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碩士學位論文

Tropical Derivatives and Anti-derivatives

熱帶導數與熱帶反導數

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致 謝

畢業後，直接分發入國中教書已十餘年，深覺自己學有不足，於是立定目標想要再回去唸個研究所。綜觀各校有開數學在職專班的北部國立大學，只有政治大學和師範大學，當下決定目標鎖定嚮往已久的政治大學。當放榜考上的那一刹那，真是無比開心! 能重拾課本對我來說，無比可貴。第一年暑假，認識了來自四面八方的好朋友，最遠的是在台東教書的容溶，感謝她爲本班的無悔付出，當了近三年的班代眞不容易。那年暑假尤記得大家一起準備分析及代數的小考，對我這個已經離開大學一陣子的人來說，雖已事隔久遠，但我仍認眞準備甚而比大學還用功，這之間獲得的成就感無可比擬! 上課的課程相當多元又充實，我也在那個暑假第一次接觸到Latex，而這也成爲我寫論文的工具。使用後，才發現它強大的排版功能，這要多感謝蔡炎龍老師教我們學會這個好用的軟體。

在歷經生第二個小孩，忙碌的教書工作，邊要上碩專班的課又要照顧小孩又要準備作業與小考，分身乏術的我本以爲自己不可能順利完成論文，因爲這是個艱鉅的任務，更從來不敢相信我也有能力寫論文。但是，和藹可親的蔡老師在百忙中，很有耐心的從最基本的觀念帶領我進入熱帶幾何的殿堂，一再的修正我許多錯誤之處，從來不給我任何壓力，最後我才能完成這篇論文。感謝口試委員陳天進老師和張宜武老師的指導，使得我的論文能更加完備，在此致上十二萬分的感謝。最後，也感謝我的家人一路的支持與鼓勵，讓我無後顧之憂的完成碩士學位，多謝你們!

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Abstract

In this thesis, we define the tropical derivatives and anti-derivatives. When we differentiate two identical tropical polynomials, we might get two different functions. In order to overcome the difficulties, we restrict the polynomials to largest coefficient polynomials to avoid unpredictable results when taking derivatives. The definitiion of the tropical derivatives is quite different from the definition of classical derivatives. In particular, we have $\frac{d}{dx}a_n \odot x^{\odot n} = a_n \odot x^{\odot n-1}$. To extend it linearly, we obtain $\frac{d}{dx}[a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .... \oplus a_1 \odot x \oplus a_0] = a_n \odot x^{\odot n-1} \oplus a_{n-1} \odot x^{\odot n-2} ..... \oplus a_1 \odot x \oplus -\infty$. We will explain why we use this kind of definition. The derivatives are helpful in understanding more about tropical geometry, and it carries out some information similar to classical derivatives. Finally, we discuss how to define and find tropical anti-derivatives for tropical polynomials.

Keywords : Tropical derivatives, tropical anti-derivatives, tropical polynomials.

中文摘要

在這篇論文中，我們定義了熱帶導數和熱帶反導數。當我們對兩個相同的熱帶多項式求導數時，可能會得到不同的函數。爲了克服此困難，我們限制在最大係數多項式下才求導數。熱帶導數的定義與古典導數相當不同。特別的是，我們有$\frac{d}{dx}a_n \odot x^{\odot n} = a_n \odot x^{\odot n-1}$。將它線性化，我們得到$\frac{d}{dx}[a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .... \oplus a_1 \odot x \oplus a_0] = a_n \odot x^{\odot n-1} \oplus a_{n-1} \odot x^{\odot n-2} ..... \oplus a_1 \odot x \oplus -\infty$。我們將會解釋爲什麼使用這種定義。導數對了解熱帶幾何很有幫助，它也引出了一些與古典導數相似的資訊。最後，我們討論如何定義及求熱帶多項式的熱帶反導數。

Contents

Chapter 1

Introduction

Tropical geometry is developed by the Brazilian mathematician and computer scientists Imre Simon, who pioneered the min-plus algebra in 1980. From that day, many mathematicians put into research of combinations, algebraic geometry, statistics, and other sciences such as biology. It has become a new division of mathematics. And the adjective “tropical“ is given in honor of the Brazilian mathematician Imre Simon.[1]

Along the short, tropical geometry is piecewise linear algebraic geometry and study the image of classical geometry, so we can develop some important properties as in classical geometry. In fact, it has had many corresponding versions of classical theorems in algebraic geometry.

In this thesis, we define the tropical derivatives and the tropical anti-derivatives. The derivative is useful to understand more about tropical geometry. In chapter two, we mention of the largest coefficient polynomials, which appear frequently in tropical derivatives. For example, $f(x) = x^{\odot 2} \oplus x \oplus 4$ and $g(x) = x^{\odot 2} \oplus 2 \odot x \oplus 4$ are functionally equivalent, and $g(x) = x^2 \oplus 2x \oplus 4$ is the largest coefficient polynomial. We draw some gragh about these polynomials in order to understand more about the largest coefficient polynomial. Further more, we try to judge whether it is the largest coefficient polynomial or not.

In chapter three, we differentiate the Puiseux series which is known as an algebraically closed field and define the tropical derivatives. It is amazed that some properties of the tropical derivatives are satisfied in tropical derivatives as in classical derivatives, such as the product rule and the chain rule.

In chapter four, we integrate tropical polynomials, and we define the tropical anti-derivatives. It has some difference between classical and tropical anti-derivatives. We might restrict some conditions to obtain a largest coefficient polynomial when integrating tropical polynomials.

Chapter 2

Arithmetic of the Max-plus Semiring

In tropical geometry, we deal with the semiring ($\mathbb{R} \bigcup$ {-∞}, $\bigoplus$, $\bigodot$). As we see, it is a semiring over the union of real numbers and -∞ equipped with two binary operations, maximum and additions. We will denote the semiring by $\mathbb{T}$.

Definition 2.1 *Let a and b be scalars. Then we redefine the basic arithmetic operations of addition and multiplication for this scalars as follows:*

$$\begin{aligned} a \oplus b &= \max\{a, b\} \\ a \odot b &= a + b \end{aligned}$$

In words, the tropical sum of two numbers is their maximum, and the tropical product of two numbers is their sum. These two operations also satisfy the commmutative law, associative law, and distributive law. We will introduce these properties in the preceding article. Here are some examples of how to arithemetric in this number system.

Example 2.1

$$\begin{aligned} 2 \oplus 5 &= \max\{2, 5\} = 5 \\ 2 \odot 5 &= 2 + 5 = 7 \end{aligned}$$

We find many of the familiar axioms of arithmetic remain valid in tropical mathematics:

- **associativity**:

$$\begin{aligned}(a \oplus b) \oplus c &= a \oplus (b \oplus c)\\ (a \odot b) \odot c &= a \odot (b \odot c)\end{aligned}$$

- **commutativity**:

$$\begin{aligned}a \oplus b &= \max\{a, b\} = b \oplus a\\ a \odot b &= a + b = b + a = b \odot a\end{aligned}$$

- **distributivity**:

$$\begin{aligned}a \odot (b \oplus c) &= a \odot \max\{b, c\}\\ &= \max\{a + b, a + c\}\\ &= a \odot b \oplus a \odot c\\ (a \oplus b) \odot c &= \max\{a, b\} \odot c\\ &= \max\{a + c, b + c\}\\ &= a \odot c \oplus b \odot c\end{aligned}$$

Here are some numerical examples to show distributivity:

Example 2.2

$$\begin{aligned}2 \odot (5 \oplus 9) &= 2 \odot 9 = 11\\ 2 \odot 5 \oplus 2 \odot 9 &= 7 \oplus 11 = 11\end{aligned}$$

Example 2.3

$$(7 \oplus 13) \odot 8 = 13 \odot 8 = 21$$
$$7 \odot 8 \oplus 13 \odot 8 = 15 \oplus 21 = 21$$

Besides, we can easily find out the additive identity for $\oplus$ and the multiplicative identity for $\odot$.

• **Neutral element of tropical addition**:

The additive identity for $\oplus$ is $-\infty$, which is called $0_{\mathbb{T}} = -\infty$. The reason is for any a $\in \mathbb{R}$, max{a,$-\infty$ }= a if and only if a $\oplus - \infty$= a.

• **Neutral element of tropical multiplication**:

The multiplicative identity for $\odot$ is 0, which is called $1_{\mathbb{T}} = 0$. The reason is for any a $\in \mathbb{R}$, a $\odot$ 0 =a + 0 = a.

Remark 2.1 *Note that there is no tropical subtraction, which is why $\mathbb{T}$ is a semi-ring. Because for $a \neq -\infty$, there does not exist b$\in \mathbb{T}$ such that $a \oplus b = -\infty$. For example, the equation $2 \oplus x = 1$ has no solutions x at all. However, there do exist multiplicative inverse in $\mathbb{T}$. We shall define the tropical division $a \oslash b = a - b$.*

Above all, we also can define the tropical semiring in different ways. For examples, $a \oplus b = \min\{a, b\}$, which is called the min-plus tropical semiring. In this paper, we will focus on the max-plus semiring.

And we can discuss the tropical monomial in one variable :

$$a \odot x^{\odot n} = a \odot \underbrace{x \odot x ...... \odot x}_{n\ times}$$

A tropical polynomial is the tropical sum of a collection of tropical monomials.

Definition 2.2 *[3](Tropical Polynomials).A tropical polynomial $f(x)$ is of the form*

$$f(x) = a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \oplus ...... \oplus a_{n-1} \odot x^{\odot n-1} \oplus a_n \odot x^{\odot n},$$

where n is a positive integer, and $a_0, ...a_n \in \mathbb{T}$

Eveluate $f(x)$, we obtain

$$f(x) = \max\{a_0, a_1 + x, a_2 + 2x, ..., a_{n-1} + (n-1)x, a_n + nx\}$$

Remark 2.2 *In classical polynomials, x means $1_{\mathbb{T}} \cdot x$, but $1_{\mathbb{T}} = 0$, so $x = 0 \odot x$.*

Example 2.4

$$\begin{aligned} f(x) &= x^{\odot 3} \oplus 4 \odot x^{\odot 2} \oplus 7 \odot x \oplus 5 \\ &= \max\{0 + 3x, 4 + 2x, 7 + x, 5\} \end{aligned}$$

The graph of $f(x)$ is drawn as Figure2.1.

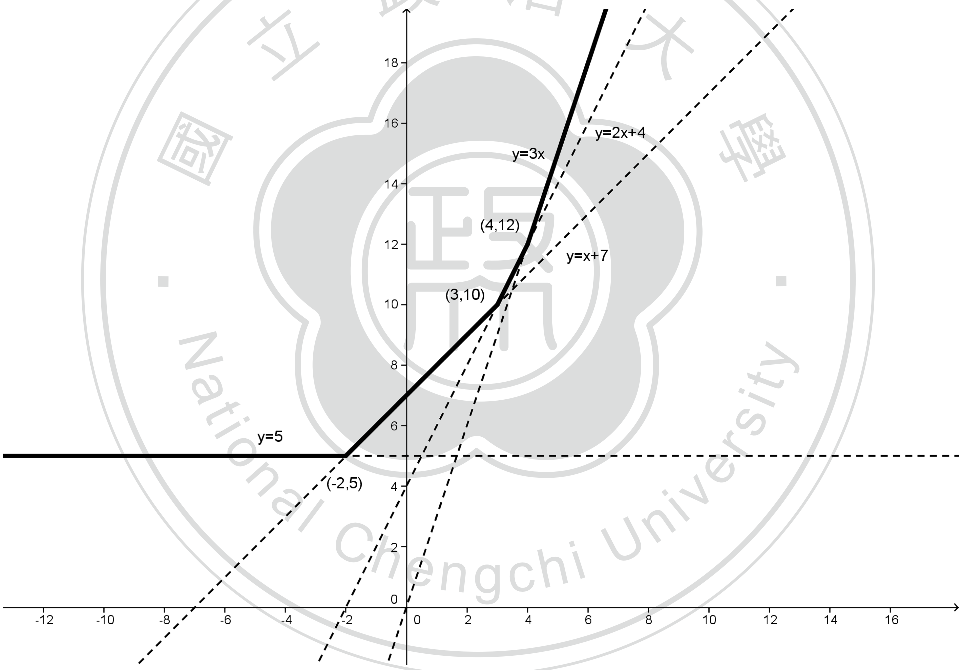

Figure 2.1: The graph of $f(x) = x^{\odot 3} \oplus 4 \odot x^{\odot 2} \oplus 7 \odot x \oplus 5$

2.1 Largest Coefficient Polynomials

In classical algebra, we all understand two distinct polynomials are certainly different, that is, if $f(x) \neq g(x)$, $f(x) - g(x) \neq 0$. However, in tropical algebra, two distinct tropical polynomials may define the same function . We say them functionally equivalent under the idea of largest coefficients.

Definition 2.3 *[3] Let $f(x)$ and $g(x)$ are two tropical polynomials. If $f(x)$ and $g(x)$ define the same function, we say that $f(x)$ and $g(x)$ are functionally equivalent.*

We refer to [2] and consider two one-variable polynomials as follows :

Example 2.5 $f(x) = x^{\odot 2} \oplus x \oplus 4$ *and* $g(x) = x^{\odot 2} \oplus 2 \odot x \oplus 4$ *are functionally equivalent. when* $x \geq 2$,

$$\begin{aligned} f(x) &= x^{\odot 2} \oplus x \oplus 4 \\ &= \max\{2x, x+0, 4\} \\ &= 2x \\ &= x \odot x \\ g(x) &= x^{\odot 2} \oplus 2 \odot x \oplus 4 \\ &= \max\{2x, x+2, 4\} \\ &= 2x \\ &= x \odot x \end{aligned}$$

when $x \leq 2$,

$$\begin{aligned} f(x) &= x^2 \oplus x \oplus 4 \\ &= \max\{2x, x+0, 4\} \\ &= 4 \end{aligned}$$

$$
\begin{aligned}
g(x) &= x^2 \oplus 2 \odot x \oplus 4 \\
&= \max\{2x, x+2, 4\} \\
&= 4
\end{aligned}
$$

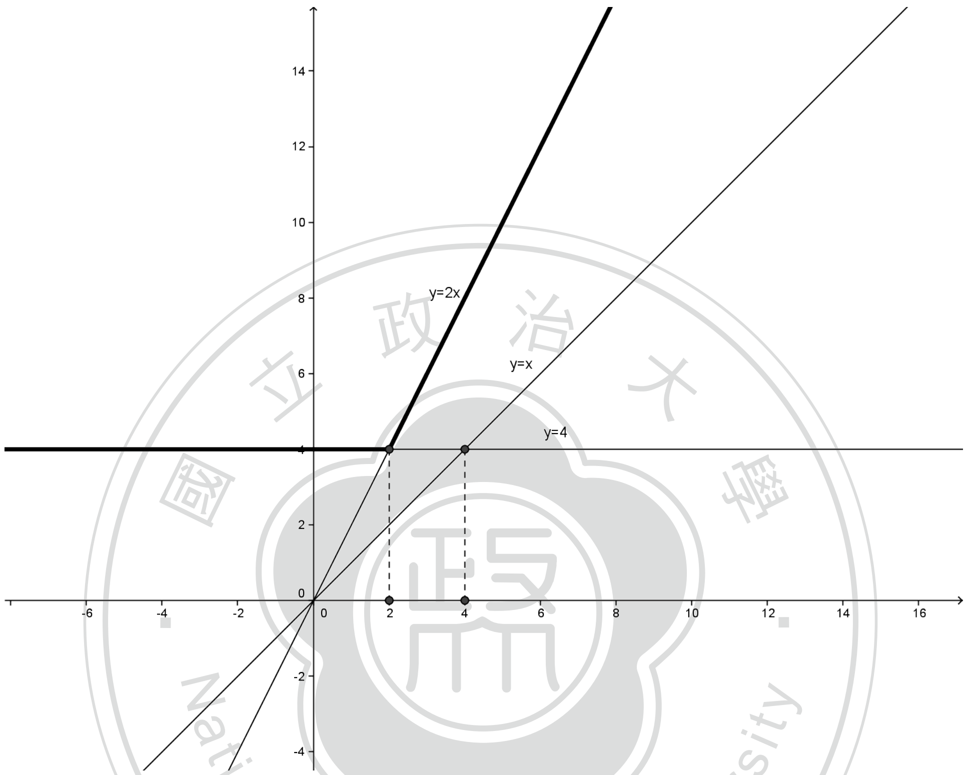

Figure 2.2: The graph of $f(x) = x^{\odot 2} \oplus x \oplus 4$

From figure 2.2, we observe the line $y = x$ is under the graph of $f(x)$. So we can move the line $y = x$ up to intersect the graph of $f(x)$ at exactly one point. This point is the intersection of $x^{\odot 2}$ and 4. And the slope of the line $y = x$ is less than $y = 2x$ and greater than $y = 4$. We recognize $f(x)$ and $g(x)$are functionally equivalent. Now we are going to use such ideas of largest coefficient polynomials to simply the work in tropical derivative. At the same time, we find that $f(x) = x^{\odot 2} \oplus a \odot x \oplus 4$, $a \leq 2$ and $g(x) = x^{\odot 2} \oplus b \odot x \oplus 4$, $b \leq 2$ are functionally equivalent.

Lemma 2.1 *Two tropical polynomials are functionally equivalent if and only if they represent the same under the idea of largest coefficients.*

Next, we are going to introduce the coefficient of $x^{\odot 2}$ term is not 0.

Example 2.6 *$f(x) = 2 \odot x^{\odot 2} \oplus 3 \odot x \oplus 4$ and $g(x) = 2 \odot x^{\odot 2} \oplus x \oplus 4$, when $x \geq 1$,*

$$\begin{aligned} f(x) &= 2x^{\odot 2} \oplus 3 \odot x \oplus 4 \\ &= \max\{2x+2, x+3, 4\} \\ &= 2x+2 \\ &= 2 \odot x^{\odot 2} \\ g(x) &= 2 \odot x^{\odot 2} \oplus x \oplus 4 \\ &= \max\{2x+2, x+0, 4\} \\ &= 2x+2 \\ &= 2 \odot x^{\odot 2} \end{aligned}$$

when $x \leq 1$,

$$\begin{aligned} f(x) &= 2 \odot x^{\odot 2} \oplus 3 \odot x \oplus 4 \\ &= \max\{2x+2, x+3, 4\} \\ &= 4 \\ g(x) &= 2 \odot x^{\odot 2} \oplus x \oplus 4 \\ &= \max\{2x+2, x+0, 4\} \\ &= 4 \end{aligned}$$

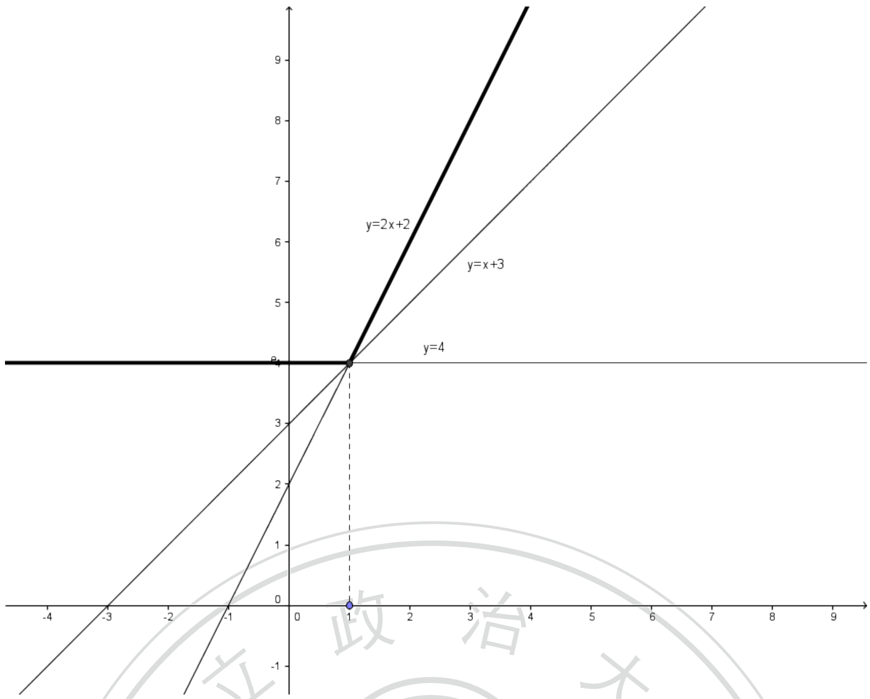

Figure 2.3: The graph of $f(x) = 2 \odot x^{\odot 2} \oplus 3 \odot x \oplus 4$

As example 2.6, to graph $f(x)$(see figure 2.3), we draw three lines in the (x, y) plane : $y = 2x + 2$, $y = x + 3$, and the horizontal line $y = 4$. The value of $f(x)$ is the largest y-value such that (x, y) is one of these three lines, i.e., the graph of f(x) is the higher envelop of the lines. So we can judge that $f(x) = 2x^{\odot 2} \oplus 3 \odot x \oplus 4$ is the largest coefficient polynomial. At the same time, we find the coefficients of these three terms $2x^{\odot 2}$, $3 \odot x$ and 4 satisfying 4-3 = 3-2. It encourges us to investigate how to judge whether the tropical polynomials is the largest coefficient one or not.

Definition 2.4 *$f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .....a_1 \odot x \oplus a_0$ is the largest coefficient polynomial, if for all $a_i, 0 \leq i \leq n$, there doesn't exist any functionally equivalent polynomials whose coefficients can replace a_i with larger numbers.*

Two distinct largest coefficient polynomails are certainly not functionally equivalent.

Example 2.7 *$f(x) = 0 \odot x^{\odot 2} \oplus 5 \odot x \oplus 7$ and $g(x) = 3 \odot x^{\odot 2} \oplus 4 \odot x \oplus 5$,* See Figure2.4 and 2.5 as bellows.

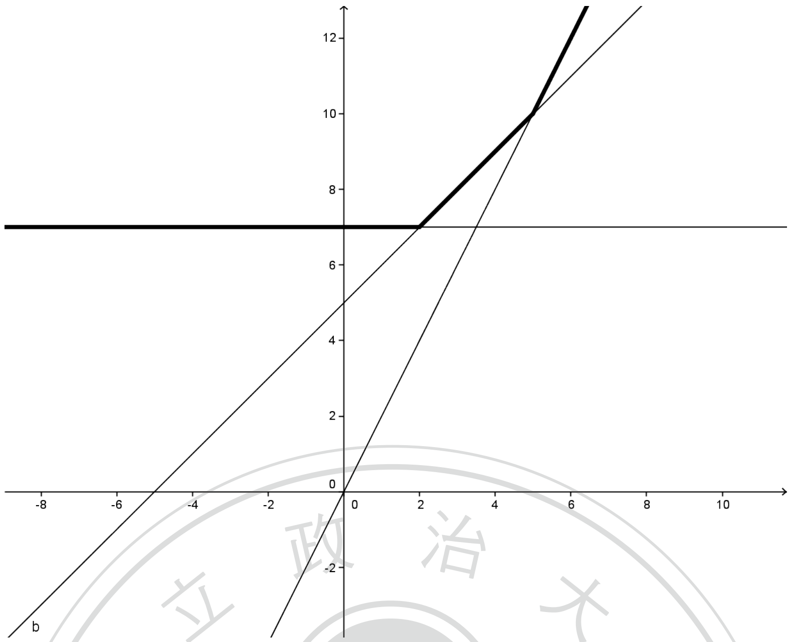

Figure 2.4: The graph of $f(x) = x^{\odot 2} \oplus 5 \odot x \oplus 7$

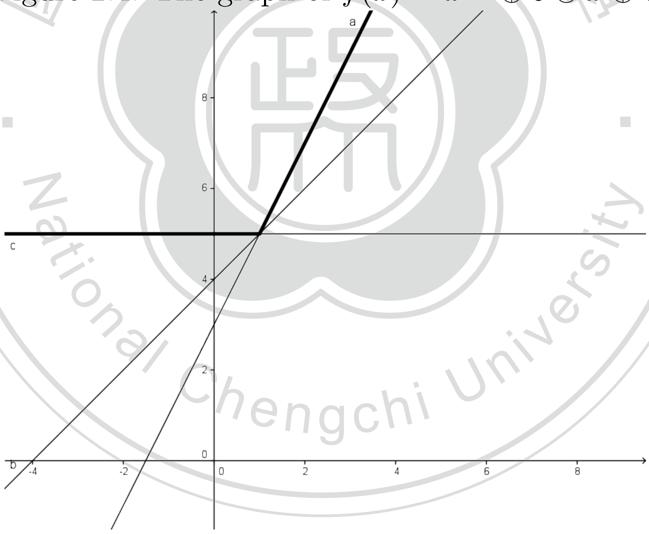

Figure 2.5: The graph of $f(x) = 3 \odot x^{\odot 2} \oplus 4 \odot x \oplus 5$

Lemma 2.2 *[5][Another definition of largest coefficient]Let* $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_r \odot x^{\odot r}$ *be a tropical polynomial, where* $a_i \neq -\infty, i = r, r+1, ..., n$*. Then* a_i *is a largest coefficient of* $f(x)$ *if and only if there exists some* $x_0 \in \mathbb{R}$ *such that* $f(x_0) = a_i \odot x_0^{\odot i}$*.*

Moreover, how do we determine if a largest coefficient polynomial is a largest coefficient one or not? We refer to[5] and solve this problem by the following theorem.

Theorem 2.1 *[5] Let* $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_r \odot x^{\odot r}$ *be a tropical polynomial, then* $g(x) = b_n \odot x^{\odot n} \oplus b_{n-1} \odot x^{\odot n-1} \oplus .... \oplus b_r \odot x^{\odot r}$ *is the largest coefficient polynomial of f(x), where*

$$b_i = \max\{\{a_i\} \cup \{\frac{a_j(k-i) + a_k(i-j)}{k-j} \mid r \leq j < i < k \leq n\}\}.$$

Example 2.8 *Let* $f(x) = 5 \odot x^{\odot 4} \oplus x^{\odot 2} \oplus 1$. *By theorem2.1, we get the corresponding largest coefficient polynomial* $g(x) = 5 \odot x^{\odot 4} \oplus 4 \odot x^{\odot 3} \oplus 3 \odot x^{\odot 2} \oplus 2 \odot x \oplus 1$

Example 2.9 *Let* $f(x) = 4 \odot x^{\odot 3} \oplus 5 \odot x^{\odot 2} \oplus 7 \odot x \oplus 5$, *the largest coefficient polynomial of f(x) is* $g(x) = 4 \odot x^{\odot 3} \oplus 5.5 \odot x^{\odot 2} \oplus 7 \odot x \oplus 5$.

Chapter 3

Tropical Derivatives

3.1 Differentiating the Puiseux Series

Let $\mathcal{K} = \overline{\mathbb{C}(t)}$ be the algebraic closure of the field of rational functions with coefficients from the field of complex numbers. An element $a(t)$ in $\mathcal{K}$ can be expressed as a Puiseux series.

Definition 3.1 *[2](Puiseux series)*
A Puiseux series $a(t)$ is of the form :

$$\sum_{i=k}^{\infty} C_i t^{\frac{i}{n}},\ k \in \mathbb{Z},\ n \in \mathbb{N},\ C_k \in \mathbb{C}$$

we define the field of Puiseux series $\mathcal{K}$ to be the collection of all Puiseux series.

Example 3.1 $a(t) = 3t^{\frac{-1}{2}} + t^{-1} + 5t^{-3} \in \mathcal{K}$.

Definition 3.2 *[5] Define a order*

$$\mathbf{Ord} : \mathcal{K} \longmapsto \mathbb{Q}$$

as followings. Let a be a nonzero element in $\mathcal{K}$*, for all* $a = \sum_{i=k}^{\infty} C_i x^{\frac{i}{n}} \in \mathcal{K}$

$$\mathbf{Ord}(a) := \min_i \{\frac{i}{n}\} = \frac{k}{n}$$

If $a = 0$*,*

$$\mathbf{Ord}(a) := -\infty.$$

Example 3.2 $a = 3t^{\frac{-1}{2}} + t^{-1} + 5t^{-3} \in \mathcal{K}$, $\mathbf{Ord}(a) = \min\{\frac{-1}{2}, -1, -3\} = -3$.

Definition 3.3 *Let* $f \in \mathcal{K}[x]$*,* $f = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$*, we define the tropicalization of* f *to be the tropical polynomial* $\overline{f}$*, such that*

$$\overline{f} = ord(a) \odot y^{\odot n} \oplus ord(a_{n-1}) \odot y^{\odot n-1} \oplus ... \oplus ord(a_1) \odot y \oplus ord(a_0)$$

Remark 3.1 *For any tropical polynomial* $g \in \mathbb{T}[y]$*, there exists at least one* $f \in \mathcal{K}[x]$ *such that* $g = \overline{f}$*.*

Example 3.3

$$\begin{aligned}
a &= 5t^{\frac{-1}{2}} + t^{-2} + 5t^{-4} \in \mathcal{K}, Ord(a) = -4 \\
b &= 8t^6 + t^2 + 5t^{-1} \in \mathcal{K}, Ord(b) = -1 \\
c &= t^2 + t \in \mathcal{K}, Ord(c) = 1 \\
f(x) &= ax^5 + bx^3 + cx \\
\overline{f(x)} &= Ord(a) \odot x^{\odot 5} \oplus Ord(b) \odot x^{\odot 3} \oplus Ord(c) \odot x \\
&= -4 \odot x^{\odot 5} \oplus (-1) \odot x^{\odot 3} \oplus (1) \odot x
\end{aligned}$$

Definition 3.4 *[2] Let* $q \odot y^{\odot n}$ *be a tropical monomial, we define the tropical derivative of* $q \odot y^{\odot n}$ *as the following :*

$$\frac{d}{dy} q \odot y^{\odot n} = q \odot y^{\odot n-1}$$

Remark 3.2 *We explain the reason we give this definition. Soppose* $q \odot y^{\odot n}$ *is the tropicalization of* $f(x) = a(t)x^n$*. That is* $q = Ord(a(t))$*.*

$$f' = \frac{da(t)x^n}{dx} = na(t)x^{n-1}$$

The tropicalization of f' *is* $Ord(na(t)) \odot x^{\odot n-1}$ *which is just* $Ord(a(t)) \odot x^{\odot n-1}$*. Because* n *is a constant, it will have no effect on the order of derivative.*

Obviously, the tropical derivative is quite different from the classical derivative. To extend it linearly, we will have the next section.

3.2 The Definition of Tropical Derivatives

Since we get the conclusion in last section, we extend a tropical monomial linearly and this will be the definition of the tropical derivative for the rest of the paper.

Definition 3.5

Given $f(y) = a_n \odot y^{\odot n} \oplus a_{n-1} \odot y^{\odot n-1} \oplus .....a_2 \odot y^{\odot 2} \oplus a_1 \odot y \oplus a_0$ *a largest coefficient polynomial, where* $a_i \in \mathcal{K}$*,* $0 \leq i \leq n$

$$\frac{d}{dy} f(y) = a_n \odot y^{\odot n-1} \oplus a_{n-1} \odot y^{\odot n-2} ..... \oplus a_2 \odot y \oplus a_1 \oplus -\infty$$

Example 3.4

$$\begin{aligned}
a &= 3t^{\frac{-3}{2}} + t^{-2} + 5t^{-6} \in \mathcal{K}, Ord(a) = -6 \\
b &= 5t^5 + t^2 + 5t^{-1} \in \mathcal{K}, Ord(b) = -1 \\
c &= 6t^3 + 4t^2 \in \mathcal{K}, Ord(c) = 2 \\
f(x) &= ax^4 + bx^3 + cx \\
f'(x) &= 4ax^3 + 3bx^2 + c \\
\overline{f(x)} &= ord(a) \odot x^{\odot 4} \oplus ord(b) \odot x^{\odot 3} \oplus ord(c) \odot x \\
&= (-6) \odot x^{\odot 4} \oplus (-1) \odot x^{\odot 3} \oplus 2 \odot x \\
\overline{f'(x)} &= ord(4a) \odot x^{\odot 3} \oplus ord(3b) \odot x^{\odot 2} \oplus ord(c) \\
&= (-6) \odot x^{\odot 3} \oplus (-1) \odot x^{\odot 2} \oplus 2
\end{aligned}$$

Since we restrict our polynomials to largest coefficient polynomials, the derivative of a largrst coefficient polynomial must be a largest coefficient polynomial. We refer to [2] and use a lemma from it.

Lemma 3.1 *Let* $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_0$*. Let* $d_i = a_{i-1} - a_i$*, then* $d_i \geq d_{i-1}$ *,i.e.* $a_{i-1} - a_i \geq a_{i-2} - a_{i-1}$*, for all* $1 \leq i \leq n \Longleftrightarrow$ *f(x) is a largest coefficient polynomial.*

Example 3.5 $f(x) = x^{\odot 3} \oplus 3 \odot x^{\odot 2} \oplus 5 \odot x \oplus 6$

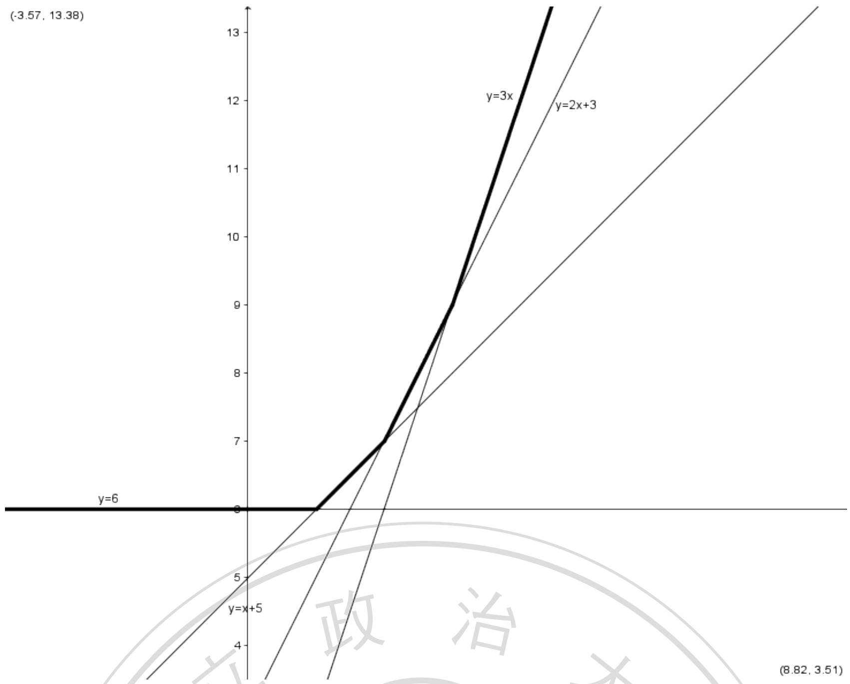

Figure 3.1: The graph of $f(x) = x^{\odot 3} \oplus 3 \odot x^{\odot 2} \oplus 5 \odot x \oplus 6$ is a largest coefficient polynomial

Corollary 3.1 *[2] Let* $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_1 \odot x \oplus a_0$. *If f(x) is a largest coefficient polynomial, then* $\frac{d}{dx} f(x) = a_n \odot x^{\odot n-1} \oplus a_{n-1} \odot x^{\odot n-2} ..... \oplus a_2 \odot x \oplus a_1 \oplus -\infty$ *is a largest coefficient polynomial.*

Proof. By Lemma2.1, $d_i \geq d_{i-1}$, for all $1 \leq i \leq n$. The derivative of f(x) is $a_n \odot x^{\odot n-1} \oplus a_{n-1} \odot x^{\odot n-2} ..... \oplus a_2 \odot x \oplus a_1$, trivial, we can get $d_i \geq d_{i-1}, \forall 1 \leq i \leq n-1$. The derivative is also in largest coefficients.

Example 3.6 *Let* $f(x) = x^{\odot 2} \oplus 3 \odot x \oplus 6$ *is the largest coefficient polynomial, then* $\frac{d}{dx} f(x) = x^{\odot 1} \oplus 3$ *is also in largest coefficients.*

3.3 Properties of the Tropical Derivatives

In classical derivative, the product rule is the quite fundamental property. Refering to[2], as below, we will show that the product rule are also hold in tropical derivatives.

3.3.1 Product Rule

Let f(x) and g(x) be tropical polynomials of degree n and m respectively.

$$f(x) = a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \ldots \oplus a_n \odot x^{\odot n}$$
$$g(x) = b_0 \oplus b_1 \odot x \oplus b_2 \odot x^{\odot 2} \ldots \oplus b_m \odot x^{\odot m}$$

Now, before we start to check the product rule, we must confirm that the product of these two largest coefficient polynomials is still a largest coefficient polynomial.

Lemma 3.2
The product of f(x) and g(x) is a largest coefficient polynomial.

Proof. Without loss of generality, we assume that the degree of $f(x)$ is greater than the degree of $g(x)$, $f(x) \odot g(x) = \sum_{k=0}^{n+m} (\sum_{i+j=k} a_i \odot b_j x^{\odot k})$,the coefficient of the $x^{\odot r}$ term is $\max_{i+j=r}\{a_i \odot b_j\}$,

$$\begin{aligned} Suppose \quad i_u + j_u &= r - 1 \\ i_v + j_v &= r \\ i_w + j_w &= r + 1 \end{aligned}$$

$$
\begin{aligned}
a_{i_v} \odot b_{j_v} - a_{i_u} \odot b_{j_u} &= a_{i_v} + b_{j_v} - a_{i_u} - b_{j_u} \\
&= a_{i_v} - a_{i_u} + b_{j_v} - b_{j_u} \\
&\geq a_{i_w} - a_{i_v} + b_{j_w} - b_{j_v} \\
&= a_{i_w} + b_{j_w} - a_{i_v} - b_{j_v} \\
&= a_{i_w} \odot b_{j_w} - a_{i_v} \odot b_{j_v}
\end{aligned}
$$

We get $\max\limits_{i+j=r}\{a_ib_j\} - \max\limits_{i+j=r-1}\{a_ib_j\} \geq \max\limits_{i+j=r+1}\{a_ib_j\} - \max\limits_{i+j=r}\{a_ib_j\}$, by Lemma 3.1, we get $f(x) \odot g(x)$ is a largest coefficient polynomial.

Now, we begin to check the product rule is also hold in tropical derivatives.

Theorem 3.1 $(f(x) \odot g(x))' = f'(x) \odot g(x) \oplus f(x) \odot g'(x)$

Proof. By induction on m, m = 0

$$
\begin{aligned}
f(x) \odot g(x) &= (a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \oplus \ldots \oplus a_n \odot x^{\odot n}) \odot b_0 \\
&= a_0 \odot b_0 \oplus a_1 \odot b_0 \odot x \oplus a_2 \odot b_0 \odot x^{\odot 2} \oplus \ldots \oplus a_n \odot b_0 \odot x^{\odot n} \\
(f(x) \odot g(x))' &= -\infty \oplus a_1 \odot b_0 \oplus a_2 \odot b_0 \odot x \oplus \ldots \oplus a_n \odot b_0 \odot x^{\odot n-1} \\
&= b_0 \odot (a_1 \oplus a_2 \odot x \oplus \ldots \oplus a_n \odot x^{\odot n-1}) \\
&= b_0 \odot f'(x) \\
&= -\infty \odot f(x) \oplus b_0 \odot f'(x) \\
&= g'(x) \odot f(x) \oplus g(x) \odot f'(x) \\
&= f'(x) \odot g(x) \oplus f(x) \odot g'(x)
\end{aligned}
$$

Suppose this is true for m = k

When m = k+1, deg(g(x)) = k+1,

$$
\begin{aligned}
f(x) \odot g(x) &= f(x) \odot (b_0 \oplus b_1 \odot x \oplus b_2 \odot x^{\odot 2} \oplus \ldots \oplus b_k \odot x^{\odot k}) \oplus f(x) \odot b_{k+1} \odot x^{\odot k+1} \\
&= f(x) \odot (b_0 \oplus b_1 \odot x \oplus b_2 \odot x^{\odot 2} \oplus \ldots \oplus b_k \odot x^{\odot k}) \oplus \\
&\quad (a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \oplus \ldots \oplus a_n \odot x^{\odot n}) \odot b_{k+1} \odot x^{\odot k+1}
\end{aligned}
$$

$$
\begin{aligned}
&= f(x)(b_0 \oplus b_1 \odot x \oplus b_2 \odot x^{\odot 2} \oplus \ldots \oplus b_k \odot x^{\odot k}) \oplus a_0 \odot b_{k+1} x^{\odot k+1} \\
&\quad \oplus a_1 \odot b_{k+1} \odot x^{\odot k+2} \oplus a_2 \odot b_{k+1} \odot x^{\odot k+3} \oplus \ldots \oplus a_n \odot b_{k+1} \odot x^{\odot k+n+1} \\
(f(x) \odot g(x))' &= f(x) \odot (b_1 \oplus b_2 \odot x \oplus \ldots \oplus b_k \odot x^{\odot k-1}) \oplus f'(x)(b_0 \oplus b_1 \odot x \ldots \oplus b_k \odot x^{\odot k}) \\
&\quad \oplus a_0 \odot b_{k+1} \odot x^{\odot k} \oplus a_1 \odot b_{k+1} \odot x^{\odot k+1} \oplus a_2 \odot b_{k+1} \odot x^{\odot k+2} \\
&\quad \oplus \ldots \oplus a_n \odot b_{k+1} \odot x^{\odot k+n} \\
&= f(x) \odot (b_1 \oplus b_2 \odot x \oplus \ldots \oplus b_k \odot x^{\odot k-1}) \oplus f'(x)(b_0 \oplus b_1 \odot x \ldots \oplus b_k \odot x^{\odot k}) \\
&\quad \oplus b_{k+1} \odot x^{\odot k}(a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \ldots \oplus a_n \odot x^{\odot n}) \oplus b_{k+1} \odot x^{\odot k+1}(a_1 \oplus a_2 \odot x \\
&\quad \ldots \oplus a_n \odot x^{\odot n-1}) \\
&= f(x)(b_1 \oplus b_2 \odot x \oplus \ldots \oplus b_k \odot x^{\odot k-1}) \oplus f'(x)(b_0 \oplus b_1 \odot x \ldots \oplus b_k \odot x^{\odot x}) \\
&\quad \oplus b_{k+1} \odot x^{\odot k} \odot f(x) \oplus b_{k+1} \odot x^{\odot k+1} f'(x) \\
&= f'(x) \odot g(x) \oplus f(x) \odot g'(x)
\end{aligned}
$$

Example 3.7 *Let* $f(x) = x \oplus 4$ *and* $g(x) = x \oplus 6$,

$$
\begin{aligned}
f(x) \odot g(x) &= x \odot x \oplus x \odot 6 \oplus 4 \odot x \oplus 4 \odot 6 \\
(f(x) \odot g(x))' &= x \odot 0 \oplus 0 \odot 6 \oplus 0 \odot 4 \\
&= max\{x, 6, 4\} \\
f'(x) \odot g(x) \oplus f(x) \odot g'(x) &= 0 \odot x \oplus 0 \odot 6 \oplus x \odot 0 \oplus 4 \odot 0 \\
&= max\{x, 6, 4\} \\
(f(x) \odot g(x))' &= f'(x) \odot g(x) \oplus f(x) \odot g'(x)
\end{aligned}
$$

In tropical derivatives, the derivative of the sum of two tropical polynomials is the sum of their derivatives. We can check this as follows :

Theorem 3.2 *(The Sum Rules)* $(f(x) \oplus g(x))' = f'(x) \oplus g'(x)$

Proof.
$$
\begin{aligned}
(f(x) \oplus g(x))' &= (a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2} \oplus \ldots \oplus a_n \odot x^{\odot n} \oplus b_0 \oplus b_1 \odot x \oplus \ldots \oplus b_m \odot x^{\odot m})' \\
&= a_1 \oplus a_2 \odot x \oplus \ldots \oplus a_n \odot x^{\odot n-1} \oplus b_1 \oplus b_2 \odot x \ldots \oplus b_m \odot x^{\odot m-1} \\
&= (a_1 \oplus a_2 \odot x \oplus \ldots \oplus a_n \odot x^{\odot n-1}) \oplus (b_1 \oplus b_2 \odot x \ldots \oplus b_m \odot x^{\odot m-1}) \\
&= f'(x) \oplus g'(x)
\end{aligned}
$$

Example 3.8 *Let* $f(x) = x^{\odot 2} \oplus 3 \odot x \oplus 2$ *and* $g(x) = x^{\odot 2} \oplus 2 \odot x \oplus 1$

$$
\begin{aligned}
(f(x) \oplus g(x)) &= (x^{\odot 2} \oplus 3 \odot x \oplus 2) \oplus (x^{\odot 2} \oplus 2 \odot x \oplus 1) \\
&= x^{\odot 2} \oplus 3 \odot x \oplus 2 \oplus x^{\odot 2} \oplus 2 \odot x \oplus 1 \\
(f(x) \oplus g(x))' &= x \oplus 3 \oplus 2 \\
&= x \oplus 3 \\
f'(x) \oplus g'(x) &= (x \oplus 3) \oplus (x \oplus 2) \\
&= x \oplus 3 \\
(f(x) \oplus g(x))' &= f'(x) \oplus g'(x)
\end{aligned}
$$

3.3.2 Chain Rule

We have yet to discuss one of the most powerful differentiation rules : the chain rule. The rule deals with composite functions, and is also hold in tropical derivatives.

Theorem 3.3 $(f(g(x)))' = f'(g(x)) \odot g'(x)$

Proof.

$$
\begin{aligned}
f(g(x)) &= a_0 \oplus a_1 \odot g(x) \oplus a_2 \odot (g(x))^{\odot 2} \oplus \ldots \oplus a_n \odot (g(x))^{\odot n} \\
(g(x)^{\odot n})' &= g'(x) \odot (g(x))^{\odot n-1} \oplus g'(x) \odot (g(x))^{\odot n-1} \oplus \ldots \\
&= g'(x) \odot (g(x))^{\odot n-1} \\
(f(g(x)))' &= a_1 \odot g'(x) \oplus a_2 \odot g'(x) \odot g(x) \oplus a_3 \odot g'(x) \odot (g(x))^{\odot 2} \ldots \\
&\quad \oplus a_n \odot g'(x) \odot (g(x))^{\odot n-1} \\
&= (a_1 \oplus a_2 \odot g(x) \oplus a_3 \odot (g(x))^{\odot 2} \ldots \oplus a_n \odot (g(x))^{\odot n-1}) g'(x) \\
&= f'(g(x)) \odot g'(x)
\end{aligned}
$$

Chapter 4

Tropical Anti-derivatives

4.1 Integrating Tropical Polynomials

In classical calculus, intergration is the inverse of differentiation. Given a function f to find a F such that $F'(x) = f(x)$. If such a function exists, it is called an anti-derivative function of f. In tropical anti-derivatives, we have the motivate to see the property of it.

In Section 2.2, we mentioned $f(x) = x^{\odot 2} \oplus 2 \odot x \oplus 4$ and $g(x) = x^{\odot 2} \oplus x \oplus 4$ are functionally equivalent. When we differentiate them, $f'(x) = x \oplus 2$ and $g'(x) = x \oplus 0$.

$$\begin{aligned} When \quad x \geq 2, \quad f'(x) &= x \oplus 2 \\ &= \max\{x, 2\} \\ &= x \\ g'(x) &= x \oplus 0 \\ &= \max\{x, 0\} \\ &= x \\ When \quad x \leq 2, \quad f'(x) &= x \oplus 2 \\ &= \max\{x, 2\} \\ &= 2 \end{aligned}$$

$$
\begin{aligned}
g'(x) &= x \oplus 0 \\
&= \max\{x, 0\} \\
&= 2
\end{aligned}
$$

We find $f'(x) = g'(x)$. That's why we use the idea of the largest-coefficient polynomials. Because if we wouldn't use it, we will get unpredictable results.

Now, if we use the integral symbol, $\int x \oplus 2 = x^{\odot 2} \oplus 2 \odot x \oplus c$, c is an arbitrary constant. But only when $2 - 0 \geq c - 2$, $4 \geq c$, $x^{\odot 2} \oplus 2 \odot x \oplus c$ is a largest coefficient polynomial.

Definition 4.1 *(Basic Intergration rules)*
We say $F(x)$ is an anti-derivative of $f(x)$, if $\frac{dF(x)}{dx} = f(x)$. Let $G(x)$ be another anti-derivative, $G(x)$ and $F(x)$ differ by a constant c. Thus, we define $\int f(x)dx = F(x) + c$.

Definition 4.2
Let $f(x)$ be a tropical polynomial. We say a tropical polynomial $F(x)$ is a tropical anti-derivative of $f(x)$, if $\frac{dF(x)}{dx} = f(x)$.

Remark 4.1 *Let $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .....a_1 \odot x \oplus a_0$, then*

$$F(x) = a_n \odot x^{\odot n+1} \oplus a_{n-1} \odot x^{\odot n} ..... \oplus a_1 \odot x^{\odot 2} \oplus a_0 \odot x,$$

which is an anti-derivative of $f(x)$.

Remark 4.2 *Obviously, if $F(x)$ is an anti-derivative, then $F(x) \oplus c$ is also an anti-derivative.*

Definition 4.3 *Let $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .....a_1 \odot x \oplus a_0$,*

$$\int f(x)dx = a_n \odot x^{\odot n+1} \oplus a_{n-1} \odot x^{\odot n} ..... \oplus a_1 \odot x^{\odot 2} \oplus a_0 \odot x \oplus c,$$

where c is a constant.

Theorem 4.1 *Let $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_1 \odot x \oplus a_0$, where $a_i \in \mathcal{K}, 0 \leq i \leq n$. If f(x) is a largest coefficient polynomial, then*

$$\int f(x)dx = a_n \odot x^{\odot n+1} \oplus a_{n-1} \odot x^{\odot n} ..... \oplus a_1 \odot x^{\odot 2} \oplus a_0 \odot x \oplus c$$

is also a largest coefficient polynomial. And c must satisfy $a_0 - a_1 \geq c - a_0$.

Proof. By Lemma2.1, let $d_i = a_{i-1} - a_i$, $d_i \geq d_{i-1}$, for all $1 \leq i \leq n$. The anti-derivative of f(x) is $a_n \odot x^{\odot n+1} \oplus a_{n-1} \odot x^{\odot n} ..... \oplus a_1 \odot x^{\odot 2} \oplus a_0 \odot x \oplus c$, trivial, we can get $d_i \geq d_{i-1}$, for all $1 \leq i \leq n-1$ and $a_0 - a_1 \geq c - a_0$ is known. So the anti-derivative is also in largest coefficients.

Example 4.1 $f(x) = 5 \odot x^{\odot 3} \oplus 6 \odot x^{\odot 2} \oplus 7 \odot x \oplus 8$ *is a largest coefficient polynomial, then* $\int f(x)dx = 5 \odot x^{\odot 4} \oplus 6 \odot x^{\odot 3} \oplus 7 \odot x^{\odot 2} \oplus 8 \odot x \oplus c$,
where c is a constant satisfying $8 - 7 \geq c - 8$, *i.e.* $c \leq 9$. *It is a largest coefficient polynomial.*

Remark 4.3

- $\int kdx = k \odot x \oplus c$, *where c is a constant.*
- $\int x^{\odot n}dx = x^{\odot n+1} \oplus c$, *where c is a constant.*
- *trivial,* $k \odot x \oplus c$ *and* $x^{\odot n+1} \oplus c$ *are largest coefficient polynomials.*

Chapter 5

Conclusion

In summary, tropical geometry is defined in the semiring $\mathbb{T} = (\mathbb{R} \bigcup \{\text{-}\infty\}, \bigoplus, \bigodot)$. As we see, it is a semiring over the union of real numbers and -∞ equipped with two binary operations, maximum and additions. At the meantime, the commmutative law, associative law, and distributive law are also hold under the basic arithmetic operations of addition and multiplication. The additive identity for $\oplus$ is $-\infty$, which is called $0_{\mathbb{T}} = -\infty$. And the multiplicative identity for $\odot$ is 0, which is called $1_{\mathbb{T}} = 0$. There is no tropical subtraction, which is why $\mathbb{T}$ is a semi-ring.

In proceeding, we discuss the tropical monomial in one variable. To extend it linearly, a tropical polynomial is the tropical sum of a collection of tropical monomials. In tropical algebra, two distinct tropical polynomials may define the same function . We say them functionally equivalent under the idea of largest coefficients. $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus .....a_1 \odot x \oplus a_0$ is the largest coefficient polynomial, if for all $a_i, 0 \leq i \leq n$, there doesn't exist any functionally equivalent polynomials whose coefficients can replace a_i with larger numbers. Another definition of largest coefficient is let $f(x) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus ... \oplus a_r \odot x^{\odot r}$ be a tropical polynomial, where $a_i \neq -\infty, i = r, r+1, ..., n$. Then a_i is a largest coefficient of $f(x)$ if and only if there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) = a_i \odot x_0^{\odot i}$. However, how do we determine if a largest coefficient polynomial is a largest coefficient one or not? we refer to [5] and solve this problem by theorem 2.1.

In chapter three, we begin to differentiate the Puiseux series and obtain the definition of the tropical derivatives. In classical derivatives, the product rule is the quite fundamental property. Refering to[2], we show that the product rule and the chain rule are also hold in tropical derivatives.

In chapter four, we discuss the anti-derivatives by integrating tropical polynomials and define the tropical anti-derivatives. It has some restriction when integrating tropical polynomials.

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