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碩士學位論文

A Study on Strict d-box Representations of Planar Graphs

探討平面圖的 **d** 維矩形表示法

碩專班學生：劉淑慧 撰

指導教授：張宜武 博士

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Abstract

We study strict d-box representations of planar graphs. We prove that a 4-connected planar triangulation graph G has a strict 2-box representation. We extend this result to that every planar graph has a strict 3-box representation. Our goal is to provide some fresh insights into the current status of research in the area while suggesting directions for the future.

keywords: interval graphs; 4-connected planar triangulation graph; strict d-box representation.

中文摘要

本文我們探討平面圖形的嚴格 d 維矩形表示法。我們證明了四連通三角平面圖有嚴格的二維矩形表示法，而且我們推廣到每一個平面圖都有嚴格的三維矩形表示法。我們的目標是希望能在平面圖矩形表示法的現今地位上，提供新的洞悉,並給未來學習者一個方向。

關鍵詞：區間圖，四連通三角平面圖，嚴格 d 維矩形表示法。

Contents

1 Introduction

In graph theory, an *intersection graph* of a set system is the graph whose vertices are the sets such that two vertices are adjacent if and only if the sets intersect. The *interval graphs* were introduced to study intersecting intervals on the real line. Each vertex v in an interval graph $G = (V, E)$ is associated with an interval I_v , and if two vertices are connected by an edge in G , then the intersection of their associated intervals is nonempty. Some of the intervals may have nonempty intersection, and the others have empty intersection.

W. T. Trotter [6] tells us that interval graphs have been characterized completely. Instead of representing a vertex by one interval it may be represented by a specified number of intervals or an interval of higher dimensions. In the case of a specified number of intervals, Scheinerman and West [4] proved that every planar graph is an intersection graph such that each vertex is represented by at most three intervals on the real line. In the case of higher dimensions, Scheinerman [3] proved that in two dimensions two rectangles for each vertex are sufficient to represent any planar graph. And Melinkov [2] charactherizes the planar graphs whose vertices can be represented by horizontal intervals in the plane such that two interval are adjacent if they can be joined by a vertical line not intersecting any other interval. Duchet, Hamidoune, Las Vergnas, and Meyniel [1] proved that every maximal planar graph has such a representation.

We adopt that a vertex is represented in higher dimensions. So we say that a graph G has a *d-box representation* (i.e. d-dimensional closed intervals) if G is an intersection graph such that the sets are closed d-boxes in R^d. In this paper, we want to discuss the *strict d-box representation*, it means that a graph is a d-box representation and no two of which have an interior point in common and such that two boxes which intersect have precisely a $(d-1)$-*box* in common. If a graph G is a strict 2-box representation($d = 2$) , then all vertices of G are drawn as rectangles and all edges are drawn as a horizontal or a vertical line segment. If there four edges form the four "outer corners", then the representation is called *rectangular representation*.

In 1953, Ungar [7] proved that every cubic cyclically 4-edge-connected graph has a plane representation such that each face is bounded by a rectangle. And Ungar showed that any plane embedding T of a cyclically 4-edge-connected planar cubic graph G has a rectangular drawing if four vertices of degree 2 are inserted on some edges on the outer face.

Carsten Thomassen [5] generalizing the results of Ungar, C. Thomassen obtains a necessary and sufficient condition for a plane graph T with the maximum degree $\triangle \leq 3$ to have a rectangular representation when a quadruplet of vertices of degree 2 on F_0(T) are designated as corners for a rectangular representation. He prove that every planar graph is the intersection graph of a collection of three-dimensional boxes, with intersections occurring only in the boundaries of the boxes. In 1984, C. Thomassen [8] characterizes the graphs that have such representations in the plane, called *strict d-boxes graph* if G is an intersection graph such that the sets are closed d-boxes in R^d. These are precisely the proper subgraphs of 4-connected planar triangulations, which we characterize by forbidden subgraphs. He prove that every planar graph is a strict 3-box graph. And E. R. Scheinerman [3] showed that every planar graph has a strict representation using at most two rectangles per vertex.

In our article, we discuss a strict 2-box representation and a strict 3-box representation .

In Chapter 1, we introduce intersection graphs, interval graphs, and the definition of a strict d-boxes representation. In Chapter 2, we introduce the definitions, propositions and the theorem of strict 2-boxes graphs. In Chapter 3, we show a strict 2-box representation for a 4-connected planar triangulation and then we extend to a strict 3-box representation for planar graphs. In Chapter 4, we mention some open problems and further directions of research.

2 Strict 2-box representation

2.1 Definitions and theorems of cyclically 4-edge-connected planar graphs and 4-connected planar triangulation graphs

In this section we give some definitions and present preliminary results. Because Theorem 2.13(Ungar, 1953 [7]) showed that a cyclically 4-edge-connected planar graph G has a strict 2-box representation.

Definition 2.1. The graphs can be drawn in the plane without crossing edges, such graphs are *planar*.

Definition 2.2. A graph G is called *cubic* if the degree of v is 3 for every vetex v.

Definition 2.3. A graph G is called 4-*edge-connected* if the removal of at least 4 edges leaves a graph such that the graph has more than one component.

Definition 2.4. A planar graph G is called *cyclically* 4-*edge-connected planar graph* if G is cubic and G is 4-edge-connected graph.

Example 2.5. Consider the graph in Figure 1 is cyclically 4-edge-connected graph, But the graph in Figure 2 is not cyclically 4-edge-connected graph, since the removal of the three edges drawn by thick dotted lines leave a graph such that the graph has more than one component.

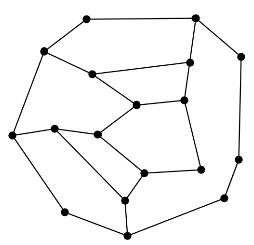

Figure 1: A cyclically 4-edge connected graph

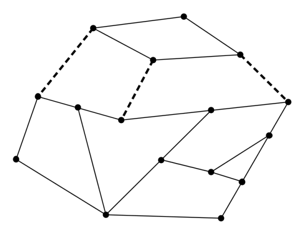

Figure 2: **not** a cyclically 4-edge connected graph

We will discuss the strict 2-box representation for 4-connected planar triangulation in section 3.1. So we gives some of the definitions for 4-connected planar triangulation.

Definition 2.6. A graph G is called 4-*connected* if the removal of at least 4 vertices leaves a graph such that the graph has more than one component.

Definition 2.7. A *triangulation* is a simple plane graph where every face boundary is 3-cycle.

Definition 2.8. A graph is called 4-*connected planar triangulation* if the planar graph is 4-connected and every face is 3-cycle.

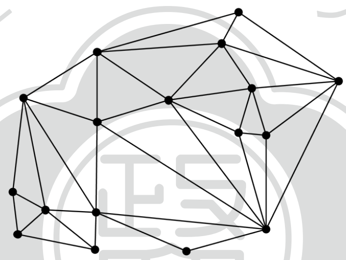

Figure 3: 4-connected planar triangulation graph

Example 2.9. G is a *cyclically* 4-*edge-connected graph* and hence the dual graph H of G is a 4-*connected planar triangulation*.

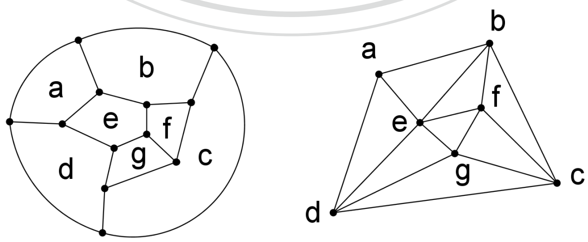

Figure 4: The left is a 4-edge-connected graph and the right is a 4-connected planar triangulation

The following theorem could help us to check if a given graph is 4-connected planar triangulation. It also gives us a way to construct graphs that are not 4-connected planar triangulation from the forbidden graphs.

Theorem 2.10 (Carsten Thomassen, Interval of Planar Graphs)**.** *The graphs which are not proper subgraphs of 4-connected planar triangulations and which are edge-minimal with that property are precisely the following:*

(a) The triangle-free subdivisions of the Kuratowski graphs K_5 and $K_{3,3}$.

(b) The planar triangulations with no separating K_3. (i.e. K_4 and the 4-connected planar triangulations)

(c) The graphs obtained from a wheel W_n of order $n \geq 5$ by adding a vertex and joining it to the center of the wheel(see Figure 5).

(d) Any graph obtained from K_4 by subdividing one edge xy and adding an additional path of length at least 2 between the two other vertices of the K_4(see Figure 6).

(e) Any graph which is obtained from K_5 with one missing edge by subdividing edges incident with the missing edge(in such a way that the graph does not contain a graph described under(d), see Figure 7).

(f) The graph K_2 v $\bar{K}_3$ (see Figure 8).

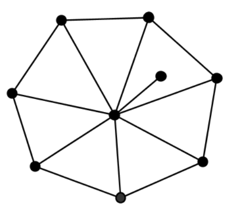

Figure 5: A W_n wheel

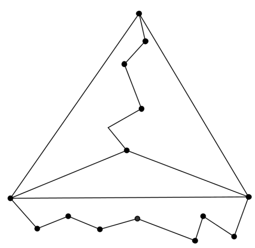

Figure 6: the graph in Theorem 2.10 (d)

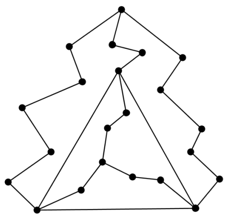

Figure 7: the graph in Theorem 2.10 (e)

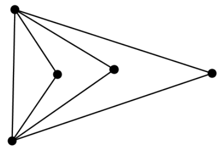

Figure 8: K_2 v $\bar{K_3}$

Definition 2.11. A triangle *xyzx* in a graph *G* is *separating* if $G-\{x,y,z\}$ has more components than *G*.

Example 2.12. If we remove the thick solid triangle, then the graphs(see Figure 9, 10, 11, 12) have more components. This contradicts the assumption that the graphs are 4-connected. So the graphs are forbidden graphs of 4-connected planar triangulation.

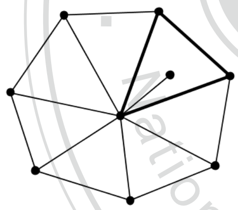

Figure 9: A W_n wheel

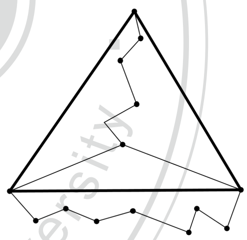

Figure 10: the graph in Theorem 2.10 (d)

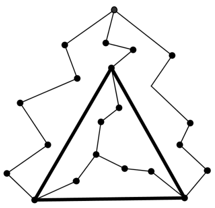

Figure 11: the graph in Theorem 2.10 (e)

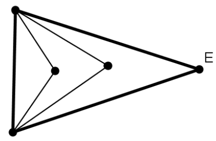

Figure 12: K_2 v $\bar{K_3}$

2.2 Planar graphs have strict 2-box representations by at least two boxes

Theorem 2.13. *(Ungar, 1953 [7]) A cyclically* 4*-edge-connected planar graph* G *has a plane representation(strict* 2*-box representation).*

Because a cyclically 4-edge-connected planar graph has no separating triangle, So we have the corollary as following:

Corollary 2.14. *(C. Thomssen, 1986 [8]) Any planar graph with no separating triangle has a strict rectangle representation.*

Theorem 2.15. *(C. Thomssen, 1986 [8]) A graph* G *is a strict rectangle graph if and only if* G *is a proper subgraph of some* 4*-connected planar triangulation* H*.*

We also obtain an extension of the result of Scheinerman that any planar graph has a rectangle representation(2-box representation) such that each vertex is represented by at most two rectangles.

Corollary 2.16. *(E. R. Scheinerman, 1984 [3]) Every planar graph has a strict rectangle representation in* R^2 *such that each vertex is represented by at most two rectangles.*

Proposition 2.17. *(C. Thomssen, 1986 [8]) Every planar graph* G *is the union of two triangle free graphs.*

Lemma 2.18. *If* G *is a planar graph not containing any subgraph described in Theorem2.12(c), then* G *contains a* 4*-connected planar triangulation if and only if some component of* G *is a* 4*-connected planar triangulation.*

3 Some results on d-box representation

3.1 A strict 2-box representation for 4-connected planar triangulation graphs

In this section we use two ways to draw the strict 2-box representation for 4-connected planar triangulation graph, and give some examples to show how to draw them. We try to extended the way to draw a strict 3-box representation in next section.

Definition 3.1. A graph G is a *strict d-box representation* if G is an intersection graph such that the sets are closed d-boxes in R^d , no two of which have an interior point in common and such that two boxes which intersect have precisely a $(d-1)$-box in common.

Example 3.2. If a graph G is a strict 1-box representation, then the vertex can be drawn by a 1-dimensional closed interval and the two intervals which intersect have precisely a 0-box in common(see Figure 13).

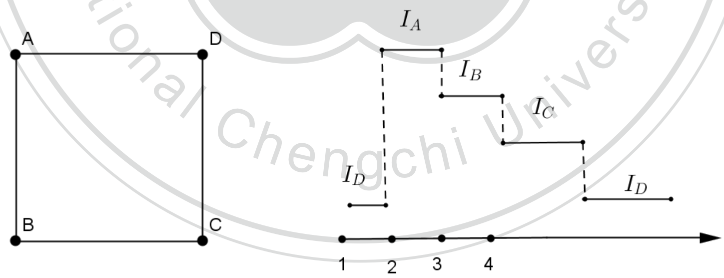

Figure 13: 1-box representation

Example 3.3. A graph G is a strict 2-box representation if all vertices of G are drawn as rectangles, and all edges are drawn as a horizontal or a vertical line segment(see Figure 14). If there four edges form the four "outer corners", then the representation is called rectangular representation(see Figure 15).

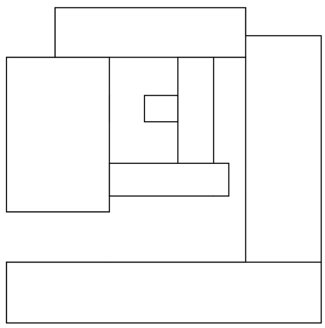

Figure 14: A strict 2-box representation

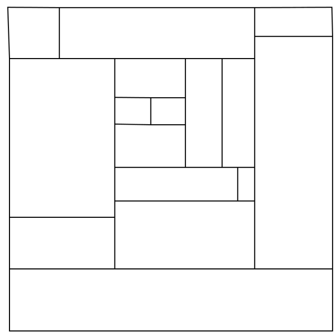

Figure 15: a correspond rectangular representation

We adopt that strict 2-box representation in this article is the graph in Figure 14.

Remark 3.4. *By Theorem2.13, we know that a cubic* 4*-edge-connected planar graph* G *has a strict* 2*-box representation. And we can think of* G *as a "geometric dual graph" of interval system. That means that the strict* 2*-box representation is precisely the proper subgraphs of the* 4*-connect planar triangulations.*

Example 3.5. Consider the graph G in Figure 16 is a cubic 4-edge-connected planar graph, and G has a strict 2-box representation. Then the graph H in Figure 17 is a dual graph of G, and we can see H is a 4-connected planar triangulation graph.

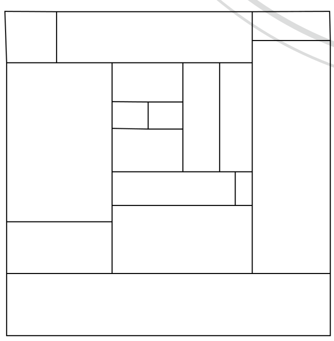

Figure 16: G is a cubic 4-edge connected planar

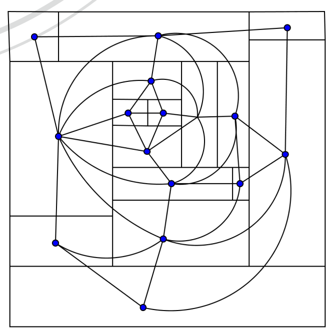

Figure 17: H is a 4-connected planar triangulation

The following theorem could help us to check that if a given graph has a strict 2-box representation. It also gives us the forbidden configurations for strict 2-box representations.

Theorem 3.6. *(Carsten Thomassen, 1986, [6]) If we draw a facial cycle C of a cubic planar G as a rectangle R such that none of its corners are vertices of C, then by Carsten Thomassen [5] this can be extended to a rectangular representation(a strict 2-box representation) of G inside R if and only if*

(a) For each vertex x not in C, there is only one set of three edges that separates x from C, namely the set of edges incident x(see Figure 18);

(b) Each connected component of $G - V(C)$ is joined to two opposite sides of R and each chord of C(if any) joins two opposite sides of R (see Figure 19);

(c) For each connected component H of $G - V(C)$ and each edge e of G, H is in $G - e$ joined to at least two sides of R (see Figure 20).

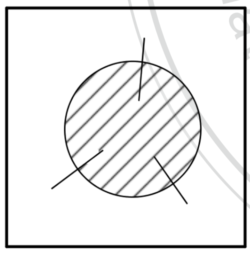

Figure 18: (a)

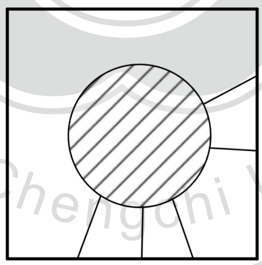

Figure 19: (b)

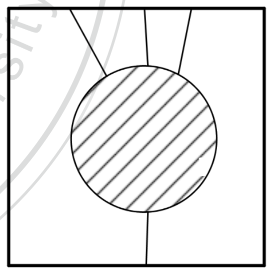

Figure 20: (c)

Theorem 3.7. *(Carsten Thomassen extension of Ungar's result) Let G be a 4-connected planar triangulation graph and let $zyxz$ and $zyvz$ be two triangles of G(see Figure 21). Then the dual graph H of G has a strict 2-box representation such that z corresponds to the unbounded face and such that x, y, and v correspond to rectangles as shown in Figure 22.*

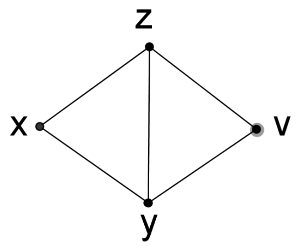

Figure 21: two triangles $zyxz$ and $zyvz$

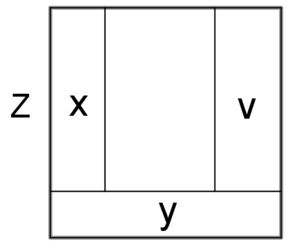

Figure 22: x, y, z, v correspond to rectangle

Proof. In order to show that the configuration in Figure 22 can be extend to a rectangular representation of H we apply Theorem 3.6(Carsten Thomassen, 1986, [6]). It suffices to show that none of the forbidden configurations indicated in Theorem 3.6 occurs.

case1: One such configuration is indicated in Figure 23.Clearly, if the removal of the three thick solid lines leaves the graph, the shaded area will be a component. Obviously, this contradicts the assumption that G is 4-connected.

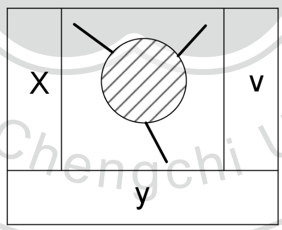

Figure 23: the removal of three thick solid lines

case2: One such configuration is indicated in Figure 24 and 25. If the removal of the three or four thick solid lines leaves the graph, then the shaded area will be a component.Obviously, this contradicts the assumption that G is 4-connected.

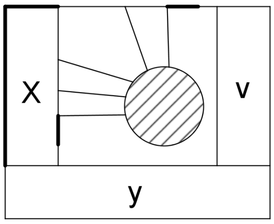

Figure 24: the removal of four thick solid lines

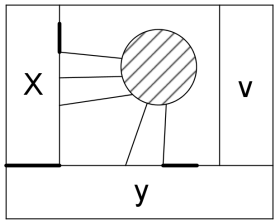

Figure 25: the removal of three thick solid lines

case3: Another configuration is indicated in Figure 26 and 27. If the removal of the three or four thick solid lines leaves the graph, then the shaded area will be a component. Clearly, this contradicts the assumption that G is 4-connected.

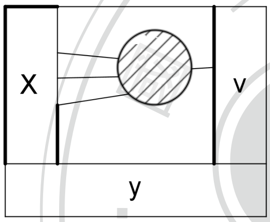

Figure 26: the removal of four thick solid lines

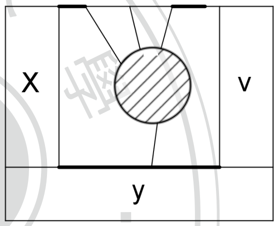

Figure 27: the removal of three thick solid lines

Finally the forbidden configuration in Theorem 3.6 do not occur and the proof is complete. □

Moreover, we try to look at two examples(Example 3.8 and 3.9). It gives us clear picture how to draw a strict 2-box representation, and show that the two triangles we choose are drawn by Theorem3.7.

Example 3.8. There is a 4-connected triangulation graph in Figure 28. We choose two triangles $cdfc$, and $cfhc$. Let the rectangle which c corresponds to be the unbounded face, and draw d, f and h correspond to rectangles as shown in Figure 29. By Theorem 3.7, we get a strict 2-box representation.

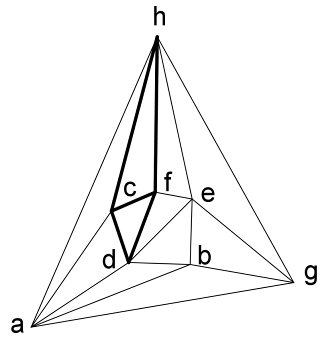

Figure 28: A 4-connected triangulation graph and we choose two triangles $cdfc$, and $cfhc$

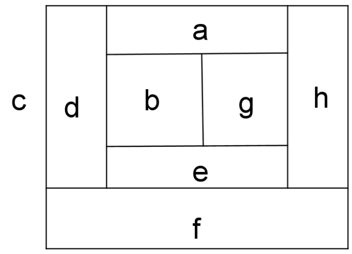

Figure 29: A graph in Figure 28 corresponding strict 2-box representation

We choose another two triangles and get another strict 2-box representation in the following example.

Example 3.9. There is a 4-connected triangulation graph in Figure 30. We choose two triangles $cdac$, and $cdfc$. Let the rectangle which c corresponds to be the unbounded face, and draw a, d and f correspond to rectangles as shown in Figure 31. By Theorem 3.7, we get a strict 2-box representation.

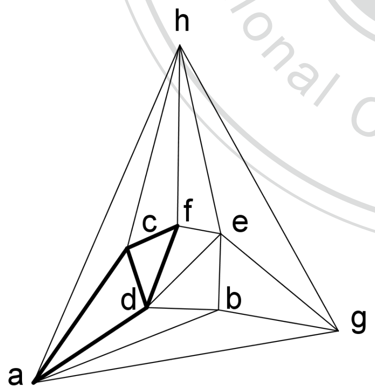

Figure 30: A 4-connected triangulation graph and we choose two triangles $cdac$, and $cdfc$

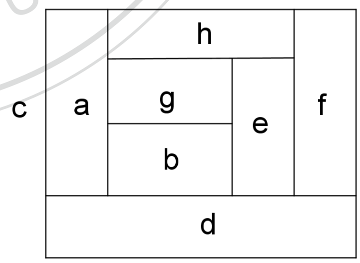

Figure 31: A graph in Figure 30 corresponding strict 2-box representation

Consider G is 4-*connected triangulation grpah*. If we can't find two triangles $zyxz$ and $zuvz$ which is described by Theorem 3.7 in G. We can choose a triangle $zxyz$ in G to be a unbounded face, and draw a strict 2-box representation. That means we can choose any triangle in G, and we have the corresponding *strict* 2-*box representation*.

Corollary 3.10. *Let G be a 4-connected planar triangulation graph and $zxyz$ a triangle in G(see Figure 32). Then $G - z$ has a strict 2-box representation such that x and y are represented as shown in Figure 33 and all other rectangles are in the shaded square.*

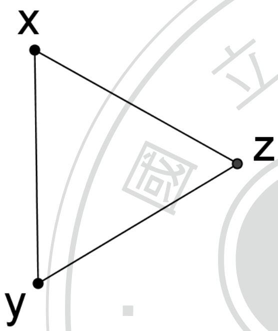

Figure 32: A triangle xyz

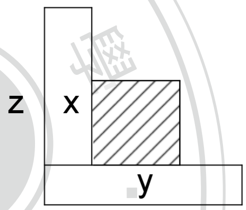

Figure 33: Strict 2-box representation of $G - z$ with prescribed x and y

Proof. This follows from Theorem 3.7 simply by letting v be the vertex distinct from x, y, z such that $vyzv$ is a triangle.

□

Moreover, we try to look at two examples(Example 3.11 and 3.12). It gives us clear picture how to draw a strict 2-box representation.

Example 3.11. There is a 4-connected triangulation graph in Figure 34. We choose $agha$, and let the rectangle which h corresponds to be the outer face, and draw the strict rectangle representation of $G - h$ with prescribed a and g by Corollary 3.10, and that is a strict 2-box representation in Figure 35. We get a strict 2-box representation.

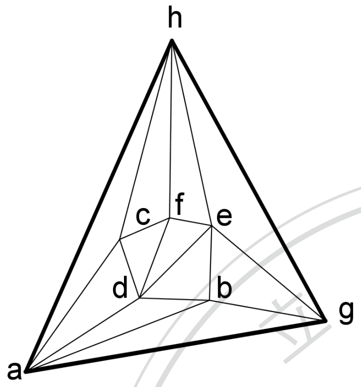

Figure 34: we choose a triangle $agha$

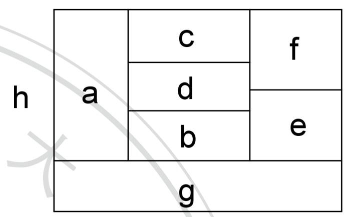

Figure 35: strict 2-box representation

Example 3.12. There is a 4-connected triangulation graph in Figure 36. We choose $cdfc$, and let the rectangle which f corresponds to be the outer face, and draw the strict rectangle representation of $G - f$ with prescribed c and d by Theorem 3.9, and that is a strict 2-box representation in Figure 37. We get another strict 2-box representation.

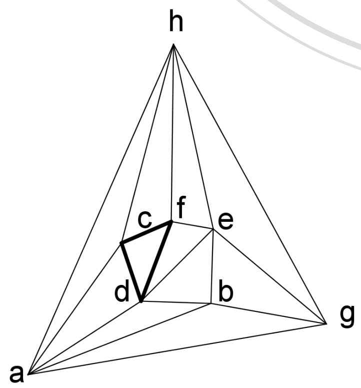

Figure 36: we choose a triangle $cdfc$

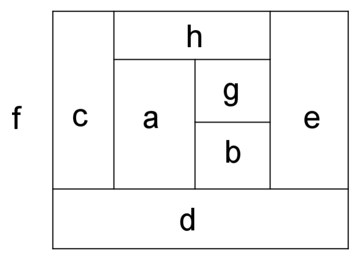

Figure 37: strict 2-box representation

3.2 A strict 3-box representation for planar graphs

In this section we mention a strict 3-box representation by extending Corollary 3.10. Now we provide some definitions and short proofs of 3-box representation for planar graph.

Definition 3.13. If a graph G is a *strict* 3-*box representation*, then the vertex can be drawn by a 3-dimensional closed interval and the two box which intersect have precisely a 2-box in common(see Figure 38).

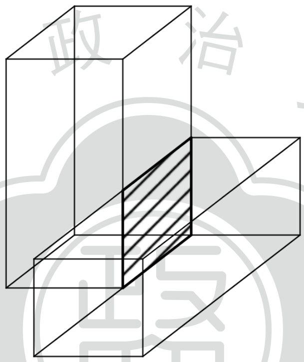

Figure 38: strict 3-box representation

By Theorem 2.10(c), we know that the graph wheel W_5 is the forbidden graph of strict 2-box representation. By Corollary 2.16, E. R. Scheinerman tells us that every planar graph has a strict rectangle representation in R^2 (strict 2-box representation) such that each vertex is represented by at most two rectangles. We show that the graph wheel W_5 has a strict 2-box representation such that each vertex is presented by at most two rectangles, and it can be drawn by a strict 3-box representation.

Example 3.14. A planar graph wheel W_5(see Figure 39) which is the forbidden graph of a strict 2-box representation. It can be drawn by a strict 2-box representation, and each vertex is represented by at most two rectangles(see Figure 40). We also draw it by a strict 3-box representation, and each vertex is represented by one box(see Figure 41).

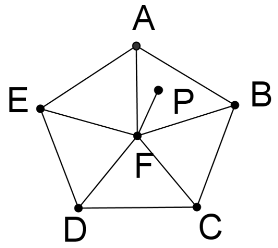

Figure 39: A planar graph wheel W_5

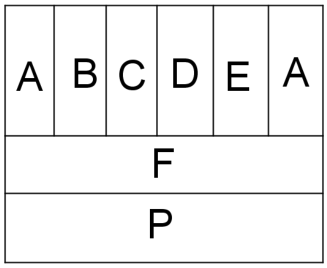

Figure 40: strict 2-box representation

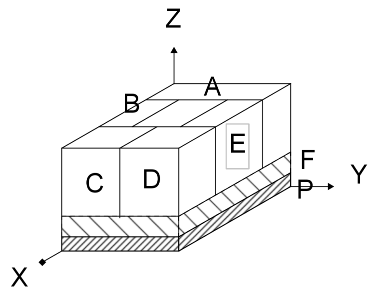

Figure 41: strict 3-box representation

We give some definition for the strict 3-box representation.

Definition 3.15. Suppose the boxes B_x ,B_y , and B_z in R^3 such that $B_x \supset \{0\} \times [0,k] \times [0,k]$, By $B_y \supset [0,k] \times \{0\} \times [0,k]$, $B_z \supset [0,k] \times [0,k] \times \{0\}$, and for i $\in \{x, y, z\}$ and k is positive integer, B_i contains no point with positive i-coordinate. Then we say the origin is an *inner corner* of the boxes B_x , B_y , and B_z(see Figure 42).

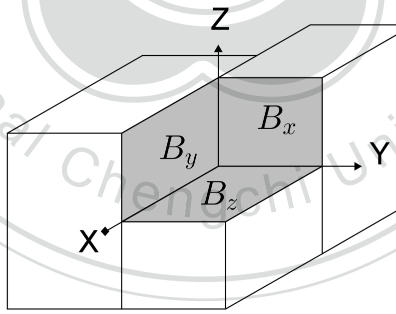

Figure 42: inner corner

Definition 3.16. If there exists a positive real number ε such that the box $(0,\varepsilon] \times (0,\varepsilon] \times (0,\varepsilon]$ intersects no box in the box system, then we say it is an *empty inner corner*.

We recall the Corollary 3.10, and we can extend to the strict 3-box representation. The following Corollary help us to check that if a given graph has a strict

3-box representation.

Corollary 3.17. *Let G be a 4-connected triangulation and $xyzx$ a triangle in G. Suppose the boxes B_x , B_y , and B_z is an original inner corner.Then B_x, B_y, B_z can be extended to a strict box representation of G in R^3 such that all other boxes are in the box* $[0,1] \times [0,1] \times [0,1]$*, and all vertices not adjacent to z are in the box* $[0,1] \times [0,1] \times [2/3,1]$*(see Figure 43).*

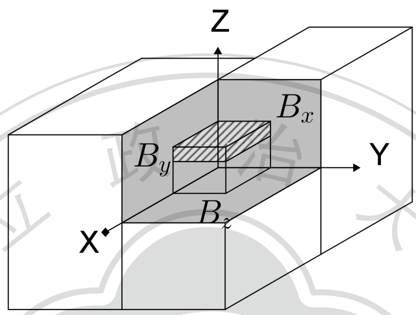

Figure 43: all vertices not adjacent to z are in the box $[0,1] \times [0,1] \times [2/3,1]$

We want to discuss every planar graph has a strict 3-box representation, so we need to prove that every planar triangulation has a strict 3-box representation.

Definition 3.18. If there is no vertex in the triangle $xyzx$ of the planar triangulation graph G, then the triangle $xyzx$ is called a $facial\ triangle$.

Theorem 3.19. *Every planar triangulation G has a strict box representation in R^3 such that every facial triangle except possibly one prescribed facial triangle $x^{'}y^{'}z^{'}x^{'}$ has an empty inner corner.*

Proof. Let $x^{'}y^{'}z^{'}x^{'}$ be any triangle that we choose.If G has a separating triangle $xyzx$ we choose it such that the number of vertices in the component of $G-\{$x,y,z$\}$ not intersecting $x^{'}y^{'}z^{'}x^{'}$ is smallest possible.

Suppose $G = G_1 \cup G_2$ where G_1 and G_2 are planar triangulations with precisely $xyzx$ in common, and G_2 contains $x^{'}y^{'}z^{'}x^{'}$, G_1 is 4-connected or isomorphic to K_4.

case1: G_1 is isomorphic to K_4

Let $G_2 = xyzx = x'y'z'x'$(see Figure 44). We can choose the separating triangle $xyvx$. Clearly, the vertex v can be drawn the box B_v and the triangle $xyvx$ and $yzvy$ have empty inner corners(see Figure 45).

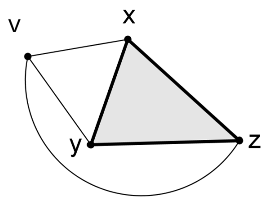

Figure 44: G_1 is a K_4

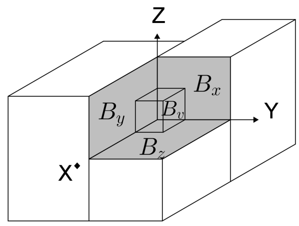

Figure 45: The strict 3-box representation of the graph in Figure 44.

case2: G_1 is 4-connected, not k_4

If G is 4-connected, we put $G = G_1$ and let $G_2 = xyzx = x'y'z'x'$. Then we represent G_2 such that x, y, and z are encoded as in Corollary 3.17 (If G_2 has more than three vertices, this can be done by the induction hypothesis). And we modify the representation in Corollary 3.17.

For each vertex u in G_1 $-${x,y,z} which is adjacent to z we cut off from its box a box of the form $(1-\varepsilon,1] \times I_1 \times I_2$ or $I_1 \times (1-\varepsilon,1] \times I_2$, where I_1, I_2 are intervals and ε is a small positive number such that different values for ε are used for different vertices u.

In other words, every vertex u can be drawn in the B_1 or B_2(see Figure 46).

In this way we create an inner empty conner(close to the plane with equation x=1 or y=1)for each facial triangle that contaions z. In order to achieve the same for all other facial triangles as well we translate all other boxes corresponding to vertices of G_1 $-${x,y,z} a little *upwards*, i.e.,the vector of translation is of the form (0,0,ε)where ε is a small positive number and different values for ε are used for different vertices.

In this way each facial triangle in G_1 not containing z gets an inner empty corner close to one of the planes with equation z=1, z=1/2.

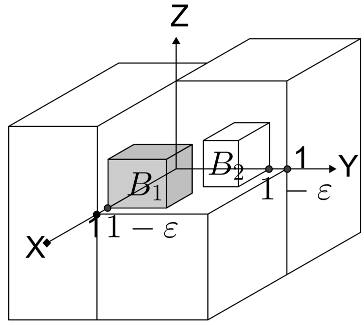

Figure 46: B_1 is the box $(1-\varepsilon,1] \times I_1 \times I_2$ and B_2 is the box $I_1 \times (1-\varepsilon,1] \times I_2$

□

Since every planar graph is an induced subgraph of a planar triangulation, then we have the following theorem.

Theorem 3.20. *Every planar graph has a strict box representation in R^3.*

Proof. Suppose G is a planar graph, and given a vertex v in any face which is not triangle. Drawing some edge from v to the vertices of the face, then we get a new planar triangulation graph G'(see Figure 47). By Theorem 3.19, G' has a strict 3-box representation. If the removal of the box corresponding to the vertex v leaves the graph G, then we get the strict 3-box representation of G. So the proof is complete. □

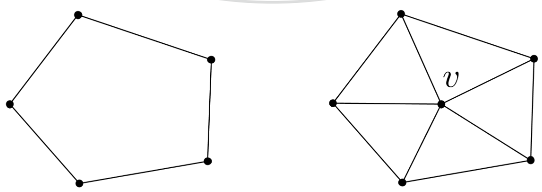

Figure 47: draw a vertex v in the face which is not triangle

4 Open problems and further directions of study

In our article, we have presented the forbidden subgraph of 4-connected planar triangulation and the strict 2-box representation. We also show that every planar has a strict 3-box representation. There are still some open problems from our article and we mention them in the following.

1 Does every 3-connected planar graph have a strict 2-box representation?

2 Does every 2-connected planar graph have a strict 2-box representation?

References

[1] P. Duchet, Y. Hamidoune, M. Las Vergas, and H. Meyniel, Representing a planar graph by vertical lines joing different levels, Discrete Math. 46(1983), 221-332.

[2] L. A. Melnikov, Problem at the "Sixth Hungar. Colloq. on Combinatorics", Eger, 1981.

[3] E. R. Scheinerman, "Intersection Classes and Multiple Intersection Parameters", Ph. D. thesis, Princetion Uni., 1984.

[4] E. R. Scheinerman and D. B. West, "The interval number of a planar graph: Three intervals suffice, J. Combin. Theory Ser. B 35 (1983), 224-239.

[5] C. Thomassen, Plane representations of graphs, in "Progress in Graph Theory", (J. A. Bondy and U. S. Murty, Eds.), pp.43-69, Academic Press, Toronto, 1984.

[6] W. T. Trotter, Graphs and partially ordered sets, in "Selected Topics in Graph Theory 2", (L. W. Beineke and R. J. Wilson, Eds.), pp.237-268, Academic Press, London, 1983.

[7] P. Unger, On diagrams representing maps, J. London Math. Soc. 28 (1953), pp.336-342

[8] C. Thomassen, "Interval representations of planar graphs, Journal of Combinatorial Theory, Series B, pp.9-20, 1986.