{(109-25+1)}\sum_{x=25}^{109}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{9\times(109-25+1)}\sum_{A=1951}^{1959}\sum_{x=25}^{109}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 4: Fitting Errors by Ages

4a: RMSE

	25-34			35-44			45-64			65-74			75-84			85-109		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	0.000371	**0.000173**	0.000389	0.000358	**0.000212**	0.000496	**0.000424**	0.000478	0.001374	**0.001072**	0.001465	0.003182	**0.002017**	0.002486	0.004863	**0.003162**	0.012218	0.022106
		(214.45%)	(95.37%)		(168.87%)	(72.18%)		(88.70%)	(30.86%)		(73.17%)	(33.69%)		(81.13%)	(41.48%)		(25.88%)	(14.30%)
US-Female	0.000226	**0.000042**	0.000129	0.000209	**0.000088**	0.000093	**0.000292**	0.000333	0.001047	**0.000705**	0.000842	0.003811	**0.001398**	0.001860	0.005437	**0.002279**	0.010120	0.022100
		(538.10%)	(175.19%)		(237.50%)	(224.73%)		(87.69%)	(27.89%)		(83.73%)	(18.50%)		(75.16%)	(25.71%)		(22.52%)	(10.31%)
UK-Male	0.000528	**0.000129**	0.000223	0.000514	**0.000157**	0.000503	**0.000604**	0.000605	0.001869	**0.001619**	0.002303	0.006466	**0.003192**	0.003784	0.011095	**0.005202**	0.013855	0.038131
		(409.30%)	(236.77%)		(327.39%)	(102.19%)		(99.83%)	(32.32%)		(70.30%)	(25.04%)		(84.36%)	(28.77%)		(37.55%)	(13.64%)
UK-Female	0.000279	**0.000076**	0.000101	0.000263	**0.000091**	0.000106	**0.000351**	0.000491	0.000973	**0.000868**	0.001108	0.002269	**0.002164**	0.002243	0.005652	**0.003294**	0.007803	0.023691
		(367.11%)	(276.24%)		(289.01%)	(248.11%)		(71.49%)	(36.07%)		(78.34%)	(38.25%)		(96.48%)	(38.29%)		(42.21%)	(13.90%)

4b: MAE

	25-34			35-44			45-64			65-74			75-84			85-109		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	0.000312	**0.000138**	0.000335	0.000301	**0.000164**	0.000431	**0.000295**	0.000357	0.001091	**0.000753**	0.001142	0.002495	**0.001557**	0.001988	0.003932	**0.002120**	0.009596	0.017745
		(226.09%)	(93.13%)		(183.54%)	(69.84%)		(82.63%)	(27.04%)		(65.94%)	(30.18%)		(78.32%)	(39.60%)		(22.09%)	(11.95%)
US-Female	0.000175	**0.000033**	0.000117	0.000165	**0.000071**	0.000075	**0.000190**	0.000243	0.000750	**0.000472**	0.000670	0.003585	**0.001036**	0.001351	0.004763	**0.001495**	0.007731	0.018753
		(530.30%)	(149.57%)		(232.39%)	(220.00%)		(78.19%)	(25.33%)		(70.45%)	(13.17%)		(76.68%)	(21.75%)		(19.34%)	(7.97%)
UK-Male	0.020213	**0.009934**	0.013829	0.019920	**0.011316**	0.021390	0.020668	**0.020037**	0.037010	**0.033942**	0.042684	0.074840	**0.047765**	0.056018	0.099856	**0.058144**	0.106319	0.167909
		(203.47%)	(146.16%)		(176.03%)	(93.13%)		(103.15%)	(55.84%)		(79.52%)	(45.35%)		(85.27%)	(47.83%)		(54.69%)	(34.63%)
UK-Female	0.015156	**0.007407**	0.008619	0.014559	**0.008344**	0.008687	0.016017	**0.018944**	0.025911	**0.024832**	0.029499	0.044682	**0.039515**	0.040400	0.063578	**0.046615**	0.078832	0.141334
		(204.62%)	(175.84%)		(174.48%)	(167.60%)		(84.55%)	(61.82%)		(84.18%)	(55.57%)		(97.81%)	(62.15%)		(59.13%)	(32.98%)

Note:

1. The cells in Tables 4a and 4b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD for different age groups. For instance, the mean RMSEs and MAEs for males with ages 25-34 are obtained by $\frac{1}{57}\sum_{A=1951}^{2007}\sqrt{\frac{1}{(34-25+1)}\sum_{x=25}^{34}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{57\times(34-25+1)}\sum_{A=1951}^{2007}\sum_{x=25}^{34}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 5: Summary Statistics of Our-of-Sample Forecasting Errors

5a: RMSE

	Mean			Median			Std. Deviation			Min			Max		
Country	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US	**0.001067**	0.008338	0.015240	**0.001034**	0.008389	0.015738	**0.000336**	0.004345	0.002972	**0.000399**	0.001077	0.008428	**0.001872**	0.015735	0.020640
		(12.80%)	(7.00%)		(12.32%)	(6.57%)		(7.73%)	(11.30%)		(37.05%)	(4.73%)		(11.90%)	(9.07%)
UK	**0.002050**	0.006185	0.013344	**0.001959**	0.004565	0.012770	**0.000664**	0.003635	0.003253	**0.000908**	0.001498	0.008600	**0.003603**	0.016084	0.023546
		(33.15%)	(15.36%)		(42.90%)	(15.34%)		(18.27%)	(20.41%)		(60.63%)	(10.56%)		(22.40%)	(15.30%)

5b: MAE

	Mean			Median			Std. Deviation			Min			Max		
Country	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US	**0.000655**	0.004393	0.008708	**0.000616**	0.004707	0.009068	**0.000239**	0.002152	0.001506	**0.000275**	0.000668	0.005222	**0.001252**	0.008083	0.011335
		(14.91%)	(7.52%)		(13.08%)	(6.79%)		(11.11%)	(15.88%)		(41.08%)	(5.26%)		(15.49%)	(11.05%)
UK	**0.001124**	0.003583	0.006950	**0.001036**	0.003016	0.007019	**0.000385**	0.001873	0.001608	**0.000520**	0.001002	0.004166	**0.002215**	0.008369	0.010343
		(31.36%)	(16.17%)		(34.33%)	(14.75%)		(20.57%)	(23.96%)		(51.85%)	(12.48%)		(26.46%)	(21.41%)

Note:

1. The mean, median, standard deviation, min, and max of RMSE and MAE are the statistics of $\left\{ \sqrt{\frac{1}{2\times(109-25+1)}\sum_{s\in\{m,f\}}\sum_{x=25}^{109}(q_{s,x,A}-\hat{q}_{s,x,A})^2} \,\middle|\, A=1990,\cdots,2007 \right\}$ and $\left\{ \sum_{s\in\{m,f\}}\sum_{x=25}^{109}\left|q_{s,x,A}-\hat{q}_{s,x,A}\right| \,\middle|\, A=1990,\cdots,2007 \right\}$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 6: Forecasting Errors by Genders

	6a: RMSE						6b: MAE					
	Male			Female			Male			Female		
Country	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD	LHT	Lee-carter	CBD
US	**0.001320**	0.009447	0.014650	**0.000814**	0.007230	0.015831	**0.000823**	0.005104	0.008694	**0.000486**	0.003682	0.008722
		(13.97%)	(9.01%)		(11.26%)	(5.14%)		(16.13%)	(9.47%)		(13.21%)	(5.58%)
UK	**0.002533**	0.008015	0.014532	**0.001567**	0.004355	0.012155	**0.001404**	0.004654	0.007682	**0.000843**	0.002511	0.006218
		(31.60%)	(17.43%)		(35.99%)	(12.90%)		(30.17%)	(18.28%)		(33.58%)	(13.56%)

Note:

1. The cells in Tables 6a and 6b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD for a specific gender. For instance, the mean RMSEs and MAEs for males are obtained by $\frac{1}{18}\sum_{A=1990}^{2007}\sqrt{\frac{1}{(109-25+1)}\sum_{x=25}^{109}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{18\times(109-25+1)}\sum_{A=1990}^{2007}\sum_{x=25}^{109}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT’s errors to those of Lee-Carter and CBD.

Table 7: Forecasting Errors by Periods

	7a: RMSE						7b: MAE					
	1990s			2000s			1990s			2000s		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	**0.001346**	0.010225	0.013945	**0.001345**	0.010223	0.016010	**0.000858**	0.005115	0.008082	**0.000780**	0.005092	0.009458
		(13.16%)	(9.65%)		(13.16%)	(8.40%)		(16.78%)	(10.62%)		(15.32%)	(8.25%)
US-Female	**0.000812**	0.007379	0.015309	**0.000856**	0.009724	0.017105	**0.000481**	0.003102	0.008127	**0.000493**	0.004409	0.009466
		(11.00%)	(5.30%)		(8.81%)	(5.01%)		(15.52%)	(5.92%)		(11.18%)	(5.21%)
UK-Male	**0.002398**	0.009952	0.016858	**0.002807**	0.007241	0.012138	**0.001465**	0.005153	0.008528	**0.001328**	0.004030	0.006624
		(24.10%)	(14.23%)		(38.76%)	(23.13%)		(28.43%)	(17.18%)		(32.95%)	(20.04%)
UK-Female	**0.001411**	0.005750	0.012013	**0.001828**	0.003743	0.012880	**0.000839**	0.002904	0.005770	**0.000849**	0.002021	0.006777
		(24.54%)	(11.74%)		(48.84%)	(14.20%)		(28.89%)	(14.54%)		(42.00%)	(12.52%)

Note:

1. The cells in Tables 7a and 7b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD during different periods. For instance, the mean RMSEs and MAEs for males during 1990s are obtained by $\frac{1}{10}\sum_{A=1990}^{1999}\sqrt{\frac{1}{(109-25+1)}\sum_{x=25}^{109}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{10\times(109-25+1)}\sum_{A=1990}^{1999}\sum_{x=25}^{109}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 8: Forecasting Errors by Ages

8a: RMSE

	25-34			35-44			45-64			65-74			75-84			85-109		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	0.000315	**0.000245**	0.000430	0.000319	0.000305	**0.000286**	**0.000343**	0.000502	0.001944	**0.000564**	0.001718	0.004979	**0.001632**	0.003361	0.007137	**0.002184**	0.018690	0.026839
		(128.57%)	(73.26%)		(104.59%)	(111.54%)		(68.33%)	(17.64%)		(32.83%)	(11.33%)		(48.56%)	(22.87%)		(11.69%)	(8.14%)
US-Female	0.000167	**0.000048**	0.000136	0.000168	0.000110	**0.000109**	**0.000196**	0.000387	0.000853	**0.000355**	0.000759	0.003574	**0.000907**	0.001407	0.007264	**0.001383**	0.015639	0.029278
		(347.92%)	(122.79%)		(152.73%)	(154.13%)		(50.65%)	(22.98%)		(46.77%)	(9.93%)		(64.46%)	(12.49%)		(8.84%)	(4.72%)
UK-Male	0.000448	**0.000122**	0.000262	0.000445	**0.000158**	0.000346	**0.000402**	0.000736	0.001296	**0.001300**	0.002491	0.003087	**0.004063**	0.005487	0.006525	**0.003868**	0.015847	0.027148
		(367.21%)	(170.99%)		(281.65%)	(128.61%)		(54.62%)	(31.02%)		(52.19%)	(42.11%)		(74.05%)	(62.27%)		(24.41%)	(14.25%)
UK-Female	0.000269	**0.000048**	0.000073	0.000245	**0.000076**	0.000060	**0.000273**	0.000603	0.000768	**0.000826**	0.001419	0.002631	0.002534	**0.002262**	0.003635	**0.002405**	0.008972	0.022684
		(560.42%)	(368.49%)		(322.37%)	(408.33%)		(45.27%)	(35.55%)		(58.21%)	(31.39%)		(112.02%)	(69.71%)		(26.81%)	(10.60%)

8b: MAE

	25-34			35-44			45-64			65-74			75-84			85-109		
Country-Gender	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD	LHT	Lee-Carter	CBD
US-Male	0.000261	**0.000205**	0.000389	0.000263	0.000252	**0.000200**	**0.000261**	0.000412	0.001632	**0.000448**	0.001463	0.004951	**0.001193**	0.002722	0.006538	**0.001696**	0.015150	0.023427
		(127.32%)	(67.10%)		(104.37%)	(131.50%)		(63.35%)	(15.99%)		(30.62%)	(9.05%)		(43.83%)	(18.25%)		(11.19%)	(7.24%)
US-Female	0.000138	**0.000037**	0.000131	0.000140	**0.000092**	0.000093	**0.000151**	0.000300	0.000618	**0.000286**	0.000601	0.003599	**0.000613**	0.001031	0.007170	**0.001031**	0.011548	0.024774
		(372.97%)	(105.34%)		(152.17%)	(150.54%)		(50.33%)	(24.43%)		(47.59%)	(7.95%)		(59.46%)	(8.55%)		(8.93%)	(4.16%)
UK-Male	0.000366	**0.000104**	0.000234	0.000366	**0.000136**	0.000289	**0.000322**	0.000500	0.001125	**0.000944**	0.002280	0.002721	**0.002619**	0.004778	0.004726	**0.002753**	0.012462	0.021861
		(351.92%)	(156.41%)		(269.12%)	(126.64%)		(64.40%)	(28.62%)		(41.40%)	(34.69%)		(54.81%)	(55.42%)		(22.09%)	(12.59%)
UK-Female	0.000217	**0.000039**	0.000063	0.000194	**0.000060**	0.000044	**0.000204**	0.000427	0.000504	**0.000610**	0.001167	0.002302	**0.001571**	0.001677	0.003209	**0.001635**	0.006978	0.018494
		(556.41%)	(344.44%)		(323.33%)	(440.91%)		(47.78%)	(40.48%)		(52.27%)	(26.50%)		(93.68%)	(48.96%)		(23.43%)	(8.84%)

Note:

1. The cells in Tables 8a and 8b represent the mean RMSEs and MAEs of LHT, Lee-Carter, and CBD for different age groups. For instance, the mean RMSEs and MAEs for males with ages 25-34 are obtained by $\frac{1}{18}\sum_{A=1990}^{2007}\sqrt{\frac{1}{(34-25+1)}\sum_{x=25}^{34}(q_{m,x,A}-\hat{q}_{m,x,A})^2}$ and $\frac{1}{18\times(34-25+1)}\sum_{A=1990}^{2007}\sum_{x=25}^{34}\left|q_{m,x,A}-\hat{q}_{m,x,A}\right|$ respectively.
2. Boldface denotes the smallest number in each comparison among LHT, Lee-Carter, and CBD.
3. The numbers in parenthesis are the ratios of LHT's errors to those of Lee-Carter and CBD.

Table 9: Dollar Durations of the Reserves at time 0 ($x = 45$, $n = 20$, and $i = 3\%$)

Products	DD_α	DD_β
Single-payment, *n*-year deferred whole life annuity-due	3.16	183.86
n-payment, *n*-year deferred whole life annuity-due	2.84	131.43
Single-payment whole life insurance	-0.11	-8.87
n-payment whole life insurance	-0.13	-12.11
Single-payment, *n*-year endowment	-0.02	-3.51
n-payment, *n*-year endowment	-0.05	-8.28
Single-payment, *n*-year term life insurance	-0.10	-12.86
n-payment, *n*-year term life insurance	-0.11	-13.76
Single-payment, *n*-year pure endowment	0.08	9.35
n-payment, *n*-year pure endowment	0.06	5.48

Table 10: Immunized Portfolios with the Constraint of Positive Weights ($x = 45$, $n = 20$, and $i = 3\%$)

Portfolios	DD_α	DD_β	Weight
Portfolio 1			
n-payment, *n*-year term life insurance	-0.11	-13.76	7.24%
n-payment whole life insurance	-0.13	-12.11	23.25%
n-payment, *n*-year pure endowment	0.06	5.48	69.51%
Portfolio 2			
Single-payment, *n*-year deferred whole life annuity-due	3.16	183.86	1.59%
Single-payment whole life insurance	-0.11	-8.87	66.61%
Single-payment, *n*-year pure endowment	0.08	9.35	31.79%
Portfolio 3			
n-payment, *n*-year deferred whole life annuity-due	2.84	131.43	1.34%
Single-payment whole life insurance	-0.11	-8.87	60.32%
Single-payment, *n*-year pure endowment	0.08	9.35	38.34%

Table 11: Immunized Portfolios Allowing Negative Weights ($x = 45$, $n = 20$, and $i = 3\%$)

Portfolios	DD_α	DD_β	Weight
Portfolio 4			
n-payment, *n*-year term life insurance	-0.11	-13.76	-191.48%
n-payment whole life insurance	-0.13	-12.11	285.26%
n-payment, *n*-year deferred whole life annuity-due	2.84	131.43	6.23%
Portfolio 5			
Single-payment, *n*-year deferred whole life annuity-due	3.16	183.86	-0.21%
Single-payment, *n*-year term life insurance	-0.10	-12.86	40.45%
Single-payment, *n*-year pure endowment	0.08	9.35	59.76%

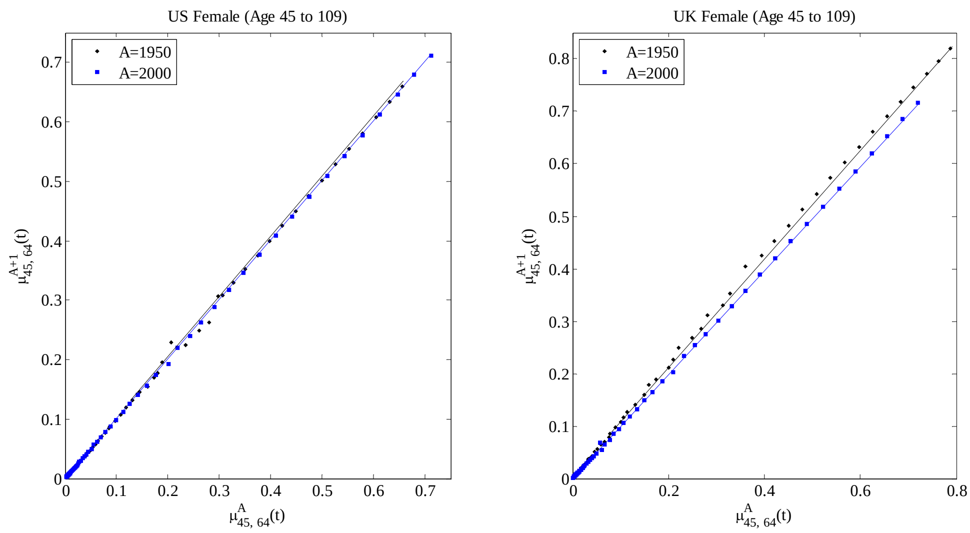

Figure 1: Sample Relations between the Forces of Mortality

國科會補助專題研究計畫項下赴國外(或大陸地區)出差或研習心得報告

日期：100 年 9 月 2 日

計畫編號	NSC 99-2410-H-004 -063 -MY3		
計畫名稱	精算與財務方法在壽險保單定價、準備金估計、以及風險管理之運用		
出國人員姓名	蔡政憲	服務機構及職稱	政治大學風險管理與保險學系
出國時間	100 年 6 月 18 日至 100 年 8 月 15 日	出國地點	美國 Kansas City

一、 國外(大陸)研究過程

個人於 6/18（六）搭乘長榮航空的飛機出發。因為需要轉機，在 Los Angel (LA)停一夜。隔天一早(日)搭乘Delta直飛Kansas City (KC)。星期一開始就開始進行一篇論文的新寫作，以及七篇論文的修改或起草，直到 8/3 從KC飛回LA為止。那篇新寫出之論文附加於「二、研究成果」中[1]。期間和訪問對象有多次一對一的討論，

[1] 此論文將在 the Seventh International Longevity Risk and Capital Market Solutions Conference（9/8-9/9）中發表。*Journal of Risk and Insurance* 將從該會議的論文挑選數篇集成一本特刊。

還有透過電話與電郵的溝通。由於出發前就已經寄給對方研究大綱與初步的內容，討論均能直接切入重點。

8/3 飛抵 LA 預備參加 2011 年 ARIA（American Risk and Insurance Association）的年會（8/7-8/10)。由於本人先前投稿的大綱未被接受，僅擔任一篇論文的評論人，因此這段行程不符合受國科會補助的規定。

參加完年會後，個人又另外和敝系的同事從 LA 飛往費城，參訪美國最大的 Life Settlement 公司-Coventry。參訪完後，於 8/13 從紐約飛回台灣，8/15 抵達。這段行程不在研究計畫的範圍內。

二、研究成果

Modeling the Dynamics of Mortality Rates as Transformations

Fang-Shu Linus Chan
Assistant Professor, Department of Financial Engineering and Actuarial Mathematics
Soochow University
No. 56, Section 1, Kuei-Yang Street, Taipei City, Taiwan10048
(Tel) +886-2-2311-1531#3628; (Fax) +886-2-2381-2510; E-mail: linuschan@scu.edu.tw

Cary Chi-Liang Tsai
Associate Professor, Department of Statistics and Actuarial Science
Simon Fraser University
Burnaby, BC V5A, Canada
(Tel) +778-7827044; (Fax) +778-7824368; E-mail: cltsai@sfu.ca

Chenghsien Tsai*
Professor, Department of Risk Management and Insurance
Research Fellow, Risk and Insurance Research Center
National Chengchi University
No. 64, Section 2, ZhiNan Road, Taipei City, Taiwan 11605
(Tel) +886-2-2936-9647; (Fax) +886-2-2939-3864; E-mail: ctsai@nccu.edu.tw

*Corresponding author. The author is grateful to the Risk and Insurance Research Center of National Chengchi University and the National Science Council of Taiwan for their financial supports (Project numbers NSC 99-2410-H-004 -063 -MY3) and to the National Association of Insurance Commissioners (NAIC) for the kind supports during the visit of the author. A major portion of the research was completed while the author was visiting NAIC.

ABSTRACT

Modeling and forecasting mortality rates are crucial to life insurers, social benefits programs, and the society as a whole. The vast literature developed four methods to model and/or forecast mortality rates. We propose a new way in this paper by modeling the dynamics of mortality rates as the transformation from one mortality curve to another. Such a proposal is reasonable since mortality rates changed gradually due to biological reasons and the rigidity of the social system.

We use empirical data to test the relative performance of this new modeling to the renowned Lee-Carter model. The tests cover both in-sample fitting and out-of-sample forecasting on the US and UK mortality rates from 1951-2007. We find that the linear hazard transform dominates the Lee-Carter model.

The new model further provides better ways in generating immunization strategies than Wang et al. (2010) did. Our model is more general and can produce explicit formulas for mortality durations. The potential of this new thought is thus confirmed and deserve further pursuit.

Keywords: mortality rates, transform, fitting, forecasting, hedging, duration

INTRODUCTION

Modeling the changes/dynamics of mortality rates is critical to the solvency of life insurers and social retirement programs. Mortality rates are one of the key factors in pricing and reserving for life and annuity products. Taking no account of possible mortality rate changes in the pricing and reserving might lead to significant under-pricing and under-reserving that would impair the profitability and solvency of a life insurer. The retirement programs sponsored by governments need to understand the dynamics of mortality rates as well since the incomes and benefit outgoes depend on mortalities. Under-estimating the improvements of mortality rates could jeopardize the solvency and continuity of the programs. Mortality rates are also important to the long-term care systems and the population structure of a country that in turn will affect the growths or declines of many industries. Therefore, mortality rate modeling is important.

The literature recognized the importance of the mortality rate modeling and tackled the problem in four different ways. Demographers and sociologists developed explanatory models to understand which factors affected the mortality rates of some populations with respect to ages, genders, regions, races, periods, etc. (please see Stallard, 2006 and the references therein.)These models helped us understand the determinants of mortality rates but lacked of forecasting capabilities.

The earliest and still the most popular model that could be used to forecast mortality rates was established by Lee and Carter (1992). Their model was a two-component model in which one parameter is used to indicate the improvement rate for each age and a single random process drives all the dynamics. Later extensions focused on how to better estimate parameters (e.g., Delwarde et al., 2007) and the incorporation of the cohort effect (e.g., Renshaw and Haberman, 2006). Some developed multifactor age-period models such as Renshaw and Haberman (2003), Carins, Blake, and Dowd (2006b), and Cairns et al. (2007).

The third way was to fit mortality rate curves to certain functions (e.g., Currie et al., 2004) and then established time-series models for the function parameters to perform forecasting. The underlying rationale is that the mortality rate curves themselves held the information on how the mortality rates of different ages relate to each other. These relations might result from biological reasons (e.g., new-born babies have higher mortality rates; mortality rates increase with ages for matured people) or social reasons (e.g., speed driving of young adults). The Lee-Carter line of papers did not capture such information.

The fourth approach developed mortality rate models using the framework of financial valuation models for the term structure of interest rates (e.g., Dahl, 2004; Dahl and Møller, 2005; Biffis, 2005; Cairns, Blake, and Dowd, 2006a).They usually specified the dynamics of one or few mortality rates and then imposed certain relations on the mortality rates across ages so that the model could depict the changes of the entire curve. For more detailed literature review, please refer to Cairns, Blake, and Dowd(2008) and the references therein.

We propose a new way to model the dynamics of mortality rates in this paper. Instead of modeling the curve first and then modeling the dynamics of the curve factors, we model the relations between two curves. More specifically, the mortality rates on a later curve are regarded as a transformation of those on an earlier curve. The justification for our new way is that mortality rate curves shifted slowly with small changes and shifted stably in terms of the shape. The small and stable changes might result from biological constraints and/or the rigidity of social changes (e.g., health care systems, living habits, improvements of medical technologies and public health, and inventions of medicines). We therefore may be able to model the changes in mortality rates between two curves as transformations from the earlier age-specific mortality rates to the later corresponding ones.

To investigate the potential of our initiative, we conducted in-sample fitting and out-of-sample forecasting tests. We first assume that there exists a linear relation, called linear hazard transform (LHT), between the forces of mortality (hazard rates) of two curves.[2] Then we derived the corresponding relations between the survival probabilities of two curves so that we might estimate the parameters of the LHT using the empirical data of US and UK from 1950 to 2007 that cover both genders. To evaluate the performance the LHT, we

[2]Jiang and Tsai (2011) were indeed the first to apply LHT to mortality fitting and forecasting. However, their paper focuses on the techniques of applying LHT in alternative ways and did not use empirical data to test the performance of the method relative to that of an existing model.

chose the 1992 Lee-Carter model as the benchmark and performed in-sample fitting and out-of-sample forecasting comparisons. The Lee-Carter model has been popular since published and was used as comparison benchmark in papers like Brouhns, Denuit, and Vermunt (2002) as well as Czado, Delwarde, and Denuit (2005).

The statistical comparisons show that our LHT model dominates the Lee-Carter model. In the fitting tests our model produces lower RMSE (root of mean square error) and MAE (mean absolute error) than the Lee-Carter model does by 71.8% and 70.5% on average for the US and by 55.4% and 58.4% on average for the UK. Our model outperforms the Lee-Carter model by 87.2% and 85.9% for the US and by 66.9% and 68.6% for the UK in the forecasting tests.

We further illustrate one advantage of our LHT method in constructing the insurance portfolios immunized from the mortality rate risk. Since our LHT requires merely two parameters to depict the dynamics of the entire mortality rate curves, we can utilize the durations with respect to these two parameters to construct the immunized portfolio using three types of life insurance / annuity products. Our method is more general than that of Wang et al. (2010) in which they calculated the optimal life insurance–annuity product mix ratio by assuming the force of mortality shifted proportionally. Indeed, their method is a special case of ours that one of the parameters in our model is set to zero. Another advantage of our method is that we could have the explicit formulas for the mortality durations, which makes the risk management more easily and accurately.

This paper contributes to the literature in several aspects. It proposes a new method in modeling the dynamics of mortality rate curves, and demonstrates the potential idea of transformation using empirical data. The merits of our method include: easy to understand and implement, fewer parameters to be estimated, good accuracies in both fitting and forecasting, and easy to generate hedging strategies. We seem to find a new avenue to model and manage the mortality/longevity risk.

The remainder of this paper is organized as follows. Section 2 specifies the assumed relations between two mortality rate curves used in this paper. It explains the economic meanings of the parameters and how we can estimate parameters using empirical data. Section 3 delineates how we conduct statistical tests in comparing our model with a benchmark model. It describes the data, the benchmark model, the two accuracy measures, and how the in-sample fitting and out-of-sample forecasting are done. Section 4 applies our model to risk management. It demonstrates how our method can generate the immunization strategies using the durations with respect to the parameters to hedge the mortality/longevity risk. In Section 5 the paper is summarized and conclusions are drawn.

RELATIONS BETWEEN TWO MORTALITY CURVES

We propose in this paper to model the dynamics of mortality rates across time as the transformations from one curve to another. More specifically, we assume that there is a linear relation plus an error term between the forces of mortality (hazard rates) of two mortality rate curves in years A and $B = A + a$, where $a \in Z$ is the difference of two years A and B. The mathematical form is:

$$\mu_{x,n}^{B}(t) = (1+\alpha_{x,n}^{A,B}) \times \mu_{x,n}^{A}(t) + \beta_{x,n}^{A,B} + \varepsilon_{x,n}^{A,B}(t), \quad t \in [0,n], \tag{1}$$

where x denotes age, ε indicates the error, $n = \omega - x$, and ω represents the ultimate age of the mortality table. Year A is called the base year in the following while year $B = A + a$ is called the target year.

Since ${}_t p_x = e^{-\int_0^t \mu_x(s)ds}$, equation (1) implies the following relation between the corresponding survival probabilities of curves B and A:

$$\begin{aligned} {}_k p_x^B &= e^{-\int_0^k \mu_{x,n}^{B}(t)dt} = e^{-(1+\alpha_{x,n}^{A,B})\times\int_0^k \mu_{x,n}^{A}(t)dt - \int_0^k \beta_{x,n}^{A,B}dt - \int_0^k \varepsilon_{x,n}^{A,B}(t)dt} \\ &= ({}_k p_x^A)^{1+\alpha_{x,n}^{A,B}} \times e^{-\beta_{x,n}^{A,B}\times k} \times e^{-\int_0^k \varepsilon_{x,n}^{A,B}(t)dt} \end{aligned}. \tag{2}$$

Taking the natural logarithm on both sides yields:

$$\ln {}_k p_x^B = (1+\alpha_{x,n}^{A,B}) \times (\ln {}_k p_x^A) - \beta_{x,n}^{A,B} \times k - \int_0^k \varepsilon_{x,n}^{A,B}(t)dt . \tag{3}$$

Then we would like to minimize the sum of squared errors across the ages from x to ω by choosing the parameter pair ($\alpha_{x,n}^{A,B}, \beta_{x,n}^{A,B}$):

$$\begin{aligned} SSE &\triangleq \sum_{k=1}^{n} \left[\int_0^k \varepsilon(t)dt \right]^2 \\ &= \sum_{k=1}^{n} \left\{ (\ln {}_k p_x{}^B) - \left[(1+\alpha_{x,n}^{A,B}) \times (\ln {}_k p_x^A) - \beta_{x,n}^{A,B} \times k \right] \right\}^2 \end{aligned}. \tag{4}$$

The solutions of ($\alpha_{x,n}^{A,B}, \beta_{x,n}^{A,B}$)can be obtained by the regular regression analysis. More specifically, take the derivatives of equation (2.4) with respect to $\alpha_{x,n}^{A,B}$ and $\beta_{x,n}^{A,B}$:

$$\frac{\partial SSE}{\partial \alpha_{x,n}^{A,B}} = -2\sum_{k=1}^{n}\left\{\ln {}_k p_x^A \times \left[(\ln {}_k p_x{}^B) - (1+\alpha_{x,n}^{A,B}) \times (\ln {}_k p_x^A) + \beta_{x,n}^{A,B} \times k\right]\right\} = 0 \quad \text{and} \qquad (5)$$

$$\frac{\partial SSE}{\partial \beta_{x,n}^{A,B}} = 2\sum_{k=1}^{n}\left\{k \times \left[(\ln {}_k p_x{}^B) - (1+\alpha_{x,n}^{A,B}) \times (\ln {}_k p_x^A) + \beta_{x,n}^{A,B} \times k\right]\right\} = 0. \qquad (6)$$

Solving for the $\hat{\alpha}_{x,n}^{A,B}$ and $\hat{\beta}_{x,n}^{A,B}$ that make the two equations above equal to 0 gives:

$$\hat{\alpha}_{x,n}^{A,B} = \frac{\left[\sum_{k=1}^{n}(\ln {}_k p_x{}^A)(\ln {}_k p_x{}^B)\right]\left[\sum_{k=1}^{n}k^2\right] - \left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^A)\right]\left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^B)\right]}{\left[\sum_{k=1}^{n}(\ln {}_k p_x{}^A)^2\right]\left[\sum_{k=1}^{n}k^2\right] - \left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^A)\right]^2} - 1 \quad \text{and} \qquad (7)$$

$$\hat{\beta}_{x,n}^{A,B} = \frac{\left[\sum_{k=1}^{n}(\ln {}_k p_x{}^A)(\ln {}_k p_x{}^B)\right]\left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^A)\right] - \left[\sum_{k=1}^{n}(\ln {}_k p_x{}^A)^2\right]\left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^B)\right]}{\left[\sum_{k=1}^{n}(\ln {}_k p_x{}^A)^2\right]\left[\sum_{k=1}^{n}k^2\right] - \left[\sum_{k=1}^{n}k(\ln {}_k p_x{}^A)\right]^2}. \qquad (8)$$

We may grab the economic meaning of the transformation as well as the meanings of α and β from equation (3).[3] The transformation decomposes the changes of the mortality rates (in their logarithm forms) from an earlier curve to a later curve into two components: a proportional change reflected by α and a parallel shift determined by β.[4] Assuming $\alpha = 0$ and $\beta = 0$ means no changes in mortality rates across time. Assuming $\beta = 0$ implies that the force-of-mortality curve shifts proportionally to the mortality rates. Higher forces of mortality will have larger improvements or deteriorations, depending on whether α is negative or positive. This type of curve changing behavior is also called proportional hazard transform. Assuming $\alpha = 0$ but $\beta \neq 0$ corresponds to the cases of parallel shifts of the force-of-mortality curves.

[3] We omit superscript and/or subscript whenever the omission causes no confusions for the easiness of reading.
[4] If the regressions are run on ${}_k p_x$, the shifts will be the product of β and k. In other words, the shift increases with the number of the surviving years.

STATISTICAL TESTS

To investigate whether our initiative works, we conduct two types of statistical tests: in-sample fitting and out-of-sample forecasting. We first estimate the parameters of the transformation function (equation (3)) year by year. Applying the estimated parameters to the base-year mortality rate curve would give us the estimated / fitted target-year curve. Then we compare the fitting errors of our LHT to those of a benchmark model.

With regard to the forecasting tests, we first assume that the parameters of the LHT followed random walks with drifts. Then we add the drifts estimated using a moving window of 40 years to the "current "estimated parameters to obtain a pair of forecasted parameters. Applying this pair of forecasted parameters to that "current" mortality rate curve would give us a forecasted curve. The performance of our LHT models could then be assessed by comparing our forecasting errors with those of the benchmark model.

Data, Benchmark, and Measures

We draw historical mortality rates from the Human Mortality Database. The drawn data cover both genders of US and UK, the countries that were studied the most probably. The sampling period is from 1950 to 2007, a few years after the World War II to the most recent ones available.[5]

Since the majority of the persons purchasing life insurance and annuities are young

[5] As of July of 2011, the most recent mortality rates of US are those of 2007 while the data of UK are updated to 2009. We preferred the same length of history for both countries and thus ended up with the sampling period of 1951-2007.

adults and older, we focus on the mortality rates of ages 25 and above. More specifically, we test our method on the section of the mortality rate curves: 25+.[6] The ultimate age of the US and UK mortality tables during the sampling period is 110. p_{110} is thus set as 0, which causes abnormal shapes of the curves between ages 109 and 110. Therefore, the sections of the mortality rate curves that are tested in this paper are those between ages 25-109. This corresponds to the case of $\omega = 109$.

We choose the well-known Lee-Carter model as the benchmark to be compared with our LHT. The Lee-Carter model has been studied extensively in the literature and served as a benchmark model in many papers as well (e.g., Booth et al., 2005; Lee and Miller, 2001). The Lee-Carter model is in essence a relational model assuming that:

$$\log q_{x,A} = a_x + b_x K_A + \varepsilon_{x,A}, \tag{9}$$

where $q_{x,A}$ denotes the one-year death rate of age x in year A, a_x and b_x are age-specific constants, K_A represents the time-varying levels of mortality rates, and $\varepsilon_{x,A}$ indicates the error associated with the age x in year A.[7]

To estimate the parameters of equation (9), we first estimate a_x by

$a_x = \frac{1}{T}\sum_{A=A_1}^{A_1+T-1} \log q_{x,A}$ in which A_1 denotes the first sample year and T indicates the number of

[6] We also test two other sections: 35+ and 45+. The results are consistent with those from the 25+ section, which can also be glimpsed from Table 2. We decide not to present these results for the sake of the paper length.

[7] In Lee and Carter (1992), the model is in essence a relational model assuming $\ln m_{x,A} = a_x + b_x K_A$ where $m_{x,A}$ is the central death rate of age x in year A. To make our analysis be compared with Lee-Carter model, we substitute the one-year death rate for the central death rate in their approach.

sample years $(T \in N)$. Then we apply singular value decomposition (SVD) technique to obtain an OLS estimate for b_x and K_A with the constraints of $\sum_x b_x = 1$ and $\sum_A K_A = 0$,as Lee and Carter (1992) did. They further assumed that K_A followed the process of the random walk with a drift when forecasting mortality rates.

We adopt two accuracy measures: RMSE and MAE. Their definitions are:

$$RMSE = \sqrt{\frac{1}{T(\omega - x_I)} \sum_{A=A_1+1}^{A_1+T} \sum_{x=x_I}^{\omega-1} (q_{x,A} - \hat{q}_{x,A})^2} \text{ and}$$

$$MAE = \frac{1}{T(\omega - x_I)} \sum_{A=A_1+1}^{A_1+T} \sum_{x=x_I}^{\omega-1} \left| q_{x,A} - \hat{q}_{x,A} \right|,$$

in which $q_{x,A}$ indicates the observed one-year death rate of age x in year A, $\hat{q}_{x,A}$ represents the fitted/forecasted value, x_I is the starting age of the mortality rate curve. These two measures are used in many papers (e.g., Gakidou and King, 2006).

In-Sample Fitting

The in-sample fitting is done by fitting equation (3) using two mortality rate curves. We first draw the q_x of two different years from our dataset and calculate the corresponding ${}_k p_x$: ${}_k p_x^A$ and ${}_k p_x^B$. Taking the natural log of these ${}_k p_x$ and then running the regular regression analysis would render the $\hat{\alpha}^{A,a}$ and $\hat{\beta}^{A,a}$ in equation (3).Plugging the estimated $\hat{\alpha}^{A,a}$ and $\hat{\beta}^{A,a}$ into equation (3) with ${}_k p_x^A$ as input could give us ${}_k \hat{p}_x^B$ and $\hat{p}_{x+k-1}^B = {}_k \hat{p}_x^B / {}_{k-1} \hat{p}_x^B$, $k = 1, 2, 3, \cdots$.Then we calculate RMSE and MAE to measure the fitting errors. Repeating the steps for *A* = 1951-2006 with $a = 1$, we generate the following tables.[8]

[Insert Tables 1-3 Here]

As we can see from the rows titled "Overall-25" in Table 1, the fitting errors of our LHT relative to the Lee-Carter model averaged across ages, sampled years, and genders, in terms of RMSE, are28.2% (=0.00150984/0.00535141)and 44.6% for the US and UK data respectively.[9] In terms of MAE, our fitting errors are 29.5% and 41.6% of the Lee-Carter's. In addition to the significant improvements in the average of fitting errors, our method has lower standard deviation of fitting errors across the sampled years under both fitting measures. For instance, our standard deviations are 20.9% (= 0.00061883/0.00296784) and 29.7% for US and UK data, respectively, in terms of RMSE. The minimum, medium, and maximum of fitting errors during the sampling period of our method are also smaller than those of the Lee-Carter model in both countries using these two accuracy measures. The improvements of our method to the Lee-Carter model are significant and extensive.

The other rows of Table 1 further show our model performs better than the Lee-Carter model in fitting the data of both genders. Our LHT model produces mean fitting errors with respect to the data of US males and US females that were30.1% (= 0.00177571/0.00589731) and 25.9% of those by the Lee-Carter model, respectively, measured by RMSE. The fitting

[8] The sample size T in estimating the LHT is thus equal to 2006-1951+1 = 56. On the other hand, the sample size sued to estimate the Lee-Carter model is 57 since the estimation can be done using a single-year data. The LHT uses the two-year data at a time instead.

[9] Also, we can see from the rows titled "Overall-45" in Table 1, the fitting errors of our LHT relative to the Lee-Carter model averaged across ages, sampled years, and genders, in terms of RMSE, are 27.4% (= 0.00167276/0.00609781) and 41.9% for the US and UK data respectively.

errors to UK data are also smaller: 42.4% and 48.5%. The variations of our fitting errors across years are 21.6% (=0.00068953/0.00318861), 20.0%, 28.6%, and 31.4% of the Lee-Carter's for US males, US females, UK males, and UK females, respectively. The maximum errors produced by our model during the sampling period are also fractions of the errors by the Lee-Carter model: 29.4% (=0.00377734/0.01284675), 29.5%, 35.4%, and 56.8%. The superiorities of LHT model to the Lee-Carter model remain at the equivalent levels even when we switch the accuracy measure to MAE. The better performance of our model relative to the Lee-Carter model is robust across genders.

Our LHT model performs better than the Lee-Carter model in all the sub-periods of the sampling period in both US and UK, as Table 2 shows. The mean and standard deviation of the fitting errors during each decade of the sampling period produced by our LTH are all smaller than those by the Lee-Carter model. For instance, our mean fitting errors to US males in terms of RMSE for the decades of 50s, 60s, 70s, 80s, and 90s are 57.9% (= 0.00268754/0.00464334), 50.6%, 31.0%, 18.5%, and 24.4% of the Lee-Carter's, respectively. The variations within each of these decades of our method are also smaller: 34.3% (=0.00064566/0.00188334), 15.5%, 21.4%, 8.8%, and 11.5% relative to those of the Lee-Carter model, respectively. The dominance of our model over the Lee-Carter model in terms of the sub-period performance is robust across genders, countries, and accuracy measures.

The advantages of our model over the Lee-Carter model indeed lie in the better fitting to the populations 45 years older. Table 3 shows that our fitting errors are smaller than the Lee-Carter's for the age groups of 45-64, 65-74, 75-84, and 85-109 to both genders, countries, and accuracy measures.[10] For instance, the ratios of our fitting errors relative to the Lee-Carter's with respect to US male in terms of RMSE for these age groups are: 88.7% (=0.00042377/0.0004779), 73.2%, 81.1%, and 25.9%respectively. The Lee-Carter model outperforms our model for the age groups of 24-34 and 35-44, and this out-performance is consistent across genders, countries, and accuracy measures. Since we provide better fitting to most ages (45-109) than the Lee-Carter model, our overall performance in fitting the mortality rate curves of ages 25-109 is better (see Tables 1 and 2).

Table 3 implies that the performance of our model would be better than that of the Lee-Carter model on the sections of the mortality rate curves with ages greater than 25. This speculation is confirmed by replicating Tables 1-3 using the data on the age sections of 35+ and 45+.[11] Since the major customers of life insurance and annuity products are 25-years older with the annuity buyers concentrating on even older age groups, the better performance of our method relative to the Lee-Carter model as illustrated above is meaningful and has practical implications to life insurers.

Out-of-Sample Forecasting

[10]These is one exception: the 45-64 age group of UK male when measured by MAE. Our MAE is 103.2% of the Lee-Carter's.

[11]We do not present the replicated tables for the sake of paper length.

For simplicity and following Lee-Carter (1992) and Nelson and Siegel (1987), we assume that the dynamics of the two parameters follow the random walk with a drift individually. In formulas, $\alpha^{A}-\alpha^{A-1}=\Delta\alpha^{A}=\overline{\alpha}+\varepsilon_{\alpha}$ and $\beta^{A}-\beta^{A-1}=\Delta\beta^{A}=\overline{\beta}+\varepsilon_{\beta}$, where $\overline{\alpha}$ and $\overline{\beta}$ indicate the long-term means of α and β respectively, $\varepsilon_{\alpha}\sim N(0,\sigma_{\alpha})$, $\varepsilon_{\beta}\sim N(0,\sigma_{\beta})$.

We estimate the drifts using the F-year periods of data prior to the "current" year upon which the projection would be made.[12] For instance, if we have the mortality data up to 1989(i.e., $A = 1989$) and head for making projections for 1990, we will use the period of (1989-F+1) to1989 to estimate the drifts. Our estimators for the drifts are simply the averages of the changes in the parameter values over the corresponding F-year period:

$$\frown{\alpha}^{A-1}=\frac{1}{F-1}\sum_{i=A-F+1}^{A-1}\Delta\hat{\alpha}^{i} \quad \text{and} \quad \frown{\beta}^{A-1}=\frac{1}{F-1}\sum_{i=A-F+1}^{A-1}\Delta\hat{\beta}^{i}, \tag{10}$$

in which $\Delta\alpha^{i}$ and $\Delta\beta^{i}$ are calculated using the $\alpha^{i,1}$ and $\beta^{i,1}$ estimated in the in-sample fitting. We set F = 40 for the out-of-sample forecasting tests.

The projected parameters are equal to:

$$\tilde{\alpha}^{A}=\hat{\alpha}^{A-1}+\frown{\alpha}^{A-1} \quad \text{and} \quad \tilde{\beta}^{A}=\hat{\beta}^{A-1}+\frown{\beta}^{A-1}. \tag{11}$$ [13]

Using equation (3) to apply the projected parameters to the mortality rates of year A would render the projected mortality rates for a person aged x at the beginning of year A (i.e.,

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[12] Using Dickey-Fuller test, we confirm that there is no unit root in the time-series of $\alpha^{i,1}$ and $\beta^{i,1}$, $i=A-F+1,\cdots,A-1$.

[13] We use the top script ˜ to indicate a projected value, ˆ to denote an estimated value, and ⌢ for an averaged value.

$\tilde{p}_x^{A+1}$). Then we calculate RMSE and MAE to measure the forecasting errors. We repeat the procedures above for $A = 1990 - 2007$ and produce the following tables.

[Insert Tables 4-6 Here]

The rows titled "Overall-25" in Table 4 shows that the out-of-sample forecasting errors of our method are smaller than those of the Lee-Carter model in both US and UK.[14] For instance, the mean RMSE of our model are 0.00106694 and 0.00205025 while those of the Lee-Carter model are 0.00833841 and 0.00618506. Our model also produces smaller error variations. The standard deviations of MAE produced by our model are 0.00021660 and 0.00044835 in US and UK respectively, and they are smaller than 0.00419124 and 0.00306349 resulting from the Lee-Carter model. Other statistics of Table 4 also support the superiority of our method to the Lee-Carter model. An example is the much smaller maximum RMSE produced by our model: 0.00147155 vs. 0.01534837 in US and 0.00301468 vs. 0.01291569 in UK.

To each gender of US and UK, our forecasting errors are smaller as well. The ratios of our mean RMSE to Lee-Carter's mean RMSE are 14.0% (= 0.00131977/0.00944712), 11.3%, 31.6%, and 36.0% to US males, US females, UK males, and UK females respectively. The ratios with respect to median RMSE show similarly smaller errors: 12.5%, 12.3%, 28.4%, and 37.8% respectively. The standard deviations and the ranges of our forecasting errors are

[14]We calculate the forecasting errors using the same formulas to those used in calculating the in-sample fitting errors.

also smaller than those of Lee-Carter's. For instance, the ranges of our MAE to males are 0.000742 (US) and 0.001268 (UK). They are much smaller than the corresponding errors produced by the Lee-Carter model: 0.006534 and 0.006272.

We further find that the superiority of our model to the Lee-Carter model is even more significant in the forecasting than in the in-sample fitting. The error ratios of our method to Lee-Carter's with respect to all statistics in terms of both accuracy measures are consistently smaller in forecasting tests. This can be illustrated by comparing 14.0%, 11.3%, 31.6% and 36.0% (the ratios of our mean RMSE to Lee-Carter's mean RMSE presented in the previous paragraph) with the corresponding 30.1%, 25.9%, 42.4% and 48.5% presented in the previous section.

The superior forecasting performance of our LHT model to the Lee-Carter model is robust in both decades of 1990s and 2000s, as Table 5 shows. For instance, the ratios of our mean forecasting RMSE to Lee-Carter's for females are 11.0% (US) and 24.5% (UK) in 1990s and 8.8% and 48.8% in 2000s.[15] The ratios in terms of the standard deviations of MAE for males are 13.4% (US) and 22.4% (UK) in 1990s and 7.8% and 12.7% in 2000s.

Comparing Table 5 with Table 2, we also observe that the superiority of our model to the Lee-Carter model is more significant in forecasting than in in-sample fitting. All but two ratios of our errors to Lee-Carter's are smaller in forecasting than in fitting. For

[15] 11.0% is obtained by 0.00081182/0.00737924.

instance, the mean RMSE ratio of ours to Lee-Carter's in 1990s is 13.2% (= 0.00134595/0.01022495) from Table 5, and the corresponding ratio in Table 2 is 24.4% (= 0.00134275/0.00550242).

Table 6 tells similar stories to Table 3: our model performs well to the ages 45+ while the Lee-Carter model is better to ages 25-44. For instance, the ratios of mean forecasting errors between ours and Lee-Carter's to the age group of 85-109 are 11.7% (US males), 8.8% (US females), 24.4% (UK males), and 26.8% (UK females).[16] On the other hand, the corresponding ratios to the age group of 25-34 are 128.5%, 349.3%, 367.5% and 554.9% respectively. The relative performance of the two models is consistent in in-sample fitting and out-of-sample forecasting. In addition, the overall forecasting performance of our model is better since the LHT is superior to the Lee-Carter model in many more ages, and the people at these ages are the major customers of life insurers.

RISK MANAGEMENT

One major Usage of the mortality modeling/projection by life insurers is managing the mortality rate risk. Such management might involve developing internal / natural hedging portfolios of life insurance and annuity products so that reserves will not deviate from the expected to a significant extent.[17] We will illustrate in this section how our new method has

[16] 11.7% is calculated by 0.00218371/ 0.01869033.
[17] The conventional way to manage the mortality rate risk is by reinsurance. Alternative ways are to use the asset products / derivatives linked to mortalities, but only few products are available.

advantages over the existing literature in developing the hedging portfolios.

Mortality Durations

We may regard the LHT as a two-factor model on mortality improvements. The mortality improvement risks can thus be measured and managed by the "duration" with respect to the factors α and β [18].More specifically, the sensitivities of a policy's reserve R_* (based on the adjusted force of mortality $(1+\alpha)\times\mu_x+\beta$, see Tsai and Jiang (2011))to changes in α and β can be defined as

$$DD_\gamma(R_*) = -\frac{\partial R_*}{\partial \gamma}, \quad (12)$$

where *DD* denotes the dollar duration, and $\gamma=\alpha$ or β. *DD* measures the change of the reserves caused by the change of a mortality factor. It can also be deemed as the slope of the reserve-factor curve with the opposite sign.[19]

Under the LHT, the mortality durations may have explicit formulas that can facilitate the risk management. The mortality rates of a future year under the LHT are a function of the current-year mortality rates with the parameters/factors of α and β. Since appropriate reserving done today should take into account of the expected changes in mortalities, and

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[18]The idea is the same as the duration management for the interest rate risk. Many financial institutions, especially banks and life insurers, calculate the interest rate durations of individual assets and liabilities to measure their exposures to the interest rate risk. The use of the interest rate duration in finance markets is extensive (Bierwag and Fooladi, 2006). We apply the same idea of duration management but substitute the interest rate for the mortality rate as the underlying risk factor.

[19]Another popular risk measure is modified duration (*MD*) defined as: $MD_\gamma(R_*) = -\frac{\partial R_*}{\partial \gamma}\times\frac{1}{R_*} = \frac{DD_\gamma(R_*)}{R_*}$.
The economic meaning of *MD* is the percentage change of reserves caused by the change of a mortality factor. *DD* is more suitable for life insurance since it avoids the irregularities caused by small or zero reserves as Tsai (2009) identified.

thus should be based on the projected mortality rates, reserves are functions of α and β.

The partial derivatives of reserves with respect to α and β may have explicit formulas so that we may derive closed-form formulas for mortality duration.

For the original and adjusted force of mortality μ_x, we will define and derive the mortality durations for several products including n-year temporary life annuities-due, endowment, and term life insurance. Denote the net single premiums of the *m*-year deferred and *n*-year temporary life annuity-due issued to an individual aged x by:

$$ {}_{m|}\ddot{a}_{x:\overline{n}|\,i} = {}_{m|}\ddot{a}_{x:\overline{n}|\,i,(\mu_x+\delta)} = \sum_{k=m}^{m+n-1} {}_k p_x v^k = \sum_{k=m}^{m+n-1} e^{-\int_0^k [\mu_x(t)+\delta]dt} , \tag{13} $$

where $v = 1/(1+i)$ and $\delta = \ln(1+i)$. The symbol is associated with $\mu_x + \delta$ because it net single premium is based on the curve (or function) $\mu_x + \delta$. Note that $m = 0$ (*n*-year temporary life annuity-due), $n = 1$ (*m*-year pure endowment), and $n = \omega - x - m$ (*m*-year deferred whole life annuity-due) are three common special cases.

When the force of mortality μ_x is changed proportionally to $(1+\alpha)\mu_x$, the underlying curve becomes $(1+\alpha)\times\mu_x + \delta$, and the associated net single premium above is:

$$ {}_{m|}\ddot{a}_{x:\overline{n}|\,i,((1+\alpha)\times\mu_x+\delta)} = \sum_{k=m}^{m+n-1} e^{-\int_0^k [(1+\alpha)\times\mu_x(t)+\delta]dt} . \tag{14} $$

Then we can define the dollar duration of the single premium of the m-year deferred and n-year temporary life annuity-due with respect to α, the proportional shift of μ_x, by:

$$DD_\alpha({}_{m|}\ddot{a}_{x:\overline{n}|\,i}) = -\lim_{\alpha\to 0}\frac{{}_{m|}\ddot{a}_{x:\overline{n}|\,i,((1+\alpha)\times\mu_x+\delta)} - {}_{m|}\ddot{a}_{x:\overline{n}|\,i,(\mu_x+\delta)}}{\alpha}$$
$$= -\sum_{k=m}^{m+n-1} e^{-\int_0^k[\mu_x(t)+\delta]dt}\lim_{\alpha\to 0}\frac{e^{-\alpha\int_0^k\mu_x(t)dt}-1}{\alpha}\quad . \tag{15}$$

It is easy to see that

$$DD_\alpha({}_{m|}\ddot{a}_{x:\overline{n}|\,i}) = \sum_{k=m}^{m+n-1}(-\ln {}_k p_x)\,{}_k p_x v^k\,. \tag{16}$$

Similarly, we define the dollar duration with respect to β, the parallel (constant) shift of μ_x, by:

$$DD_\beta({}_{m|}\ddot{a}_{x:\overline{n}|\,i}) = -\lim_{\beta\to 0}\frac{{}_{m|}\ddot{a}_{x:\overline{n}|\,i}(\mu_x+\beta+\delta) - {}_{m|}\ddot{a}_{x:\overline{n}|\,i}(\mu_x+\delta)}{\beta}$$
$$= -\sum_{k=m}^{m+n-1} e^{-\int_0^k[\mu_x(t)+\delta]dt}\lim_{\beta\to 0}\frac{e^{-\int_0^k\beta dt}-1}{\beta} = \sum_{k=m}^{m+n-1} k\,{}_k p_x v^k\,. \tag{17}$$

Note the Equation (16) and (17) are also the duration of the reserve at time 0 with respect to α and β, respectively, for the single premium of the m-year deferred and n-year temporary life annuity-due. For the case of the level *h*-payment premium $(m \ge h)$, the dollar duration of the reserve at time 0 is

$$DD_\gamma({}_0R({}_hP({}_{m|}\ddot{a}_{x:\overline{n}|\,i}))) = DD_\gamma({}_{m|}\ddot{a}_{x:\overline{n}|\,i}) - {}_hP({}_{m|}\ddot{a}_{x:\overline{n}|\,i})\times DD_\gamma(\ddot{a}_{x:\overline{h}|\,i})\,, \tag{18}$$

where $\gamma = \alpha,\ \beta$, and ${}_hP({}_{m|}\ddot{a}_{x:\overline{n}|\,i}) = {}_{m|}\ddot{a}_{x:\overline{n}|\,i}/\ddot{a}_{x:\overline{h}|\,i}$.

For *n*-year endowment, we assume the death benefit is payable at the end of the year of death. Denote its net single premium as

$$A_{x:\overline{n}|\,i} = \sum_{k=0}^{n-1}({}_k p_x - {}_{k+1}p_x)\times v^{k+1} + {}_n p_x \times v^n = 1 - d\times\ddot{a}_{x:\overline{n}|\,i}\,, \tag{19}$$

where $d = 1 - v$, which implies $DD_\gamma(A_{x:\overline{n}|\,i}) = -d\times DD_\gamma(\ddot{a}_{x:\overline{n}|\,i})$, $\gamma = \alpha,\beta$. Setting $n = \omega - x$

yields $DD_\gamma(A_x) = -d \times DD_\gamma(\ddot{a}_x)$, a relation between the dollar durations of whole life insurance and whole life annuity. For *n*-year term life insurance, since its net single premium

$A^1_{x:\overline{n}|\,i} = A_{x:\overline{n}|\,i} - {}_n|\ddot{a}_{x:\overline{1}|\,i}$, we have $DD_\gamma(A^1_{x:\overline{n}|\,i}) = -d \times DD_\gamma(\ddot{a}_{x:\overline{n}|\,i}) - DD_\gamma({}_n|\ddot{a}_{x:\overline{1}|\,i})$.

We give some examples for illustration in the following. Assume that $i = 3\%$, $x = 45$ and ${}_k p_x$, $k = 1, 2, \cdots, n = 20$ are the out-of-sample forecasting mortality rates using our LHT with $A = 2007$ and $F = 40$. We calculate the DD_α and DD_β of reserves at time 0 for 10 products, and the results are placed in Table 7.

[Insert Table 7 Here]

We find that annuity products and pure endowment have positive mortality durations, implying exposures to the longevity risk. For instance, (DD_α, DD_β) of the reserve at time 0 for single-payment 20-year deferred whole life annuity-due and 20-payment 20-year pure endowment are (3.16, 183.86) and (0.06, 5.48) respectively. Whole life insurance, on the other hand, has negative mortality durations that implies exposures to the mortality deterioration risk.[20] For example, the DD_α and DD_β of the reserve at time 0 for the 20-payment whole life insurance is -0.13 and -12.11, respectively.

The DD_β s of reserves for life insurance and annuity products have much bigger magnitudes than the DD_α s, which means that reserves are more sensitive to shocks/changes

[20]Following Wang et al. (2010), we define that a product is subject to the longevity risk if mortality improvements would increase the reserves of the product. A product is subject to the mortality deterioration risk if increases in mortality rates would increase the product's reserves. The term "mortality rate risk" is used as a general term referring to the risks of reserve changes due to changes in mortality rates, and thus includes both the longevity risk and the mortality deterioration risk.

in β (parallel shift of the force-of-mortality curves) than in α (proportional shift). The fact that DD_β is larger than DD_α further highlights the importance of our extension to Wang et al. (2010). Their immunization strategies are based on the assumption that forces of mortality change proportionally and thus use only DD_α. Reserves of life insurance are however more sensitive to the constant changes in forces of mortality.

Reserves of whole life annuities have the largest mortality duration figures, (3.16, 183.86) and (2.84, 131.43),thus fluctuate more with shocks to mortality rates than other products and have larger mortality rate risk. Term life insurance has the second largest sensitivity to parallel shifts of forces of mortality ($DD_\beta =$ -12.86 and -13.76) while whole life insurance has the second largest sensitivity to the proportional shifts ($DD_\alpha =$ -0.11 and -0.13). The endowment is least sensitive to proportional shifts of forces of mortality ($DD_\alpha =$ -0.02 and -0.05).

Internal Hedging

After calculating the mortality durations of reserves for several life insurance products, we can measure the mortality rate risk of the portfolios consisting of these products and further construct the portfolios with minimal mortality risks. The duration of reserves of a portfolio of life insurance and annuity products with respect to a mortality factor is simply the weighted average of the mortality durations of reserves of individual products, with the weights to be determined. By combining products in different ways, we may find a portfolio with zero mortality durations and is thus “immunized” from the mortality risk. Managing the mortality rate risk by carefully constructing product portfolios is called the natural hedging strategy in the literature (e.g., Cox and Lin, 2007; Wang et al., 2010).

We construct some portfolios that have the minimal exposure to the mortality rate risk with regard to reserves at time 0. Ideally, we want to be able to construct a portfolio with zero DD_α and DD_β. This requires three products since we have three equations to solve as follows:

$$\sum_{i=1}^{3} w_i \times DD_\alpha{}^i = 0,$$
$$\sum_{i=1}^{3} w_i \times DD_\beta{}^i = 0, \quad (20)$$
$$\sum_{i=1}^{3} w_i = 1, \text{ and } w_i > 0,\ i = 1,\ 2,\ 3.$$

The solutions to Equation system (20) are:

$$w_1 = \begin{vmatrix} DD_\alpha{}^2 & DD_\alpha{}^3 \\ DD_\beta{}^2 & DD_\beta{}^3 \end{vmatrix} / D, w_2 = \begin{vmatrix} DD_\alpha{}^3 & DD_\alpha{}^1 \\ DD_\beta{}^3 & DD_\beta{}^1 \end{vmatrix} / D, w_3 = \begin{vmatrix} DD_\alpha{}^1 & DD_\alpha{}^2 \\ DD_\beta{}^1 & DD_\beta{}^2 \end{vmatrix} / D, \quad (21)$$

where $D = \begin{vmatrix} DD_\alpha{}^2 & DD_\alpha{}^3 \\ DD_\beta{}^2 & DD_\beta{}^3 \end{vmatrix} + \begin{vmatrix} DD_\alpha{}^3 & DD_\alpha{}^1 \\ DD_\beta{}^3 & DD_\beta{}^1 \end{vmatrix} + \begin{vmatrix} DD_\alpha{}^1 & DD_\alpha{}^2 \\ DD_\beta{}^1 & DD_\beta{}^2 \end{vmatrix}$.

The key differences in generating an optimal portfolio between our method and the method of Wang et al. (2010) are the calculations of the mortality durations. Our method involves calculating two mortality durations while they computed only one under an implicit assumption on how the mortality rate curve changes. They also assumed that the force of mortality within each age interval $(x,\ x+1)$ is constant, and is changed by a certain equal percentage (e.g., -10%); then the resulting changes in reserves are calculated to obtain the mortality duration (see Equation (15) with $\lim_{\alpha \to 0}$ removed). Thus, the numerical value of their mortality duration depends on the size and sign of the equal percentage. The assumption of constant force of mortality is inconsistent with the observable mortality data. Neither is the equal percentage change for all forces of mortality consistent with the observed behaviors of mortality rate curves. This assumption is indeed equivalent to $\alpha = -0.1$ and $\beta = 0$ under our LHT. Therefore, our method is more general with an extra benefit of having explicit formulas for the durations.

Illustrations

We form three portfolios with both weighted DD_α and DD_β equal to 0, as Table 8 shows. The weights are calculated using Equation (20). Portfolio P_1 consists of life insurance products: whole life, term life, and pure endowment (all 20-payment). The pure endowment accounts for 69.51% while whole life insurance makes up 23.25%. The term life insurance contributes 7.24% only. Portfolios P_2 (all single-payment) and P_3 (a mixing of 20-payment and single-payment) are examples of the natural hedging between life insurance and annuity products. The mortality deterioration risk of whole life insurance is hedged by the longevity risk of annuities and pure endowment. The whole life insurance accounts for about two third of Portfolios P_2 and P_3 (66.61% and 60.32%, respectively), and pure endowment makes up thirty some percent (31.79% and 38.34%). The weights of annuities are low due to their large mortality durations. These compositions show the substitution effect between annuities and pure endowment in the mortality rate risk, which is important for the countries with small annuity markets.

[Insert Table 8 Here]

Equation (20) might have no solutions. It can be shown that all w_1, w_2, and w_3 fall within the interval $(0,1)$ if and only if the three determinants in D are either all positive or all negative. The determinants might not have uniform signs because of close relations among the reserves of insurance and annuity products and the resulting bondages among the mortality durations. We present two portfolios that have negative weights in Table 9. Portfolio P_4 is formed by replacing the 20-payment and 20-year pure endowment of Portfolio P_1 with the 20-payment and 20-year deferred whole life annuity-due, and keeping the other two products unchanged. Portfolio P_5 is constructed with only the single-payment and 20-year term life insurance being substituted for the single-payment whole life insurance of Portfolio P_2. One would probably expect natural hedging to be feasible. However, the close relation $A_x = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} - d \times {}_{n|}\ddot{a}_x$ (where $A_{x:\overline{n}|}^{\ \ 1}$ is the net single premium of n-year pure endowment) among the net single premiums of these three underlying products prevents all weights for each of Portfolios P_4 and P_5 from being positive. The insurer has to “own” rather than “sell” the product with negative weight to hedge the mortality rate risk. These examples did not show up in the literature like Wang et al. (2010) since they calculated only one mortality duration; it needs only two products to hedge the sole duration, and the weights shall be positive as long as the durations of the reserves for these two products have different signs. When the mortality rate curve changes in more complex ways and thus demand more than one factor to model its dynamics, it takes at least three products for the immunization. The all-positive weights will appear only when the mortality durations meet the aforementioned necessary and sufficient condition. Therefore, life insurers may not be able to internally hedge the mortality rate risk to the full extent.

[Insert Table 9 Here]

External hedging arrangements are thus needed. The negative weights mean that a life insurer may have to buy life insurance products from other issuers to achieve a better hedge for the mortality rate risk. Life settlements seem to fit this demand, in addition to asset-side considerations (e.g., diversifications and/or high yields). Other mortality securities like mortality bonds and derivatives may also render hedging benefits for life insurers to hedge the mortality rate risk externally and complement internal hedging. The possible incompleteness of internal hedging found in this paper is new to the literature.

CONCLUSIONS AND REMARKS

Modeling and projecting mortality rates are essential to life insurers, social benefits programs, and the society as a whole. Future mortality rates affect the pricing and reserving of insurance/annuity products that in turn have impacts on the solvency of the insurer. Future mortality rates also affect the solvency and continuity of various social benefits programs (e.g., retirement plans and health care programs), through affecting the cash outflows as well as inflows. The population structure of a society has widespread impacts on the demands and supplies of many industries, and it is shaped by mortality rates.

The literature thus studied mortality rates extensively. Early researches of demographers and sociologists developed cross-sectional, explanatory models. Starting from the early 1990s, statisticians and actuarial scholars tried to model the dynamics of mortality rates. Lee and Carter (1992) was the pioneer and stimulated many papers along this line. Another line developed later was to use curve fitting for mortality rate curves and then built up time-series models of the function parameters to forecast mortality rates. The latest line of the literature applied the interest rate modeling method developed in the finance field to the modeling of mortality rates.

We propose a different method from the existing literature in this paper. Instead of modeling the mortality rates themselves, we model the relation between two forces of mortality. We assume that the force of mortality on a later curve is a linear transformation of one on an earlier curve. Then we establish the time-series behaviors of the linear transformation parameters to project mortality rates. This methodology might work because mortality rate curves changed in small and stable ways due to biological natures and/or the rigidity of the changes in social systems.

To investigate the potential of our new thought, we use empirical data to test the performance of the linear hazard transformation relative to that of the well-known Lee-Carter model. We conduct both in-sample fitting tests and out-of-sample forecasting tests using the data of US and UK that cover both genders from 1950 to 2007. The empirical results show that our LHT dominates the Lee-Carter model in both types of tests to significant extents. The idea of regarding changes in mortality rates across time as transformation seemed to work well and have good potential.

We further illustrate two advantages of our model in managing the mortality rate risk using product portfolios in this paper. Since our model is parsimonious with two parameters, we need to calculate only two durations with respect to these two parameters to construct an immunized portfolio consisting of life insurance and annuity products. Our model is more general than Wang et al. (2010) in which they had to assume constant forces of mortality within each age interval and proportional shifts of force of mortality to calculate a sole mortality duration. Their model is indeed more restricted than our model due to more assumptions made. Another advantage of our model is that we may have explicit formulas for the mortality durations, which facilitates the risk management.

This paper points out a new way to model and forecast mortality rates. The empirical results show that our LHT model outperforms the classic Lee-Carter model. The risk management resulting from the new method is more general, more accurate, and easier to implement than that proposed by Wang et al. (2010). The potential of the LHT model is thus confirmed. Our findings about the incompleteness of internal hedging have implications on the risk management strategies of life insurers and call for more active second markets of mortality securities.

There is much to be explored along this line though. For instance, how will the choices of a (the difference between the base year A and the target year B) and other forms of transformation affect the performance? How can this method incorporate the cohort effect? This paper initiates a first attempt to attract more research in the future.

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TABLES

Table 1: Summary Statistics of In-Sample Fitting Errors

	RMSE									
	LHT					Lee-Carter				
Country-Gender-Starting Age	Mean	Std. Deviation	Median	Min	Max	Mean	Std. Deviation	Median	Min	Max
US-Male-25	**0.00177571**	**0.00068953**	**0.00155900**	**0.00082463**	**0.00377734**	0.00589731	0.00318861	0.00495500	0.00097021	0.01284675
US-Fenmale-25	**0.00124397**	**0.00054813**	**0.00109649**	**0.00040205**	**0.00317637**	0.00480551	0.00274708	0.00453950	0.00077150	0.01077886
US-Overall-25	**0.00150984**	**0.00061883**	**0.00132775**	**0.00061334**	**0.00347685**	0.00535141	0.00296784	0.00474725	0.00087086	0.01181281
UK-Male-25	**0.00296770**	**0.00089865**	**0.00277804**	**0.00163560**	**0.00573901**	0.00699724	0.00314613	0.00625656	0.00229499	0.01619813
UK-Female-25	**0.00187019**	**0.00061562**	**0.00176119**	**0.00082951**	**0.00462851**	0.00385473	0.00196059	0.00339774	0.00079847	0.00814878
UK-Overall-25	**0.00241895**	**0.00075714**	**0.00226962**	**0.00123256**	**0.00518376**	0.00542598	0.00255336	0.00482715	0.00154673	0.01217346
US-Male-45	**0.00195973**	**0.00078344**	**0.00171994**	**0.00091554**	**0.00426869**	0.00672145	0.00363200	0.00552952	0.00142728	0.01471459
US-Fenmale-45	**0.00138579**	**0.00061616**	**0.00118361**	**0.00044332**	**0.00362277**	0.00547416	0.00323359	0.00482226	0.00090090	0.01299220
US-Overall-45	**0.00167276**	**0.00069980**	**0.00145178**	**0.00067943**	**0.00394573**	0.00609781	0.00343280	0.00517589	0.00116409	0.01385340
UK-Male-45	**0.00333705**	**0.00102594**	**0.00313008**	**0.00186796**	**0.00643850**	0.00835331	0.00405067	0.00730579	0.00284869	0.02179470
UK-Female-45	**0.00210745**	**0.00069433**	**0.00197899**	**0.00092464**	**0.00508577**	0.00463653	0.00244948	0.00457057	0.00099154	0.00991423
UK-Overall-45	**0.00272225**	**0.00086014**	**0.00255454**	**0.00139630**	**0.00576213**	0.00649492	0.00325008	0.00593818	0.00192011	0.01585447

Table 1 Continued

	MAE									
	LHT					Lee-Carter				
Country-Gender-Starting Age	Mean	Std. Deviation	Median	Min	Max	Mean	Std. Deviation	Median	Min	Max
US-Male-25	**0.00103666**	**0.00033670**	**0.00101881**	**0.00052527**	**0.00198526**	0.00331025	0.00162196	0.00297684	0.00071368	0.00674417
US-Fenmale-25	**0.00070190**	**0.00027261**	**0.00065248**	**0.00028490**	**0.00144673**	0.00258104	0.00133857	0.00255819	0.00049303	0.00527297
US-Overall-25	**0.00086928**	**0.00030465**	**0.00083565**	**0.00040508**	**0.00171600**	0.00294564	0.00148026	0.00276752	0.00060336	0.00600857
UK-Male-25	**0.00159354**	**0.00047700**	**0.00149248**	**0.00089071**	**0.00306352**	0.00402932	0.00173013	0.00377186	0.00132454	0.00955776
UK-Female-25	**0.00100769**	**0.00033373**	**0.00096951**	**0.00049886**	**0.00261748**	0.00222125	0.00106616	0.00195894	0.00047803	0.00467101
UK-Overall-25	**0.00130061**	**0.00040537**	**0.00123099**	**0.00069479**	**0.00284050**	0.00312529	0.00139814	0.00286540	0.00090128	0.00711439
US-Male-45	**0.00123166**	**0.00039592**	**0.00122610**	**0.00066729**	**0.00243652**	0.00422664	0.00209149	0.00373474	0.00096228	0.00866937
US-Fenmale-45	**0.00085479**	**0.00032511**	**0.00081749**	**0.00032320**	**0.00165701**	0.00331775	0.00181321	0.00313347	0.00061814	0.00725569
US-Overall-45	**0.00104323**	**0.00036051**	**0.00102180**	**0.00049524**	**0.00204676**	0.00377220	0.00195235	0.00343410	0.00079021	0.00796253
UK-Male-45	**0.00194069**	**0.00056712**	**0.00181149**	**0.00112349**	**0.00394262**	0.00536570	0.00248700	0.00487549	0.00181030	0.01415680
UK-Female-45	**0.00124411**	**0.00042643**	**0.00115642**	**0.00058448**	**0.00335352**	0.00299605	0.00154390	0.00301799	0.00073237	0.00630859
UK-Overall-45	**0.00159240**	**0.00049677**	**0.00148396**	**0.00085399**	**0.00364807**	0.00418088	0.00201545	0.00394674	0.00127134	0.01023270

Table 2: Fitting Errors by Periods

	RMSE											
	LHT											
	1950s		1960s		1970s		1980s		1990s		2000s	
Country-Gender-Starting Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	**0.00268754**	**0.00064566**	**0.00515583**	**0.00042748**	**0.00225550**	**0.00078059**	**0.00138438**	**0.00027115**	**0.00134275**	**0.00031779**	**0.00132201**	**0.00018418**
US-Fenmale-25	**0.00211962**	**0.00066297**	**0.00443231**	**0.00029799**	**0.00143025**	**0.00031388**	**0.00094679**	**0.00013096**	**0.00081379**	**0.00020780**	**0.00084312**	**0.00013173**
UK-Male-25	**0.00443013**	**0.00100552**	**0.00679529**	**0.00055079**	**0.00306463**	**0.00054677**	**0.00240436**	**0.00045389**	**0.00240003**	**0.00044426**	**0.00277908**	**0.00055154**
UK-Female-25	**0.00288823**	**0.00078172**	**0.00520569**	**0.00044678**	**0.00179600**	**0.00028044**	**0.00167294**	**0.00027971**	**0.00136491**	**0.00026368**	**0.00179609**	**0.00031272**
US-Male-45	**0.00300354**	**0.00073127**	**0.00619016**	**0.00047905**	**0.00250387**	**0.00089529**	**0.00148009**	**0.00025140**	**0.00142683**	**0.00030627**	**0.00146076**	**0.00019117**
US-Fenmale-45	**0.00238689**	**0.00075800**	**0.00534280**	**0.00033750**	**0.00155364**	**0.00029977**	**0.00104805**	**0.00012228**	**0.00089073**	**0.00021643**	**0.00093843**	**0.00014425**
UK-Male-45	**0.00499497**	**0.00112205**	**0.00813900**	**0.00062479**	**0.00347206**	**0.00064987**	**0.00267390**	**0.00052128**	**0.00264352**	**0.00045235**	**0.00315087**	**0.00063995**
UK-Female-45	**0.00324943**	**0.00084284**	**0.00624501**	**0.00053028**	**0.00202777**	**0.00032768**	**0.00188949**	**0.00031719**	**0.00151723**	**0.00029201**	**0.00203533**	**0.00035367**
	Lee-Carter											
	1950s		1960s		1970s		1980s		1990s		2000s	
Country-Gender-Starting Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	0.00464334	0.00188334	0.01019004	0.00275932	0.00726749	0.00365147	0.00747126	0.00309644	0.00550242	0.00276870	0.00857318	0.00324581
US-Fenmale-25	0.00420302	0.00144294	0.00961021	0.00192137	0.00490843	0.00280018	0.00555747	0.00272298	0.00500266	0.00277740	0.00768820	0.00335832
UK-Male-25	0.01074944	0.00410495	0.01109237	0.00284157	0.00730927	0.00256439	0.00773218	0.00238643	0.00673635	0.00274530	0.00545388	0.00153993
UK-Female-25	0.00513624	0.00113610	0.00978740	0.00198389	0.00334688	0.00158207	0.00328574	0.00135122	0.00428204	0.00203200	0.00340820	0.00175523
US-Male-45	0.00520444	0.00216074	0.01196434	0.00315089	0.00832156	0.00423971	0.00852663	0.00347684	0.00627201	0.00299316	0.00987142	0.00365483
US-Fenmale-45	0.00435189	0.00159538	0.01154532	0.00207994	0.00560418	0.00329449	0.00687731	0.00325584	0.00560810	0.00316702	0.00878131	0.00403268
UK-Male-45	0.01375261	0.00533022	0.01315877	0.00313382	0.00871326	0.00299586	0.00968319	0.00341369	0.00769405	0.00303335	0.00600507	0.00182729
UK-Female-45	0.00611162	0.00307967	0.01137970	0.00218446	0.00402800	0.00149553	0.00490052	0.00212041	0.00516479	0.00239665	0.00448972	0.00246475

Table 2 Continued

	MAE											
	LHT											
	1950s		1960s		1970s		1980s		1990s		2000s	
Country-Gender-Initial Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	**0.00141422**	**0.00035433**	**0.00112557**	**0.00022536**	**0.00115627**	**0.00028635**	**0.00088070**	**0.00024106**	**0.00085518**	**0.00023842**	**0.00077308**	**0.00016624**
US-Fenmale-25	**0.00100000**	**0.00026364**	**0.00082404**	**0.00014996**	**0.00086127**	**0.00025504**	**0.00054099**	**0.00011617**	**0.00048268**	**0.00015650**	**0.00048985**	**0.00008143**
UK-Male-25	**0.04724530**	**0.00059162**	**0.04117325**	**0.00040339**	**0.03843510**	**0.00016149**	**0.03706781**	**0.00030059**	**0.03838141**	**0.00038810**	**0.03630768**	**0.00021781**
UK-Female-25	**0.03771190**	**0.00046925**	**0.03397307**	**0.00024119**	**0.03065037**	**0.00015729**	**0.02970170**	**0.00017057**	**0.02856451**	**0.00021783**	**0.02898509**	**0.00016386**
US-Male-45	**0.00169460**	**0.00045561**	**0.00138150**	**0.00025988**	**0.00136623**	**0.00030468**	**0.00101035**	**0.00023034**	**0.00098561**	**0.00024621**	**0.00093957**	**0.00016352**
US-Fenmale-45	**0.00123268**	**0.00030287**	**0.00103524**	**0.00017397**	**0.00103068**	**0.00026267**	**0.00065545**	**0.00012306**	**0.00056940**	**0.00016915**	**0.00059014**	**0.00009786**
UK-Male-45	**0.05250554**	**0.00073683**	**0.04549151**	**0.00041284**	**0.04249554**	**0.00021222**	**0.04057909**	**0.00028071**	**0.04183764**	**0.00039225**	**0.04044146**	**0.00028523**
UK-Female-45	**0.04230551**	**0.00062303**	**0.03769576**	**0.00028870**	**0.03388070**	**0.00017950**	**0.03322851**	**0.00020560**	**0.03130549**	**0.00024508**	**0.03216362**	**0.00021298**
	Lee-Carter											
	1950s		1960s		1970s		1980s		1990s		2000s	
Country-Gender-Initial Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	0.00248920	0.00094400	0.00316467	0.00146667	0.00349875	0.00189441	0.00373726	0.00155548	0.00276734	0.00147658	0.00432513	0.00153663
US-Fenmale-25	0.00232526	0.00082070	0.00298914	0.00102999	0.00218867	0.00136300	0.00243990	0.00125657	0.00221144	0.00140743	0.00348757	0.00155525
UK-Male-25	0.07553601	0.00227123	0.06091544	0.00175765	0.06241321	0.00144086	0.06482175	0.00113642	0.05959783	0.00133341	0.05558933	0.00084601
UK-Female-25	0.05465173	0.00062246	0.05556498	0.00109078	0.04087144	0.00088888	0.04187274	0.00065759	0.04565214	0.00102129	0.04149722	0.00090610
US-Male-45	0.00310509	0.00122126	0.00394438	0.00185011	0.00451858	0.00252206	0.00478638	0.00194419	0.00349709	0.00180472	0.00568853	0.00191000
US-Fenmale-45	0.00264867	0.00104047	0.00391357	0.00123572	0.00276398	0.00186221	0.00351035	0.00177620	0.00277714	0.00181172	0.00445296	0.00221042
UK-Male-45	0.08993610	0.00334641	0.06937412	0.00217169	0.07259969	0.00191749	0.07448426	0.00184732	0.06832124	0.00168497	0.06129493	0.00109244
UK-Female-45	0.05759657	0.00211232	0.06167988	0.00132437	0.05007383	0.00093574	0.05412225	0.00136055	0.05405386	0.00140131	0.04910503	0.00152393

Table 3: Fitting Errors by Ages

	RMSE											
	LHT						Lee-Carter					
Country-Gender-Starting Age	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109
US-Male-25	0.00037056	0.00035783	**0.00042377**	**0.00107242**	**0.00201703**	**0.00316229**	**0.00017284**	**0.00021223**	0.00047790	0.00146537	0.00248598	0.01221766
US-Fenmale-25	0.00022627	0.00020923	**0.00029243**	**0.00070518**	**0.00139784**	**0.00227947**	**0.00004163**	**0.00008822**	0.00033335	0.00084221	0.00186031	0.01011994
UK-Male-25	0.00052814	0.00051438	**0.00060431**	**0.00161901**	**0.00319225**	**0.00520155**	**0.00012918**	**0.00015740**	0.00060470	0.00230320	0.00378444	0.01385506
UK-Female-25	0.00027876	0.00026287	**0.00035121**	**0.00086801**	**0.00216387**	**0.00329375**	**0.00007573**	**0.00009134**	0.00049070	0.00110803	0.00224293	0.00780307
US-Male-45	X	x	0.00061608	**0.00099171**	**0.00193298**	**0.00306426**	x	x	**0.00040381**	0.00137475	0.00240726	0.01218835
US-Fenmale-45	X	x	0.00040589	**0.00066090**	**0.00134502**	**0.00222486**	x	x	**0.00030581**	0.00074458	0.00149008	0.01019382
UK-Male-45	X	x	0.00087920	**0.00143764**	**0.00309423**	**0.00513956**	x	x	**0.00049261**	0.00173463	0.00327833	0.01477789
UK-Female-45	X	x	0.00049890	**0.00080628**	**0.00211339**	**0.00324889**	x	x	**0.00035550**	0.00091194	0.00220263	0.00831377

	MAE											
	LHT						Lee-Carter					
Country-Gender-Starting Age	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109
US-Male-25	0.00031152	0.00030077	**0.00029456**	**0.00075272**	**0.00155686**	**0.00212026**	**0.00013807**	**0.00016446**	0.00035741	0.00114212	0.00198834	0.00959571
US-Fenmale-25	0.00017501	0.00016529	**0.00019008**	**0.00047158**	**0.00103575**	**0.00149536**	**0.00003324**	**0.00007079**	0.00024297	0.00067023	0.00135086	0.00773112
UK-Male-25	0.02021275	0.01992017	**0.02066789**	**0.03394221**	**0.04776523**	**0.05814386**	**0.00993424**	**0.01131561**	0.02003653	0.04268431	0.05601814	0.10631947
UK-Female-25	0.01515646	0.01455873	**0.01601711**	**0.02483213**	**0.03951501**	**0.04661543**	**0.00740704**	**0.00834367**	0.01894369	0.02949906	0.04039952	0.07883175
US-Male-45	X	x	0.00048955	**0.00067796**	**0.00147851**	**0.00194810**	x	x	**0.00030283**	0.00107183	0.00193951	0.00954246
US-Fenmale-45	X	x	0.00029635	**0.00044640**	**0.00098429**	**0.00141309**	x	x	**0.00022736**	0.00057867	0.00108416	0.00777914
UK-Male-45	X	x	0.02571326	**0.03161458**	**0.04656051**	**0.05700809**	x	x	**0.01883131**	0.03686940	0.05102390	0.10991818
UK-Female-45	X	x	0.01958455	**0.02378631**	**0.03877623**	**0.04582668**	x	x	**0.01621381**	0.02692415	0.04145904	0.08125223

Table 4: Summary Statistics of Out-of-Sample Forecasting Errors

	RMSE									
	LHT					Lee-Carter				
Country-Gender-Initial Age	Mean	Std. Deviation	Median	Min	Max	Mean	Std. Deviation	Median	Min	Max
US-Male-25	**0.00131977**	**0.00026189**	**0.00130986**	**0.00086876**	**0.00187173**	0.00944712	0.00390974	0.01049683	0.00262827	0.01573486
US-Fenmale-25	**0.00081411**	**0.00017132**	**0.00079987**	**0.00039897**	**0.00107138**	0.00722970	0.00447274	0.00651801	0.00107692	0.01496187
US-Overall-25	**0.00106694**	**0.00021660**	**0.00105486**	**0.00063387**	**0.00147155**	0.00833841	0.00419124	0.00850742	0.00185260	0.01534837
UK-Male-25	**0.00253301**	**0.00053011**	**0.00235133**	**0.00181341**	**0.00360328**	0.00801532	0.00375362	0.00827227	0.00324610	0.01608440
UK-Female-25	**0.00156750**	**0.00036658**	**0.00149865**	**0.00090829**	**0.00242608**	0.00435480	0.00237336	0.00396455	0.00149803	0.00974699
UK-Overall-25	**0.00205025**	**0.00044835**	**0.00192499**	**0.00136085**	**0.00301468**	0.00618506	0.00306349	0.00611841	0.00237207	0.01291569
US-Male-45	**0.00143534**	**0.00025542**	**0.00147314**	**0.00099023**	**0.00193454**	0.01101811	0.00418484	0.01247884	0.00343653	0.01754075
US-Fenmale-45	**0.00090639**	**0.00018944**	**0.00089841**	**0.00044615**	**0.00119418**	0.00844427	0.00528367	0.00863630	0.00092319	0.01724894
US-Overall-45	**0.00117086**	**0.00022243**	**0.00118578**	**0.00071819**	**0.00156436**	0.00973119	0.00473426	0.01055757	0.00217986	0.01739485
UK-Male-45	**0.00283971**	**0.00060005**	**0.00265904**	**0.00203238**	**0.00409812**	0.00976611	0.00474127	0.00984962	0.00323957	0.02061898
UK-Female-45	**0.00177054**	**0.00041910**	**0.00169096**	**0.00105753**	**0.00275328**	0.00546563	0.00361579	0.00486669	0.00179560	0.01284077
UK-Overall-45	**0.00230513**	**0.00050958**	**0.00217500**	**0.00154496**	**0.00342570**	0.00761587	0.00417853	0.00735815	0.00251758	0.01672987

Table 4 Continued

	MAE									
	LHT					Lee-Carter				
Country-Gender-Initial Age	Mean	Std. Deviation	Median	Min	Max	Mean	Std. Deviation	Median	Min	Max
US-Male-25	**0.00082338**	**0.00020915**	**0.00077637**	**0.00050997**	**0.00125242**	0.00510444	0.00189275	0.00557389	0.00154932	0.00808325
US-Fenmale-25	**0.00048637**	**0.00011784**	**0.00045770**	**0.00027456**	**0.00072229**	0.00368249	0.00216130	0.00316540	0.00066841	0.00767814
US-Overall-25	**0.00065488**	**0.00016350**	**0.00061704**	**0.00039226**	**0.00098735**	0.00439347	0.00202703	0.00436964	0.00110886	0.00788069
UK-Male-25	**0.00140392**	**0.00032258**	**0.00135094**	**0.00094716**	**0.00221483**	0.00465403	0.00183162	0.00504375	0.00209264	0.00836900
UK-Female-25	**0.00084325**	**0.00018866**	**0.00083480**	**0.00051974**	**0.00129117**	0.00251143	0.00116782	0.00228284	0.00100242	0.00504963
UK-Overall-25	**0.00112359**	**0.00025562**	**0.00109287**	**0.00073345**	**0.00175300**	0.00358273	0.00149972	0.00366330	0.00154753	0.00670931
US-Male-45	**0.01432286**	**0.01466846**	**0.00901199**	**0.00064385**	**0.04392397**	0.02181066	0.01616575	0.01724764	0.00233446	0.04414928
US-Fenmale-45	**0.01161907**	**0.01291383**	**0.00636685**	**0.00050780**	**0.03763328**	0.01986005	0.01722995	0.01354592	0.00060508	0.05716010
US-Overall-45	**0.01297097**	**0.01379115**	**0.00768942**	**0.00057583**	**0.04077863**	0.02083536	0.01669785	0.01539678	0.00146977	0.05065469
UK-Male-45	**0.00170481**	**0.00034429**	**0.00165091**	**0.00113107**	**0.00253036**	0.00621366	0.00269946	0.00645344	0.00234979	0.01235919
UK-Female-45	**0.00103446**	**0.00022622**	**0.00098369**	**0.00066065**	**0.00151584**	0.00358369	0.00222159	0.00324162	0.00127869	0.00836609
UK-Overall-45	**0.00136964**	**0.00028525**	**0.00131730**	**0.00089586**	**0.00202310**	0.00489867	0.00246053	0.00484753	0.00181424	0.01036264

Table 5: Forecasting Errors by Periods

	RMSE							
	LHT				Lee-Carter			
	1990s		2000s		1990s		2000s	
Country-Gender-Initial Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	**0.00134595**	**0.00031484**	**0.00134494**	**0.00017338**	0.01022495	0.00367282	0.01022326	0.00418403
US-Fenmale-25	**0.00081182**	**0.00019942**	**0.00085644**	**0.00011938**	0.00737924	0.00411397	0.00972368	0.00452000
UK-Male-25	**0.00239812**	**0.00042241**	**0.00280711**	**0.00057069**	0.00995249	0.00369596	0.00724146	0.00322537
UK-Female-25	**0.00141083**	**0.00028202**	**0.00182829**	**0.00032640**	0.00575024	0.00245177	0.00374325	0.00177323
US-Male-45	**0.00143406**	**0.00029776**	**0.00148713**	**0.00018163**	0.01184287	0.00382832	0.01171471	0.00458078
US-Fenmale-45	**0.00089960**	**0.00021589**	**0.00095792**	**0.00013932**	0.00894143	0.00497786	0.01110474	0.00537895
UK-Male-45	**0.00265535**	**0.00043627**	**0.00318440**	**0.00065996**	0.01265157	0.00437059	0.00806841	0.00377105
UK-Female-45	**0.00158400**	**0.00031923**	**0.00207659**	**0.00036451**	0.00781925	0.00398794	0.00449500	0.00225171

	MAE							
	LHT				Lee-Carter			
	1990s		2000s		1990s		2000s	
Country-Gender-Initial Age	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation	Mean	Std Deviation
US-Male-25	**0.00085813**	**0.00023554**	**0.00077995**	**0.00016025**	0.00511454	0.00175431	0.00509183	0.00205265
US-Fenmale-25	**0.00048127**	**0.00014443**	**0.00049274**	**0.00007139**	0.00310153	0.00206093	0.00440871	0.00206196
UK-Male-25	**0.00146485**	**0.00036627**	**0.00132776**	**0.00023663**	0.00515323	0.00163770	0.00403004	0.00186948
UK-Female-25	**0.00083891**	**0.00020270**	**0.00084869**	**0.00016933**	0.00290413	0.00118665	0.00202056	0.00093525
US-Male-45	**0.02502240**	**0.01138836**	**0.00094842**	**0.00015554**	0.03385726	0.01177892	0.00675242	0.00253291
US-Fenmale-45	**0.02043930**	**0.01118628**	**0.00059379**	**0.00008621**	0.03124624	0.01535930	0.00562732	0.00290657
UK-Male-45	**0.00173940**	**0.00036430**	**0.00166157**	**0.00031215**	0.00718523	0.00246287	0.00499921	0.00248173
UK-Female-45	**0.00101986**	**0.00022788**	**0.00105272**	**0.00022279**	0.00432476	0.00247138	0.00265734	0.00138764

Table 6: Forecasting Errors by Ages

	RMSE											
	LHT						Lee-Carter					
Country-Gender-Initial Age	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109
US-Male-25	0.00031461	0.00031906	**0.00034294**	**0.00056355**	**0.00163195**	**0.00218371**	**0.00024474**	**0.00030468**	0.00050198	0.00171827	0.00336070	0.01869033
US-Fenmale-25	0.00016673	0.00016764	**0.00019556**	**0.00035520**	**0.00090657**	**0.00138349**	**0.00004773**	**0.00011021**	0.00038728	0.00075875	0.00140693	0.01563917
UK-Male-25	0.00044812	0.00044528	**0.00040161**	**0.00129973**	**0.00406337**	**0.00386848**	**0.00012195**	**0.00015781**	0.00073625	0.00249124	0.00548657	0.01584671
UK-Female-25	0.00026870	0.00024465	**0.00027270**	**0.00082585**	**0.00253423**	**0.00240452**	**0.00004842**	**0.00007553**	0.00060348	0.00141923	0.00226151	0.00897249
US-Male-45	X	x	0.00055118	**0.00050218**	**0.00153429**	**0.00205927**	x	x	**0.00040536**	0.001621946	0.003377216	0.018852706
US-Fenmale-45	X	x	0.00028538	**0.00032312**	**0.00085439**	**0.00135292**	x	x	**0.00032893**	0.000689609	0.001372364	0.016029648
UK-Male-45	X	x	0.00072377	**0.00139578**	**0.00399422**	**0.00378452**	x	x	**0.00048011**	0.001995892	0.004532016	0.017217237
UK-Female-45	X	x	**0.00041873**	**0.00087162**	**0.00249283**	**0.00238273**	x	x	0.000460395	0.001273194	0.002491079	0.010409714

	MAE											
	LHT						Lee-Carter					
Country-Gender-Initial Age	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109	25 ~ 34	35 ~ 44	45 ~ 64	65 ~ 74	75 ~ 84	85 ~ 109
US-Male-25	0.00026117	0.00026281	**0.00026141**	**0.00044836**	**0.00119329**	**0.00169612**	**0.00020534**	**0.00025161**	0.00041244	0.00146271	0.00272219	0.01515009
US-Fenmale-25	0.00013791	0.00013953	**0.00015137**	**0.00028572**	**0.00061268**	**0.00103138**	**0.00003713**	**0.00009172**	0.00029987	0.00060088	0.00103075	0.01154782
UK-Male-25	0.00036627	0.00036617	**0.00032164**	**0.00094374**	**0.00261908**	**0.00275347**	**0.00010386**	**0.00013600**	0.00049979	0.00228007	0.00477788	0.01246228
UK-Female-25	0.00021680	0.00019426	**0.00020397**	**0.00061040**	**0.00157054**	**0.00163514**	**0.00003873**	**0.00006016**	0.00042655	0.00116745	0.00167690	0.00697776
US-Male-45	X	x	0.03499706	**0.00665531**	**0.00940330**	**0.00384197**	x	x	**0.01474379**	0.01808440	0.01435094	0.03235439
US-Fenmale-45	X	x	0.02762891	**0.00741204**	**0.00734359**	**0.00303191**	x	x	**0.01805985**	0.01216922	0.01377453	0.02716240
UK-Male-45	X	x	0.00059343	**0.00087172**	**0.00260868**	**0.00260558**	x	x	**0.00033732**	0.00176465	0.00377325	0.01369635
UK-Female-45	X	x	0.00033675	**0.00058544**	**0.00156598**	**0.00158354**	x	x	**0.00032118**	0.00108722	0.00178873	0.00790184

Table 7: Dollar Durations of Reserves at time 0

Products	DD_α	DD_β
single-payment and *n*-year deferred whole life annuity-due	3.16	183.86
single-payment whole life insurance	-0.11	-8.87
single-payment and *n*-year endowment	-0.02	-3.51
single-payment and *n*-year term life insurance	-0.10	-12.86
single-payment and *n*-year pure endowment	0.08	9.35
n-payment and *n*-year deferred whole life annuity-due	2.84	131.43
n-payment whole life insurance	-0.13	-12.11
n-payment and *n*-year endowment	-0.05	-8.28
n-payment and *n*-year term life insurance	-0.11	-13.76
n-payment and *n*-year pure endowment	0.06	5.48

Table 8: Immunized Portfolios with All Weights Positive, $x = 45$, $n = 20$ and $i = 3\%$

Portfolio	DD_α	DD_β	Weight
Portfolio P_1			
n-payment whole life insurance	-0.13	-12.11	23.25%
n-payment and *n*-year term life insurance	-0.11	-13.76	7.24%
n-payment and *n*-year pure endowment	0.06	5.48	69.51%
Portfolio P_2			
single-payment and *n*-year deferred whole life annuity-due	3.16	183.86	1.59%
single-payment whole life insurance	-0.11	-8.87	66.61%
single-payment and *n*-year pure endowment	0.08	9.35	31.79%
Portfolio P_3			
n-payment and *n*-year deferred whole life annuity-due	2.84	131.43	1.34%
single-payment whole life insurance	-0.11	-8.87	60.32%
single-payment and *n*-year pure endowment	0.08	9.35	38.34%

Table 9:Immunized Portfolios with Negative Weights, $x = 45$, $n = 20$ and $i = 3\%$

Portfolio	DD_α	DD_β	Weight
Portfolio P_4			
n-payment whole life insurance	-0.13	-12.11	285.26%
n-payment and *n*-year term life insurance	-0.11	-13.76	-191.48%
n-payment and *n*-year deferred whole life annuity-due	2.84	131.43	6.23%
Portfolio P_5			
single-payment and *n*-year deferred whole life annuity-due	3.16	183.86	-0.21%
single-payment and *n*-year term life insurance	-0.10	-12.86	40.45%
single-payment and *n*-year pure endowment	0.08	9.35	59.76%

國科會補助專題研究計畫項下赴國外(或大陸地區)出差或研習心得報告

日期：101 年 9 月

計畫編號	NSC 99-2410-H-004 -063 -MY3		
計畫名稱	精算與財務方法在壽險保單定價、準備金估計、以及風險管理之運用		
出國人員姓名	蔡政憲	服務機構及職稱	政治大學風險管理與保險學系
出國時間	101 年 6 月 30 日至 101 年 8 月 3 日	出國地點	英國 London 與德國 Munich

二、 國外研究過程

個人於 6/30（六）搭乘中華航空的飛機出發。在香港轉機後，於同一日傍晚抵達倫敦。

7/3 星期二早上先和 Professor David Blake 及該系秘書 Jennifer Simeon 碰面寒暄後，中午就開始和 Pablo Antolin of the OECD who is heading up a longevity risk project there 正式會談。7/4 早上和該系博士 Ana 會談，下午則是和 Professor Blake 有深入的互動，從個人的論文、長壽風險領域的發展、談到研討會的舉

辦等。7/5 和 Andrew Hunt 會談。

第二週先於 7/9（一）下午和該院的 Professor Steven Haberman, Deputy Dean of Cass at that time（如今已是院長）進行深度的學術討論。7/12 早上則是到 Government Actuary's Department 和 Martin Lunnon、Adrian Gallop、and Bill Rayner 就英國以及台灣的長壽風險進行討論與意見交換。

這兩週週間的其餘時段，大多在該系給我的空間工作，主要是寫論文(請參見研究成果)，也會和文章的共同作者 Skype meetings.

7/14 清晨從倫敦飛往慕尼黑。星期一開始就「按表操課」：

Visitor Schedule **at Ludwig-Maximilians-Universität Munich**

Prepared by LMU

Professor Chenghsien Tsai, PhD

National Chengchi University, Taipei, Taiwan
Department of Risk Management and Insurance

Period: July 14 – July 27
ctsai@nccu.edu.tw

Itinerary

Arrival:	Saturday, July 14, 09:50 a.m., flight No: EZY 5381
Departure:	Friday, July 27, 04:00 p.m., flight No: HG 8389

Accommodation:	Pension Carolin, Kaulbachstr. 42, 80539 München

Activities at the MRIC:

July 27, 09:00 – 10:30 a.m.	M&M seminar: "Relational Modeling on Mortality Rates: International Tests and Hedging"

Research talks with team members

July 16, 12:00 p.m.	Lunch with Richard Peter
July 16, 03:00 p.m.	Coffee with Winnie Sun
July 17, 09:00 a.m.	Breakfast with Johannes Jaspersen
July 17, 12:00 p.m.	Lunch with Aihua Zhang
July 17, 03:00 p.m.	Coffee with Stefan Neuß
July 19, 03:00 p.m.	Coffee with Gunther Kraut
July 20, 02:00 p.m.	Coffee with Christian Knoller
July 24, 12:00 p.m.	Lunch with Christoph Lex
July 24, 03:00 p.m.	Coffee with Vijay Aseervatham

Further agenda items

July 19	Lunch with Andreas Richter

雖然表面上看起來都是在「吃吃喝喝」，但每一個會談都是在很簡短的寒暄後就進入雙方研究成果討論與心得的交換，相當「硬」，充分顯示德國人的認真。

這兩週間的其餘時間，多在該研究中心給我的研究室寫論文，直到離開慕尼黑為止。

三、 研究成果

Relational Modeling on Mortality Rates: International Tests and Hedging

Linus Fang-Shu Chan
Assistant Professor, Department of Financial Engineering and Actuarial Mathematics
Soochow University
No. 56, Section 1, Kuei-Yang Street, Taipei City, Taiwan 10048
(Tel) +886-2-2311-1531#3628; (Fax) +886-2-2381-2510; E-mail: linuschan@scu.edu.tw

Cary Chi-Liang Tsai
Associate Professor, Department of Statistics and Actuarial Science
Simon Fraser University
Burnaby, BC V5A1S6, Canada
(Tel) +778-7827044; (Fax) +778-7824368; E-mail: cltsai@sfu.ca

Chenghsien Tsai*
Professor, Department of Risk Management and Insurance
Director, Risk and Insurance Research Center
National Chengchi University
No. 64, Section 2, ZhiNan Road, Taipei City, Taiwan 11605
(Tel) +886-2-2936-9647; (Fax) +886-2-2939-3864; E-mail: ctsai@nccu.edu.tw

* Corresponding author. The author is grateful to the Risk and Insurance Research Center of National Chengchi University and the National Science Council of Taiwan for their financial support (Project numbers NSC 99-2410-H-004 -063 -MY3).

Relational Modeling on Mortality Rates: International Tests and Hedging

ABSTRACT

Modeling the changes/dynamics of mortality rates is important, and scholars have developed several types of methods/models to understand and/or forecast mortality rates. One of them is called relational model. The virtues of relational modeling include that it takes full account of the information on the relations among the mortality rates of different ages and can be applied to cross-sectional fitting/forecasting in addition to time-series modeling.

The contributions of this paper are twofold. We are the first to conduct global tests on the fitting and forecasting capabilities of a relational model relative to two well-known, different types of models. Our second contribution is investigating the efficacy of international mortality hedging using the LHT modeling. Our empirical tests show that LHT possesses excellent fitting capabilities and outstanding forecasting accuracies. Then we built several country-region LHT models to establish relevant hedging strategies. We found that the longevity bond linked to a regional mortality index were able to complement internal hedging for non-international insurers and make complete hedging of mortality rate risk possible.

Keywords: mortality rates, fitting, forecasting, hedging, duration

INTRODUCTION

Modeling the changes/dynamics of mortality rates is important. Mortality rates are a significant factor in determining the premiums and reserves of life insurance and annuity products, in determining the incomes and benefit outgoes of retirement programs and health care systems, and in shaping the population structure of a country. Ignoring possible mortality rate changes may impair the profitability and solvency of a life insurer, jeopardize social benefit programs' solvency and continuity, and produce wrong outlooks for many industries. Therefore, modeling the dynamics of mortality rates is critical to life insurers, social benefit programs, and the society as a whole.

Scholars recognized the importance of mortality rate dynamics and developed several types of methods/models to understand and/or forecast mortality rates. The major ones include: explanatory models by demographers and sociologists (please see Stallard (2006) and the references therein), factor models started by Lee and Carter (1992) and extended by Renshaw and Haberman (2003) , Renshaw and Haberman (2006), Hyndman and Ullah (2007), and many others, curve/function fitting models (e.g., McNown and Rogers, 1989; Cairns, Blake and Dowd, 2006a; Plat, 2009; Blackburn and Sherris, 2011), financial-risk types of models (interest rates: Dahl, 2004; Dahl and Møller, 2005; Biffis, 2005; Cairns, Blake and Dowd, 2006b; credit risk: Biffis, 2005; Luciano and Vigna, 2005), and relational modeling such as Brass (1971), Zaba (1979), Ewbank, De Leon, and Stoto (1983), Murray et al. (2003), Tsai and Jiang (2011), and Chan, Tsai, and Tsai (2011).

Relational modeling has its virtues. Firstly, it takes full account of the information on the relations among the mortality rates of different ages.[1] These relations may result from biological reasons (e.g., older adults have higher mortality rates) or social reasons (e.g., the spiked mortality rates of young adults caused by speed driving). Common-factor models did not incorporate such information on the other hand. Secondly, relational modeling can be applied to cross-sectional fitting/forecasting in addition to time-series modeling. For instance, mortality rates on the curve of a later year can be regarded as a transformation of those on the curve of an earlier year. Two mortality rate curves from different regions could also be related to each other in terms of their survival probabilities. This virtue can be useful for an insurer to hedge its mortality rate risk using the assets linked to foreign mortality.

The contributions of this paper are twofold. We are the first to conduct global tests on the fitting and forecasting capabilities of a relational model relative to two well-known, different types of models. More specifically, we compare the linear hazard transform (LHT) model with the Lee-Carter (LC) and CBD (Cairns, Blake and Dowd, 2006a) models in regard to their fitting and forecasting accuracies using the empirical data of 22 countries of female and male populations from 1950 to 2007.[2] These countries spread over Europe, North

[1] The term “mortality rates” in this paper is used loosely to convey the general concept of mortality and survival. Similarly, the term “mortality rate curves” may encompass survival probability curves.
[2] The exact sampling periods may vary across countries. We choose not to trim the sampling periods across

America, and Asia-Pacific with various economic status and mortality characteristics. This paper hence greatly expands the scope of Chan, Tsai, and Tsai (2011) in which the samples are US and UK only and provides a conclusive assessment on the performance of the LHT modeling.

Our second contribution is investigating the efficacy of international mortality hedging using the LHT modeling. Current undertakers of mortality risk count on the so-called internal/natural hedging (Cox and Lin, 2007; Wang et al., 2010), but Chan, Tsai, and Tsai (2011) illustrated the deficiency of internal hedging. Mortality-linked assets are thus needed. Such assets are linked to several countries to date and expose hedgers to basis risk. The literature however has not yet provided guidance on how to construct the hedging strategies and quantify the associated basis risk. Applying the LHT modeling to cross-country mortality rate curves, in addition to the aforementioned time-series applications to individual countries/regions, enables us to quantify the sensitivity of oversea mortality-linked assets to the changes of domestic mortality rates. The hedging strategies can be established accordingly. The statistics obtained from estimating the cross-country LHT models can further help us to assess the basis risk. This paper may encourage the demands for mortality-linked assets from broader regions, therefore, and facilitate mortality risk management.

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countries since the trimming involves deleting some country-year samples.

Our empirical tests show that LHT possesses outstanding fitting capabilities. It renders the best fitting results in XX out of XX gender-countries. When fitted to regional mortality indexes of Europe, North America, and Asia-Pacific, the results from LHT are even better. LHT's fitting errors are the smallest in all regions. LHT is also best fitted to the global indexes that are population-weighted indexes of XX countries.

The forecasting capabilities of LHT are equivalent to those of LC and CBD at the country-gender level. LHT, LC, and CBD produce the smallest forecasting errors in XX, XX, and XX gender-countries respectively. At the regional and global level that is more relevant to mortality-linked assets, LHT renders excellent forecasting results. Its average errors over the 10-year forecasting period are the smallest in XX regions. Its forecasting errors are also the smallest with respect to the global index. We therefore may conclude that relational modeling performs well globally and deserve more attentions from academia and mortality risk stakeholders.

We built several country-region LHT models to establish relevant hedging strategies. The hedging strategies are established using the two mortality durations of a longevity bond with respect to the parameters of the country-region LHT models. We found that the longevity bond did complement internal hedging and make complete hedging of mortality rate risk possible. The associated basis risk seems to be moderate since the error terms of the country-region LHT models have small variances.

LINEAR HAZARD TRANSFORM

Chan, Tsai, and Tsai (2011) applied the concept of linear hazard transform to the changes of mortality rates across time and confirmed the success of the application to US and UK data. More specifically, they assumed that there is a linear relation (plus an error term) between the forces of mortality (i.e., hazard rates) of two mortality rate curves from years A and $B = A + a$, where $a \in N$. One mathematical representation of such a relation is:

$$\mu_{x,n}^{B}(t) = (1+\alpha_{x,n}^{A,B}) \times \mu_{x,n}^{A}(t) + \beta_{x,n}^{A,B} + \varepsilon_{x,n}^{A,B}(t), \quad t \in [0,n], \tag{1}$$

$$\mu_x^B(t) = \left(1 + \alpha_{b,l}^{A,B}\right) \times u_x^A(t) + \beta_{b,l}^{A,B} + \varepsilon_{b,l}^{A,B}(t)$$

where μ denotes the forces of mortality, x b indicates the starting age of the mortality rate curve to be studied, $l = \omega - b$, ω represents the ending age of the studied curve section, α and β are constants to be estimated, and ε is the error term. Parameter α reflects the proportional change of the forces of mortality across time while β represents the parallel shift.

Equation (1) implies the following relation between ${}_k p_x^A$ and ${}_k p_x^B$:

$${}_k p_x^B = e^{-\int_0^k \mu_{x,n}^B(t)dt} = e^{-(1+\alpha_{x,n}^{A,B})\times\int_0^k \mu_{x,n}^A(t)dt - \int_0^k \beta_{x,n}^{A,B}dt - \int_0^k \varepsilon_{x,n}^{A,B}(t)dt} = [{}_k p_x^A]^{1+\alpha_{x,n}^{A,B}} \times e^{-\beta_{x,n}^{A,B}\times k} \times e^{-\int_0^k \varepsilon_{x,n}^{A,B}(t)dt}, \tag{2}$$

where ${}_k p_x$ denotes the probability that a person with age of x remains alive for k periods of time. Taking the natural logarithm on both sides of Equation (2) yields:

$$(-\ln {}_k p_x^B) = (1+\alpha_{x,n}^{A,B}) \times (-\ln {}_k p_x^A) + \beta_{x,n}^{A,B} \times k + \int_0^k \varepsilon_{x,n}^{A,B}(t)dt. \tag{3}$$

Then the parameter pair ($\alpha_{x,n}^{A,B}, \beta_{x,n}^{A,B}$) can be estimated by minimizing the sum of squared integrated errors $\sum_{k=1}^{n}\left[\int_0^k \varepsilon_{x,n}^{A,B}(t)dt\right]^2$ on the time-series sample set $\{(-\ln {}_k p_x^A, -\ln {}_k p_x^B)$: $k=1,2,\cdots,n$ }.

GLOBAL TIME-SERIES TESTS

Data, Benchmarks, and Measures

We draw historical one-year death rates q_x from the Human Mortality Database (HMD). The drawn data cover both genders of 23 countries[3] for the age section from 45 to 90.[4] The sampling period starts from 1950 with slightly varying ending periods due to data availability. The sampled countries with corresponding sampling periods are listed in Table 1.

[Insert Table 1 Here]

We choose two well-known models as the benchmarks to be compared with the LHT model: the Lee-Carter model and the CBD model. The Lee-Carter model is probably the most popular model. It is essentially a one-factor, linear model assuming that:

$$\log q_{x,A} = a_x + b_x K_A + \varepsilon_{x,A}, \tag{4}$$

where *a* and *b* are age-specific parameters, κ is the factor used to capture the time-varying component of mortality rates, and ε is the error term.[5] We follow Lee and Carter (1992) to estimate and forecast the age-specific parameters.

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[3] Iceland was excluded from this study because it contained too many zero mortality rates.

[4] Choosing this section of the mortality rate curve is consistent with CBD (2006a), and this section covers the underlying populations' ages of most mortality-linked assets (e.g., the longevity bonds issued by Swiss Reinsurance Company).

[5] The original Lee-Carter model was on the central death rate. We substitute the one-year death rate for the central death rate to ensure that the LHT, Lee-Carter, and CBD models use the same raw data.

The other benchmark that we choose is the CBD model that is a good, popular representation of curve/function fitting models. The model specification is:

$$\text{logit } q_{x,A} = K_A^{(1)} + K_A^{(2)}(x - \bar{x}) + \varepsilon_{x,A}, \tag{5}$$

where $\text{logit } q_{x,A} = q_{x,A} / (1 - q_{x,A})$, and both parameters $K_A^{(1)}$ and $K_A^{(2)}$ are assumed to follow random walks with drifts (Cairns et al., 2009).

We adopt two accuracy measures, RMSE and MAE, with definitions as follows:

$$RMSE = \frac{1}{T} \sum_{A=A_1+1}^{A_1+T} \sqrt{\frac{1}{(\omega - x_I)} \sum_{x=x_I}^{\omega-1} (q_{x,A} - \hat{q}_{x,A})^2} \tag{4}$$

and

$$MAE = \frac{1}{T(\omega - x_I)} \sum_{A=A_1+1}^{A_1+T} \sum_{x=x_I}^{\omega-1} \left| q_{x,A} - \hat{q}_{x,A} \right|, \tag{5}$$

where $\hat{q}$ represents the fitted/forecasted value, B_I denotes the first tested target year, and T stands for the length of the tested period.

In-Sample Fitting

In-sample fitting is done by fitting Equation (3) onto the mortality rate curves of years A and B. We first draw two series of q_x from our dataset and calculate corresponding ${}_k p_x^A$ and ${}_k p_x^B$. Taking the natural log of these ${}_k p_x$ and then running the regular regression analysis on Equation (3) yields $\hat{\alpha}$ and $\hat{\beta}$. Combining the estimated $\hat{\alpha}$ and $\hat{\beta}$ with ${}_k p_x^A$ produces ${}_k \hat{p}_{x,B} = [{}_k p_{x,A}]^{1+\hat{\alpha}} \times e^{-\hat{\beta} \times k}$. Since $\hat{q}_{x,B} = 1 - \hat{p}_{x,B}$, we are able to compute RMSE and MAE to measure fitting errors. Repeating the steps for B from B_I to

B_I+T with a = 1, we obtain the following table.[6]

[Insert Table 2 here]

Table 2 shows that the LHT model produces better global fits than both benchmark models. From Table 2a we count that LHT renders the smallest RMSE on females' mortality rates in 14 out of 23 countries, followed by Lee-Carte's 9 countries. The gap is smaller in terms of MAE: 13 by LHT vs. 10 by Lee-Carter. The LHT model performs similarly well on males' mortality rates as we can see from Table 2b. It generates the smallest RMSE and MAE in 14 and 12 countries respectively, with Lee-Carter wining in 6 and 7 and CBD's 3 and 4.

In terms of the global average fitting errors, LHT's improvements over Lee-Carter and CBD with regard to RMSE are 11.21% and 54.82% on females' data.[7] The improvement ratios in regard of MAE are 5.43% and 52.00% respectively. On males' data, LHT's fitting errors on the basis of global average are smaller than those of Lee-Carter and CBD by 14.49% and 19.84% when measured by RMSE and 9.53% and 15.28% in terms of MAE.

The improvements are robust across regions. All regional average improvement ratios of LHT over the benchmark models are positive on both genders with regard to both fitting error measures. For instance, the improvement ratios of the LHT model relative to

[6] For instance, Japan's sampling period is from 1950 to 2008. When a = 1, B_I equals to 1950+a=1951 and T = 2008-1951+1=58.

[7] The improvement percentage/ratio is defined as -(the error produced by LHT – that by a benchmark) / the error by the benchmark. Therefore, a positive/negative improvement ratio implies that LHT produces a smaller/larger fitting error.

Lee-Carter in terms of RMSE on females' data are 5.59%, 26.78%, and 32.62% in Europe, North America, and Asia-Pacific regions respectively. The improvements of LHT relative to CBD on males' data with regard to MAE are 13.58% in Europe, 27.36% in North America, and 12.91% in Asia-Pacific regions.

At the country level, the LHT model performs particularly well (with improvement ratios of 25% and above by both fitness measures) on females' mortality rates relative to the benchmark models in France, Spain, UK, US, and Japan. Its performance is relatively bad (with negative improvement ratios of 10% and worse by both measures) to Lee-Carter in Czech Republic and Slovakia. On males' mortality rates, the LHT model performs particularly well in four countries: Netherlands, UK, US, and Japan. The improvement ratios of LHT to the benchmark models are negative to some extent in terms of both fitness measures in Finland and New Zealand.

Out-of-Sample Forecasting

Following the practices of many papers including Lee-Carter (1992), Nelson and Siegel (1987), and Cairns et al. (2009), we model the dynamics of the two parameters in Equation (3) as random walks with drifts. More specifically, we assume that:

$$\gamma_{x,n}^{A,A+1} - \gamma_{x,n}^{A-1,A} = \Delta\gamma_{x,n}^{A} = \overline{\Delta\gamma_{x,n}} + \varepsilon_{x,n}^{\gamma}, \quad (6)$$

where $\overline{\Delta\gamma_{x,n}}$ denote the long-term mean change (i.e., drift) of γ, $\gamma = \alpha$ or β, and $\varepsilon_{x,n}^{\gamma} \sim N(0, \sigma_{x,n}^{\gamma})$. We estimate the drifts using the F-year data prior to a given year A with the estimators being the average changes of parameter values during this *F*-year period:

$$\overline{\Delta\hat{\gamma}}_{x,n}^{A-1,F} = \frac{1}{F-1}\sum_{i=A-F+1}^{A-1}\Delta\hat{\gamma}_{x,n}^{i} \quad , \tag{7}$$

where $\Delta\hat{\gamma}_{x,n}^{i} = \hat{\gamma}_{x,n}^{i,i+1} - \hat{\gamma}_{x,n}^{i-1,i}$ and $\hat{\gamma}_{\mathrm{b,l}}^{i-1,i}$ is estimated during in-sample fitting.[8] We set $F = 40$ for out-of-sample forecasting tests, after weighting the tradeoff between the adequacy of in-sample sizes and the number of out-of-sample tests.

The projected parameter $\tilde{\gamma}_{x,n}^{A,A+1}$ is assumed to satisfy:

$$\tilde{\gamma}_{x,n}^{A,A+1} = \hat{\gamma}_{x,n}^{A-1,A} + \overline{\Delta\hat{\gamma}}_{x,n}^{A-1,F} \quad . \tag{8}$$ [9]

Combining the projected parameters with the mortality rates of year A using Equation (3) could produce the projected mortality rates of the person aged *x* in year *A*+1 (i.e., ${}_{k}\tilde{p}_{x}^{A+1}$). Then we calculate RMSE and MAE in the same way as in the in-sample fitting section to measure the forecasting errors. Repeating the above procedures for *A* from 1990 to the most recent year available produces the following table.

[Insert Tables 3 Here]

Table 3 demonstrates that the LHT model produces the most accurate forecasting among the tested models. From Table 3a we see that the improvement ratios of LHT over Lee-Carter and CBD on RMSE when predicting females' mortality rates are 28% and 66% respectively, in terms of the global average. The improvement ratios with regard to MAE

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[8] For instance, *A* can be set as 1991 with $F = 40$. This setup would mean that the first pair of years used to estimate "historical" γ is 1951-1952. There would be 40 γ and 39 $\Delta\gamma$.

[9] We use the top script ˜ to indicate a projected value, ˆ to denote an estimated value, and ¯ for an averaged value.

are similarly significant: 33% and 59%. In predicting males' mortality rates, the global average improvement ratios of LHT upon Lee-Carter and CBD are 9% and 41% in terms of RMSE and 15% and 36% with regard to MAE, as we can see from Table 3b.

Furthermore, the LHT model renders the smallest RMSE and MAE in 18 and 20 out of the 23 sampled countries when predicting females' mortality rates, as we can count from Table 3a. In 15 countries including UK, US, Australia and Japan, the improvement ratios of LHT over both benchmarks are more than 25% in terms of both accuracy measures. From Table 3b we count that LHT wins 15 and 16 rounds in forecasting males' mortality rates in terms of RMSE and MAE respectively. It works particularly well in 9 countries including France, Italy, Sweden, Czech Republic, and Australia.

The comparative advantage of the LHT model to the benchmark models are more significant in out-of-sample forecasting than in in-sample fitting tests. The LHT model provides the most accurate results for more countries in forecasting tests than in fitting tests. More specifically, it stands out in 18 (female; RMSE), 20 (female; MAE), 15 (male; RMSE), 16 (male; MAE) during forecasting tests while wins in 14 (female; RMSE), 13 (female; MAE), 14 (male; RMSE), 12 (male; MAE) for fitting tests. The global-average improvement ratios of LHT upon benchmarks are also higher in forecasting than in fitting with one exception only:[10] 28% vs. 11% (female; RMSE; with respect to Lee-Carter), 66%

[10] The exception happens when LHT is compared with Lee-Carter using males' data judging by RMSE: 9% vs. 14%. The bad forecasting of LHT for the males of Ireland causes this exception. The improvement ratio to Lee-Carter is -162% in forecasting versus -3% in fitting.

vs. 55% (female; RMSE; to CBD), 33% vs. 5% (female; MAE; Lee-Carter), 59% vs. 52% (female; MAE; CBD), 41% vs. 20% (male; RMSE; CBD), 15% vs. 10% (male; MAE; Lee-Carter), and 36% vs. 15% (male; MAE; CBD).

Out of the solid performance of the LHT model as depicted in the above, we observe that LHT exhibits better forecasting results on females' mortality rates than on males'. The improvement ratios of LHT upon two benchmarks, on the basis of the global average with regard to both accuracy measures, are higher when predicting females' mortality rates than in predicting males'. Furthermore, all 12 regional improvement ratios[11] in predicting females' mortality rates are all positive and noteworthy (21% to 72%) while one is negative and one shows immaterial improvement (4%) when predicting males'.

Table 3 also displays some weak spots of the LHT model. For instance, it works particularly badly, compared to both benchmarks, in Ireland for both females' and males' mortality forecasting with regard to both accuracy measures. It does not work well in Bulgaria and Hungary relative to the Lee-Carter model when forecasting females' mortality rates. In forecasting males' mortality rates, LHT is inferior to CBD in Hungary and to Lee-Carter in Bulgaria, Netherlands, Norway, and Canada.

Fitting and Forecasting on Multi-Country Index

[11] The 12 regional improvement ratios are from comparing with two benchmark models by two accuracy measures in three regions.

Observing that many mortality-linked assets link to multi-country indexes rather than single-country mortality rates, we conduct in-sample fitting and out-of-sample forecasting tests on multi-country indexes. We establish equally weighted mortality indexes for European, North America, and Asia-Pacific regions and for the 23 sampled countries as a whole. The tests results are shown in Table 4.

[Insert Table 4 Here]

Table 4 demonstrates that the LHT model produces the most accurate results among the three tested models in both fitting and forecasting tests. In the fitting tests, all but one improvement ratios are positive and 24 out of 32 ratios are larger than 25%.[12] The performance of LHT is even better in forecasting. Every improvement ratio is positive, and only two of the 32 ratios are smaller than 25%.

We notice that LHT works particurlarly well for the European and global indexes. All fitting improvement ratios are larger than 20% and all forecasting ones are larger than 59%. We speculate that the cross-country averaing makes the mortality rate curve smoothier with more stable changes across time. The relations between two cross-country mortality rate curves thus come closer to linear, which gives the LHT model more edges in tests.

CROSS-COUNTRY HEDGING

[12] The exception is the case when LHT is compared with Lee-Carter in fitting the Asia-Pacific region index judged by MAE. There are 32 cases in total: comparing LHT with two benchmark models on both genders' 4 multi-country indexes with two accuracy measures.

Managing mortality rate risk will most likely involves cross-country hedging. Chan, Tsai, and Tsai (2011) demonstrated the deficiency of internal hedging and called on the development of mortality-linked assets. Mortality-linked assets are however scarce and usually tie to a multi-country mortality index. Therefore, it may be necessary for life insurers and/or social benefits programs to resort to the assets that link to foreign mortality rates.

Few papers addressed the issue of the cross-country hedging. Zhou, Li, and Tan (2011) studied the impact of population basis risk, i.e., the risk due to the mismatch in the populations of the exposure and the hedge, on prices of mortality-linked securities. We in this section demonstrate how relational modeling can be applied to such hedging.

Suppose that there is an insurer selling life insurance and annuity products in country *C*. The associated reserves will depend on future survival probabilities and can be expressed in a functional form by $V(\{{}_k p_{x,Y}^{C}\})$ in which $\{{}_k p_{x,Y}^{C}\}$ indicates the forecasted one-year survival probabilities that take possible mortality improvements into account for age x in country d. Assume that there exists an asset linked to the mortality rates of another country / region r. The value function of this asset is expressed by $A(\{q_x^r\})$. How can the insurer utilize the asset to hedge its mortality risk?

Applying LHT to the time-series mortality rates of country d and to the cross-section mortality rates between country d and region r will help. We can calculate the mortality durations of both reserves and assets and then employ the strategy of mortality duration matching that is commonly seen in interes rate risk management to hedge the insurer's mortality risk. To see this, assume that the linear relationship between the forces of mortality of two mortality rate curves for country d and region r has been obtained from (6):

$$\hat{\mu}^{r}_{x,n}(t) = (1+\hat{\alpha}^{d,r}_{x,n}) \times \mu^{d}_{x,n}(t) + \hat{\beta}^{d,r}_{x,n}, \quad t \in [0,n]. \tag{6}$$

The associated relationship for the *k*-year survival probaobilties is ${}_{k}\hat{p}^{r}_{x} = ({}_{k}p^{d}_{x})^{1+\hat{\alpha}^{d,r}_{x,n}} \times e^{-\hat{\beta}^{d,r}_{x,n} \times k}$. In other words, we take the mortality rate curves of some countries as bases and regard the curves implied by the indexes composed of the mortality rates of the countries in the same regions as targets. The base countries tested are Canada, United Kingdom, and Japan with the target indexes of North America, Europe, and Asia-Pacific respectively.

The above concept can be applied to the relations of mortality rates across countries as well. Later in the paper we will assume that the forces of mortality of a particular mortality rate curve in region *r* is a linear transform of those in country *d*. More specifically, we will implement the above procedure on forecasted survival probabilities $\{(-\ln {}_{k}p^{d}_{x}, -\ln {}_{k}p^{r}_{x})$: $k = 1, 2, \cdots, n\}$ to estimate the parameter pair $(\alpha^{d,r}_{x,n}, \beta^{d,r}_{x,n})$ that reflect the relations between the two mortality rate curves of countries d and r. For instance, the transformation decomposes the relations between the forces of mortality of countries d (domestic) and r (other region) into two components: a proportional relation reflected by $\alpha^{d,r}$ and a parallel difference determined by $\beta^{d,r}$. Assuming $\beta^{d,r}$ = 0 implies that the force-of-mortality curves of the two countries relate to each other proportionally. Assuming $\alpha^{d,r}$ = 0 corresponds to the case of a constant difference between the two curves of the countries. The fitting results on the sampling period from 1950 to 2007 are presented in Table 6.

[Insert Table 6 Here]

Denote $A(\hat{\mu}^{r}_{x,n})$ as the value of a longevity bond based on the fitted force of mortality $\hat{\mu}^{r}_{x,n}$ for region r. Then

$$DD_{\alpha^r}(A(\hat{\mu}^{r}_{x,n})) = -\lim_{\alpha^r \to 0} \frac{A((1+\alpha^r)\times\hat{\mu}^{r}_{x,n}) - A(\hat{\mu}^{r}_{x,n})}{\alpha^r} \quad (7)$$

and

$$DD_{\beta^r}(A(\hat{\mu}^{r}_{x,n})) = -\lim_{\beta^r \to 0} \frac{A(\hat{\mu}^{r}_{x,n}+\beta^r) - A(\hat{\mu}^{r}_{x,n})}{\beta^r} \quad (8)$$

are the mortality durations with respect to a proportional change and a parallel shift in the forces of mortality $\hat{\mu}^{r}_{x,n}$ for region r (see Chan, Tsai, and Tsai, 2011). Similarly, the mortality durations with respect to a proportional movement and a parallel change in the forces of mortality $\mu^{d}_{x,n}$ for country d are

$$DD_{\alpha^d}(A(\hat{\mu}^{r}_{x,n})) = -\lim_{\alpha^d \to 0} \frac{A((1+\hat{\alpha}^{d,r}_{x,n})\times(1+\alpha^d)\times\mu^{d}_{x,n}+\hat{\beta}^{d,r}_{x,n}) - A((1+\hat{\alpha}^{d,r}_{x,n})\times\mu^{d}_{x,n}+\hat{\beta}^{d,r}_{x,n})}{\alpha^d} \quad (9)$$

and

$$DD_{\beta^d}(A(\hat{\mu}^{r}_{x,n})) = -\lim_{\beta^d \to 0} \frac{A((1+\hat{\alpha}^{d,r}_{x,n})\times(\mu^{d}_{x,n}+\beta^d)+\hat{\beta}^{d,r}_{x,n}) - A((1+\hat{\alpha}^{d,r}_{x,n})\times\mu^{d}_{x,n}+\hat{\beta}^{d,r}_{x,n})}{\beta^d}, \quad (10)$$

respectively.

Future Work

We will complete the above time-series and cross-section estimations by the conference. The global, time-series analyses on fitting and forecasting capabilities of the LHT, LC, and CBD models will render conclusions on their relative performance. Furthermore, we plan to calculate the mortality durations of the mortality bond designed in Lin and Cox (2005) under various combinations of the tested base countries and target regions. The statistics obtained from estimating $\alpha^{d,r}$ and $\beta^{d,r}$ will be used to quantify the basis risk of using an asset linked to different mortality rates as a hedging tool. The calculation and estimation results will be of interest to the literature and to the undertakers of mortality risk.

Table 1 Sampled countries from the Human Mortality Database

Geographical Region (Number of Countries)	Country	Sampling Period
Asia-Pacific (3)	Australia	1950-2007
	New Zealand	1950-2008
	Japan	1950-2009
Europe (18)	Hungary; Ireland; Spain	1950-2006
	France; Italy; Switzerland	1950-2007
	Austria; Denmark; Netherlands; Norway; Sweden;	1950-2008
	Belgium; Bulgaria; Czech Republic; Finland; Portugal; Slovakia; United Kingdom	1950-2009
North America (2)	Canada; United States	1950-2007

Table 2: Descriptive Statistics on the Fitting Improvement Ratios of the LHT Model with respect to the Lee-Carter and CBD Models across 22 Countries of Male Populations

(a) Performance Relative to Lee-Carter

	No. of Samples	Mean (%)	Median (%)	S.D. (%)	Max. (%)	Min. (%)
Improvement Ratio in RMSE	22	36.91	37.45	16.53	69.89	8.02
Improvement Ratio in MAE	22	43.15	43.12	14.41	68.68	20.06

(b) Performance Relative to CBD

	No. of Samples	Mean (%)	Median (%)	S.D. (%)	Max. (%)	Min. (%)
Improvement Ratio in RMSE	22	68.86	66.61	9.46	85.59	54.46
Improvement Ratio in MAE	22	69.78	68.42	9.10	84.86	54.68

Table 3: In-Sample Fitting Results

	RMSE					MAE				
Country	LHT	LC	Improvement Ratio to LC (%)	CBD	Improvement Ratio to CBD (%)	LHT	LC	Improvement Ratio to LC (%)	CBD	Improvement Ratio to CBD (%)
Euro										
Austria	0.0059	0.0071	17.81	0.0167	64.83	0.0030	0.0040	26.29	0.0091	67.29
Belgium	0.0061	0.0088	31.29	0.0174	65.01	0.0029	0.0050	41.41	0.0092	68.23
Bulgaria	0.0082	0.0090	9.04	0.0210	60.89	0.0041	0.0051	20.06	0.0114	63.95
Czech Republic	0.0069	0.0075	8.02	0.0201	65.48	0.0034	0.0043	20.81	0.0103	66.72
Denmark	0.0059	0.0107	45.49	0.0146	59.88	0.0029	0.0061	52.12	0.0074	60.47
Finland	0.0067	0.0096	30.42	0.0154	56.33	0.0037	0.0056	33.91	0.0082	55.06
France	0.0036	0.0059	38.37	0.0158	77.19	0.0019	0.0034	44.82	0.0090	79.24
Hungary	0.0074	0.0135	45.13	0.0163	54.67	0.0037	0.0081	54.53	0.0089	58.83
Ireland	0.0068	0.0082	17.53	0.0202	66.37	0.0037	0.0050	27.24	0.0108	66.16
Italy	0.0039	0.0068	41.84	0.0158	75.11	0.0020	0.0040	50.49	0.0084	76.44
Netherlands	0.0039	0.0102	62.20	0.0176	78.10	0.0020	0.0058	65.37	0.0088	77.32
Norway	0.0067	0.0095	28.90	0.0167	59.61	0.0038	0.0055	31.23	0.0089	57.69
Portugal	0.0061	0.0078	21.80	0.0171	64.53	0.0032	0.0045	29.58	0.0102	68.61
Sweden	0.0047	0.0082	42.25	0.0171	72.30	0.0025	0.0046	45.34	0.0090	72.47
Switzerland	0.0057	0.0074	23.27	0.0172	66.86	0.0029	0.0042	31.45	0.0092	68.94
Slovakia	0.0075	0.0109	31.50	0.0164	54.46	0.0040	0.0066	38.96	0.0089	54.68
United Kingdom	0.0030	0.0070	57.59	0.0206	85.59	0.0016	0.0040	60.45	0.0105	84.86
Average	0.0058	0.0087	32.50	0.0174	66.31	0.0030	0.0051	39.65	0.0093	67.47
North America										
Canada	0.0030	0.0070	57.59	0.0206	85.59	0.0016	0.0040	60.45	0.0105	84.86
United States	0.0018	0.0059	69.89	0.0118	84.99	0.0010	0.0033	68.68	0.0063	83.60
Average	0.0024	0.0064	63.74	0.0162	85.29	0.0013	0.0037	64.57	0.0084	84.23
Asia-Pacific										
Australia	0.0047	0.0082	42.25	0.0171	72.30	0.0025	0.0046	45.34	0.0090	72.47
Japan	0.0037	0.0078	53.33	0.0158	76.81	0.0018	0.0049	62.67	0.0090	79.86
New Zealand	0.0050	0.0078	36.53	0.0156	68.00	0.0027	0.0044	38.10	0.0083	67.31
Average	0.0045	0.0080	44.04	0.0161	72.37	0.0023	0.0046	48.70	0.0088	73.21

Table 4: Descriptive Statistics on the Forecasting Improvement Ratios of the LHT Model with respect to the Lee-Carter and CBD Models across 22 Countries of Male Populations

(a) Performance Relative to Lee-Carter

	No. of Samples	Mean (%)	Median (%)	S.D. (%)	Max. (%)	Min. (%)
Improvement Ratio in RMSE	22	10.57	12.81	42.74	66.93	-72.38
Improvement Ratio in MAE	22	12.15	19.17	41.78	70.69	-70.48

(b) Performance Relative to CBD

	No. of Samples	Mean (%)	Median (%)	S.D. (%)	Max. (%)	Min. (%)
Improvement Ratio in RMSE	22	30.09	44.22	33.73	77.85	-56.01
Improvement Ratio in MAE	22	24.02	34.19	39.45	78.72	-89.04

Table 5: Out-of-Sample 5-Year Forecasting Results

	RMSE					MAE				
Country	LHT	LC	Improvement Ratio to LC (%)	CBD	Improvement Ratio to CBD (%)	LHT	LC	Improvement Ratio to LC (%)	CBD	Improvement Ratio to CBD (%)
Euro										
Austria	0.0118	0.0080	-48.78	0.0129	8.63	0.0079	0.0049	-60.61	0.0081	3.06
Belgium	0.0086	0.0196	55.99	0.0177	51.29	0.0051	0.0109	53.83	0.0100	49.47
Bulgaria	0.0371	0.0215	-72.38	0.0313	-18.70	0.0230	0.0135	-70.48	0.0178	-29.33
Czech Republic	0.0079	0.0098	19.20	0.0141	44.04	0.0046	0.0066	31.40	0.0074	38.51
Denmark	0.0056	0.0078	28.07	0.0118	52.13	0.0035	0.0058	40.06	0.0066	47.35
Finland	0.0091	0.0231	60.43	0.0202	54.90	0.0058	0.0124	53.48	0.0113	48.68
France	0.0045	0.0100	54.65	0.0204	77.85	0.0026	0.0052	49.35	0.0124	78.72
Hungary	0.0083	0.0222	62.47	0.0168	50.39	0.0043	0.0148	70.69	0.0103	58.17
Ireland	0.0119	0.0114	-4.65	0.0159	25.30	0.0074	0.0080	6.93	0.0078	4.69
Italy	0.0111	0.0094	-17.85	0.0111	0.02	0.0073	0.0060	-21.48	0.0070	-4.45
Netherlands	0.0114	0.0191	40.24	0.0158	27.74	0.0064	0.0110	41.79	0.0088	27.72
Norway	0.0286	0.0232	-23.30	0.0271	-5.51	0.0154	0.0132	-17.07	0.0152	-1.77
Portugal	0.0080	0.0057	-41.21	0.0239	66.26	0.0048	0.0040	-21.57	0.0143	66.09
Sweden	0.0112	0.0111	-0.49	0.0200	44.22	0.0064	0.0067	3.22	0.0117	45.06
Switzerland	0.0141	0.0086	-64.29	0.0130	-8.54	0.0087	0.0051	-69.39	0.0081	-6.53
Slovakia	0.0269	0.0212	-27.09	0.0232	-15.86	0.0158	0.0136	-15.75	0.0116	-36.29
United Kingdom	0.0062	0.0104	40.32	0.0114	45.59	0.0042	0.0065	34.82	0.0060	29.86
Average	0.0131	0.0142	3.61	0.0180	29.40	0.0078	0.0087	6.43	0.0103	24.65
North America										
Canada	0.0062	0.0104	40.32	0.0114	45.59	0.0042	0.0065	34.82	0.0060	29.86
United States	0.0066	0.0201	66.93	0.0204	67.49	0.0043	0.0103	58.04	0.0117	63.17
Average	0.0064	0.0152	53.63	0.0159	56.54	0.0043	0.0084	46.43	0.0089	46.51
Asia-Pacific										
Australia	0.0112	0.0111	-0.49	0.0200	44.22	0.0064	0.0067	3.22	0.0117	45.06
Japan	0.0130	0.0139	6.42	0.0084	-56.01	0.0082	0.0086	4.54	0.0043	-89.04
New Zealand	0.0070	0.0168	58.00	0.0180	60.89	0.0041	0.0096	57.57	0.0103	60.29
Average	0.0104	0.0139	21.31	0.0155	16.37	0.0062	0.0083	21.77	0.0088	5.44

Table 6: In-Sample Fitting Errors of Country to Region Index

Country / Region	RMSE	MAE
Canada / North American	0.002196	0.001438
United Kingdom / Europe	0.005409	0.003908
Japan / Asia-Pacific	0.008017	0.005669

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國科會補助計畫衍生研發成果推廣資料表

日期:2014/01/28

<table>
<tr><td rowspan="3">國科會補助計畫</td><td>計畫名稱: 精算與財務方法在壽險保單定價、準備金估計、以及風險管理之運用</td></tr>
<tr><td>計畫主持人: 蔡政憲</td></tr>
<tr><td>計畫編號: 99-2410-H-004-063-MY3 學門領域: 財務</td></tr>
<tr><td colspan="2">無研發成果推廣資料</td></tr>
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99 年度專題研究計畫研究成果彙整表

計畫主持人：蔡政憲	計畫編號：99-2410-H-004-063-MY3
計畫名稱：精算與財務方法在壽險保單定價、準備金估計、以及風險管理之運用	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	1	1	25%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	6	6	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	4	4	25%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	還有一篇審稿中的論文以及工作論文（請參看結案報告）。

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

還有六篇審稿中或即將投稿的工作論文。

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

本計畫所產生的六篇工作論文，個人預計其中的五篇應該都有不錯的機會可以刊登上保險領域的前三大期刊：Journal of Risk and Insurance, Insurance: Mathematics and Economics, 和 Geneva Risk and Insurance Review. 至少應該都可以上國科會財務學門 B+級的期刊。

此外，這六篇論文都很有實務上的意涵。Kuo et al. (2014)那一篇的模型已經被金管會保險局採用於台灣壽險業準備金的適足性測試中。兩篇 Life Settlements 的文章可以幫助壽險業者評估投資這類商品的風險溢酬以及如何用來規避死亡率風險。兩篇死亡率的模型則可以給壽險公司實際用來預測未來死亡率曲線的變化。已投稿出去的那一篇則是具體的台灣案例研究，其計算結果對壽險公司以及保險監理機關都有參考價值。