國立政治大學應用數學系

碩士學位論文

有關對立圖形的探討

Some Problems on Opposition Graphs

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Contents

Abstract

In this thesis, we use the number of vertices with degree greater than or equal to 3 as a criterion for trees being opposition graphs. Finally, we prove some families of graphs such as $\overline{P_n}$, C_n with $n \geq 3$ and $n = 4k, k \in \mathbb{N}$ are opposition graphs and some families of graphs such as $\overline{T_n}$, C_n with $n \geq 3$ and $n \neq 4k, k \in \mathbb{N}$ are not opposition graphs.

keywords: Opposition Graphs.

中文摘要

在這篇論文中，我們探討對立圖形的特性，並藉由度數大於等於三的點，判斷一樹是否為對立圖形，最後證明 $\overline{P_n}$, C_n $n \geq 3$ 且 $n = 4k, k \in \mathbb{N}$ 家族的圖是對立圖形且 $\overline{T_n}$, C_n $n \geq 3$ 且 $n \neq 4k, k \in \mathbb{N}$ 家族的圖是對立圖形。

關鍵詞：對立圖形

1 Introduction

From the book [1], they introduce many containment relationships between classes of graphs. In Figure 1, we can see the relations between opposition graphs and threshold graphs, and the relations between opposition graphs and perfect graphs. For example, P_4 is an opposition graph but not a threshold graph; C_6 is a perfect graph but not an opposition graph. Now we put our attention on the opposition graphs, we want to know what kind of graphs are opposition graphs.

By [2] and [3], we define a graph $G(V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. Therefore, in chapter 2, we introduce some basic definitions and theorems. In chapter 3, we give a set R which is the set of vertices with degrees greater than or equal to 3. In section 3.1, we discuss the case when R is empty, then we create some ways to give an orientation to a path. In section 3.2, we discuss the case that there is only one vertex in R, then we create a way to give an orientation to a rooted tree. In section 3.3, we discuss the case that there are two vertices in R. In section 3.4, we discuss that there are more than two vertices in R, and find out the minimum obstruction for the class of opposition graphs. In chapter 4, we prove some families of graphs such as $\overline{P_n}$, C_n with $n \geq 3$ and $n = 4k, k \in \mathbb{N}$ are opposition graphs and some families of graphs such as $\overline{T_n}$, C_n with $n \geq 3$ and $n \neq 4k, k \in \mathbb{N}$ are not opposition graphs. Finally, we bring up some open problems and further directions of research.

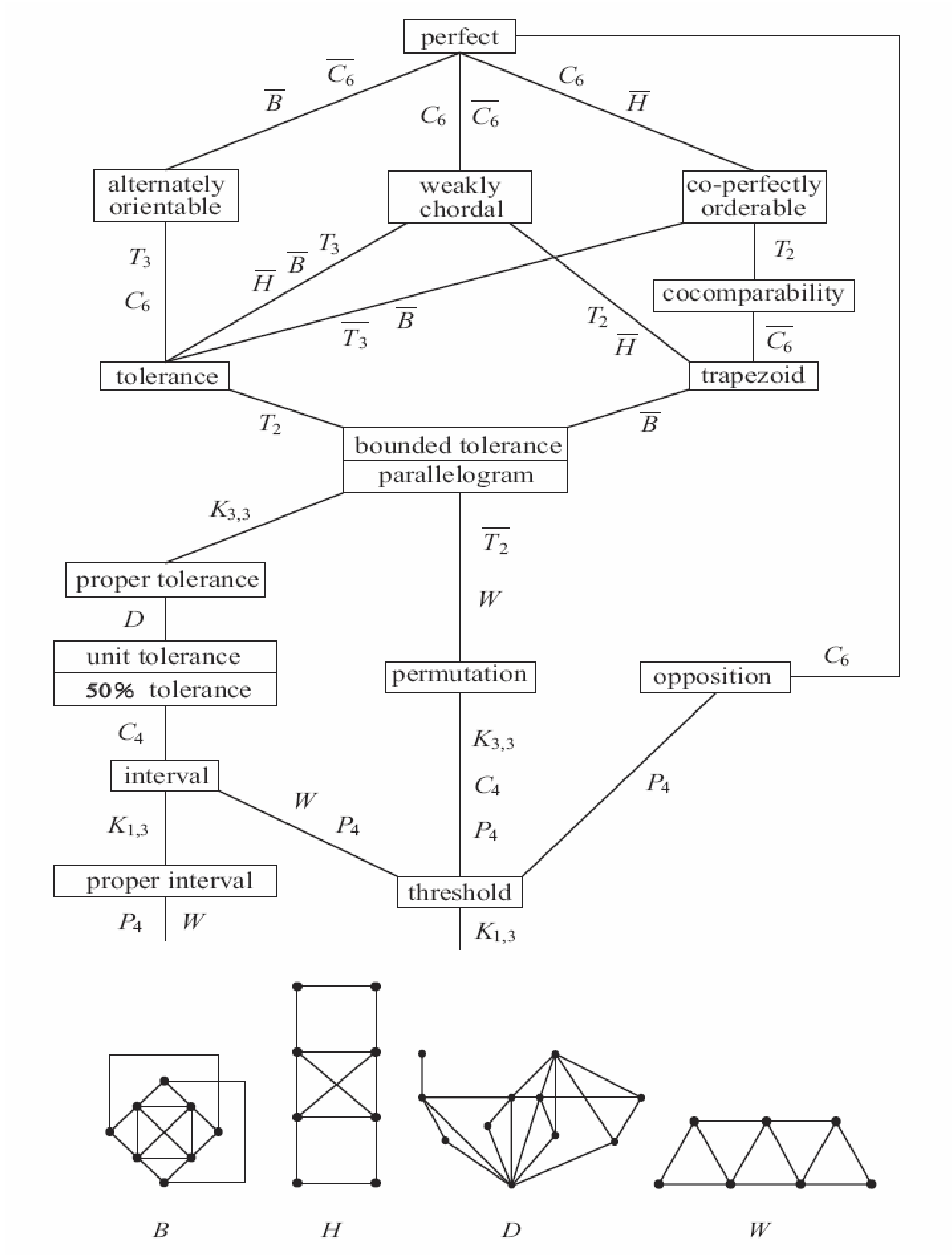

Figure 1: A complete hierarchy of classes of perfect graphs.

2 Definitions

In this chapter, we mention some basic definitions about graphs and trees.

For most of them, we follow [2] and [3]. A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices. Sometimes the edge are ordered pairs of vertices, called *directed edges*, the ordered pairs of vertices is called a *direction*.

A *directed graph* or *digraph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the *tail* of the edge, and the second is the *head*; together, they are the *endpoints*. We say that an edge is an edge from its tail to its head.An *orientation* of a graph G is a digraph D obtained from G by choosing an orientation ($u \to v$ or $v \to u$) for each edge $uv \in G$.

The *degree* of vertex v is the number of incident edges.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of end points to edges in H is the same as in G. An *induced subgraph* is a subgraph obtained by deleting a set of vertices. The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A *path* P is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path with n vertices is call P_n. A *cycle* C is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is call C_n. A graph with no cycle is *acyclic*. If G is a u, v-path, then the *distance* from u to v, written $d(u, v)$, is the least length of u, v-path. A graph G is *connected* if it has a u, v-path whenever $u, v \in V(G)$.

A *tree* is a connected acyclic graph. One can define a tree as a graph with a designated vertex called a *root* such that there is a unique path from the root to any other vertex in the tree. If a tree is unoriented, then any vertex can be the root.

A *leaf* is a vertex of degree 1. The *level number* of a vertex x in a tree T is the length from the root u to x. The *height* of a tree is the length of the longest path form root, equivalently, the largest level number of any vertex.

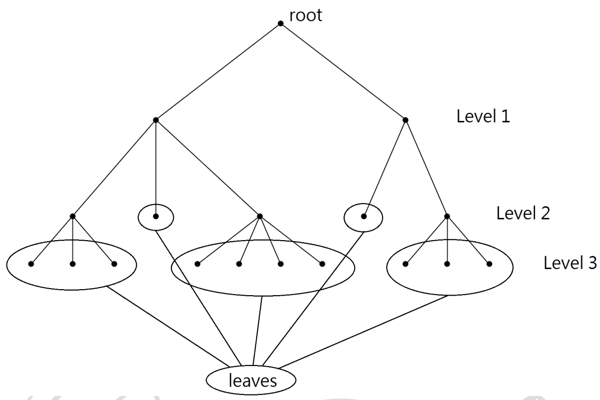

Figure 2: A tree

For any vertex x in a tree T, except the root, the *parent* of x is the vertex y with an edge from y to x, the *children* of x is the vertex z with an edge from x to z. The parent-children relationship extends to *ancestors* and *descendants*.

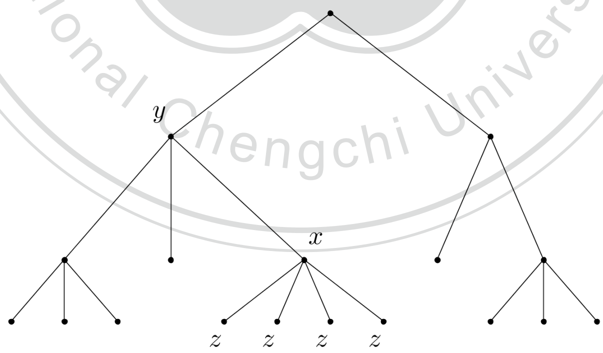

Figure 3: y is the parent of x; z are the children of x; y is an ancestor of z; z are the descendants of y.

Note the difference between "maximal" and "maximum". As adjectives, maximum means "maximum-sized", and maximal means "no larger one contains this one".

Example 2.1. In Figure 4, the path $v_1 - v_2 - v_3 - v_4$ is a maximum path and a maximal path. The path $v_1 - v_2 - v_5$ is a maximal path but not a maximum path.

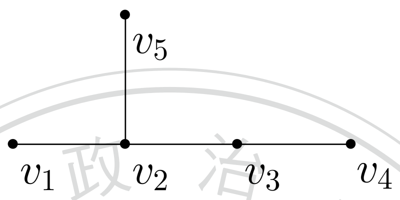

Figure 4:

Definition 2.2. A graph G is called an *opposition graph* if we can give an orientation of its edge such that in every induced P_4, the two end edges both either point inwards or outwards.

We know that if G is an opposition graph, then every induced P_4 must be shown as Figure 5

Figure 5: An orientation

Example 2.3. The graph C_8 is an opposition graph shown as Figure 6.

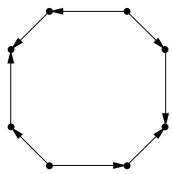

Figure 6: The graph C_8 is an opposition graph.

Example 2.4. In the graph C_5, we can give an orientation for C_5. If the direction for the edge v_1v_2 is $v_1 \to v_2$, we must have the following directions: $v_4 \to v_3, v_5 \to v_1, v_3 \to v_2$, then there are no direction for the edge v_4v_5. Similar for the direction for the edge v_1v_2 is $v_2 \to v_1$. Hence, the graph C_5 is not an opposition graph shown as Figure 7.

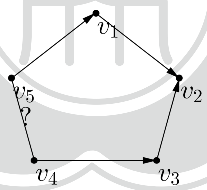

Figure 7: The graph C_5 is not an opposition graph.

Definition 2.5. A graph G is called a *threshold graph* if it does not contain a P_4, C_4, and $\overline{C_4}$ as induced subgraphs.

Proposition 2.6. *If a graph G is a threshold graph, then G is an opposition graph.*

Proof. If G is a threshold graph, then G has no induced P_4. Hence, G is an opposition graph. □

3 Some Opposition Graphs

In this chapter, we will discuss relations between opposition graphs and trees. Let T be a tree. Let $R(T) = \{x \in v(T) \mid deg(x) \geqslant 3\}$, we have the following four cases :

Case 1 $R(T) = \emptyset$.

Case 2 There is only one vertex u in $R(T)$.

Case 3 There are two vertices u,v in $R(T)$.

Case 4 There are more than two vertices in $R(T)$.

For a tree T, we will show that T is an opposition graph if and only if any two vertices in $R(T)$ have even distance. We also find the minimal obstruction for trees as opposition graphs.

3.1 $R(T) = \emptyset$

In this section, we discuss the case $R(T) = \emptyset$. Every vertex in the tree T has only degree 1 or 2, so T is a path P_n.

Theorem 3.1. *The path P_n is an opposition graph.*

Proof. Let v_1,v_2,..., v_n be the vertices of P_n. We can give an orientation of P_n as follows :

$v_i \to v_{i+1}$ for all $i = 4k$,$4k + 1$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 2$,$4k + 3$, where $k \in \mathbb{N}$ and $i < n$.

Then P_n is an opposition graph shown as Figure 8. □

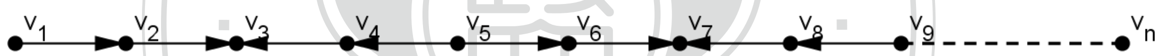

Figure 8: P_n is opposition.

Definition 3.2. Let G be an opposition graph. The orientation of G which satisfy the definition of opposition graphs is called the *oppositional orientation.*

Theorem 3.3. *There are only four oppositional orientations of P_n:*

Proof. Let v_1,v_2,...,v_n be the vertices of P_n.

Case 1 If the direction between v_1 and v_2 is $v_1 \to v_2$, then we must have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 1$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 3$, where $k \in \mathbb{N}$ and $i < n$.

Then we have two subcases:

subcase 1 The direction between v_2 and v_3 is $v_2 \to v_3$, then we have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 2$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 4$, where $k \in \mathbb{N}$ and $i < n$.

subcase 2 The direction between v_2 and v_3 is $v_3 \to v_2$, then we have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 4$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 2$, where $k \in \mathbb{N}$ and $i < n$.

Case 2 If the direction between v_1 and v_2 is $v_2 \to v_1$, then we must have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 3$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 1$, where $k \in \mathbb{N}$ and $i < n$.

Then we have two subcases:

subcase 1 The direction between v_2 and v_3 is $v_3 \to v_2$, then we have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 4$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 2$, where $k \in \mathbb{N}$ and $i < n$.

subcase 2 The direction between v_2 and v_3 is $v_2 \to v_3$, then we have the following directions:

$v_i \to v_{i+1}$ for all $i = 4k + 2$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k + 4$, where $k \in \mathbb{N}$ and $i < n$.

□

Theorem 3.3 told us that there are only four oppositional orientations D_1, D_2, $D_3$and D_4 for a path. We can choose any one of these four oppositional orientations to give an orientation for a path.

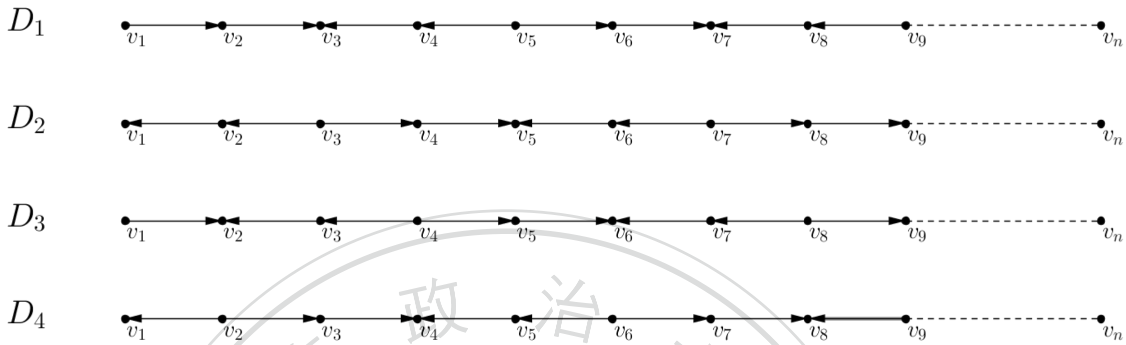

Figure 9: the orientation of P_n.

3.2 There Are Only One Vertex u in R

If there is only one vertex u in $R(T)$, then T must be the tree shown as Figure 10, we call it *sunshine graph*. We will discuss whether T is an opposition graph.

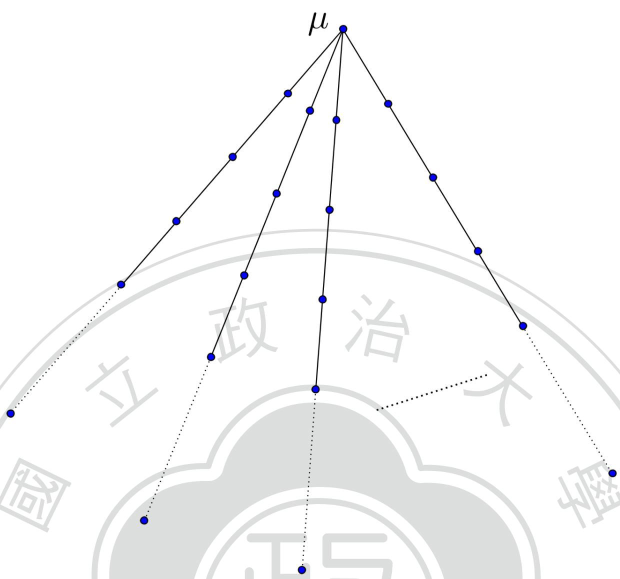

Figure 10: Sunshine graph.

Theorem 3.4. *If T is a sunshine graph, then T is an opposition graph.*

Proof. Let $u \in R(T)$ be the root of T. We can give an orientation for the edges of T as follows :

Level $i \to$ level $i+1$ for all $i = 4k$, $4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \to$ level i for all $i = 4k+2$, $4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Then T is an opposition graph shown as Figure 11. □

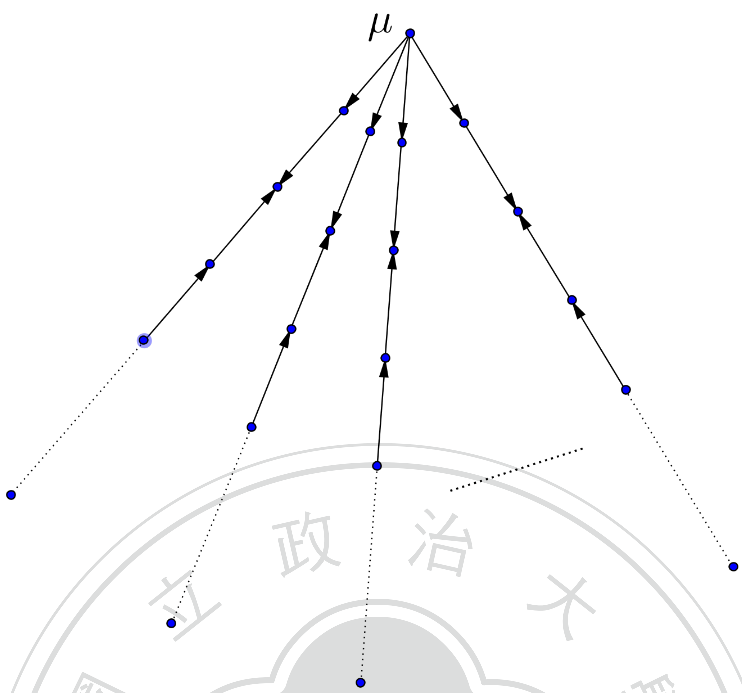

Figure 11: Sunshine graph is an opposition graph.

Theorem 3.5. *For a sunshine graph T. Let u be the root of T. If there are at least two vertices in level 2, then there are only two oppositional orientations for a sunshine graph T.*

Proof. Let T be a sunshine graph. Let $u \in R(T)$ be the root of the tree T. There are n paths from u to leaves Q_1, Q_2,...,Q_n. By Theorem 3.3, there are only four oppositional orientations for a path.

case 1 If the orientation of Q_1 is D_1, then the orientation of Q_2,...,Q_n must be D_1. Hence, the orientation of T is level $i \to$ level $i+1$ for all $i = 4k$, $4k+1$ and level $i+1 \to$ level i for all $i = 4k+2$, $4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

case 2 If the orientation of Q_1 is D_2, then the orientation of Q_2,...,Q_n must be D_2. Hence, the orientation of T is level $i+1 \to$ level i for all $i = 4k$, $4k+1$, and level $i \rightarrow$ level $i+1$ for all $i = 4k+2,\ 4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Suppose the vertices of level 1 in Q_1, Q_2, Q_3 are v_{11}, v_{12}, v_{13}, and suppose the vertices of level 2 in Q_1, Q_2, are v_{21}, v_{22}.

case 3 If the orientation of Q_1 is D_3, then the directions of T must be $v_{12} \rightarrow u$, $v_{12} \rightarrow v_{22}$, $v_{13} \rightarrow u$. Hence, the orientation of the path $v_{13}uv_{12}v_{22}$ gives us a contradiction.

case 4 If the orientation of Q_1 is D_4, then the directions of T must be $u \rightarrow v_{12}$, $v_{22} \rightarrow v_{12}$, $u \rightarrow v_{13}$. Hence, the orientation of the path $v_{13}uv_{12}v_{22}$ gives us a contradiction.

So there are only two oppositional orientations for a sunshine graph T. □

By Theorem 3.5, we can give another orientation of edges of T as follows :

Level $i \leftarrow$ level $i+1$ for all $i = 4k,\ 4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \leftarrow$ level i for all $i = 4k+2,\ 4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Then T is an opposition graph shown as Figure 12.

Corollary 3.6. *For a sunshine graph T. Let u be the root of T. If there are at least two vertices in level 2, then the orientation of T must be given as follows:*

Level $i \rightarrow$ level $i+1$ for all $i = 4k,\ 4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \rightarrow$ level i for all $i = 4k+2,\ 4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Proof. By Theorem 3.5, there are two orientations for T, these two orientations are symmetric, so we can use case1 to give the orientation for T. □

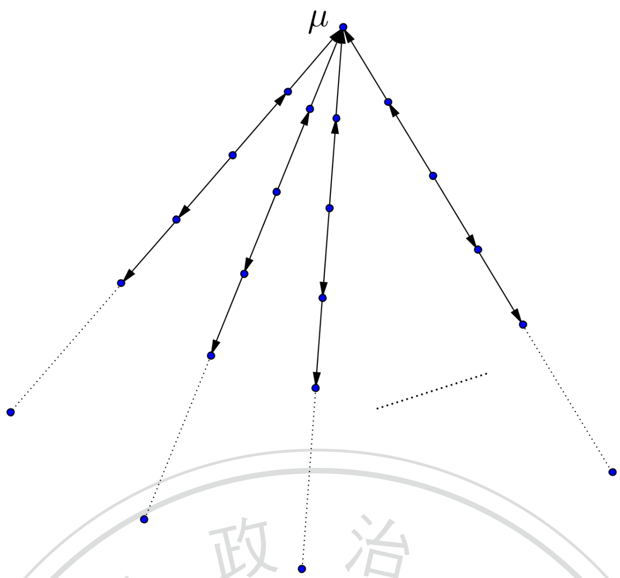

Figure 12: An sunshine graph is an opposition graph.

Theorem 3.7. *For a tree T. Let u be the root of T. If there are at least two vertices in level two and T is opposition, then the orientation of T must be given as follows:*

Level $i \to$ level $i+1$ for all $i = 4k$, $4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \to$ level i for all $i = 4k+2$, $4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Proof. Let T be a tree. Suppose $R(T) = \{u, u_1, u_2 ... u_n\}$. There is a maximal subtree T_1 containing u which is a sunshine graph. Then T can be decomposed into T_1 and some paths Q_1, Q_2, ..., Q_k with one of endpoints in $R(T)$.
Because T_1 is a sunshine graph, the orientation is given by Corollary 3.6. Now we add all paths Q_i into T_1. Suppose u_j is an endpoint of Q_i. Then uu_j union Q_i is a path, the orientation of this path is given by case 1 of Theorem 3.3. Hence, the orientation of T must be given as follows:

Level $i \to$ level $i+1$ for all $i = 4k$, $4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \to$ level i for all $i = 4k+2$, $4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

□

Now, by Theorem 3.7, when we want to determine if a tree T is an opposition graph, we can give the orientation by only one way: Let $u \in R(T)$ be the root. Level $i \to$ level $i+1$ for all $i = 4k$, $4k+1$ and level $i+1 \to$ level i for all $i = 4k+2$, $4k+3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T. When the orientation is given as above, if some induced P_4 doesn't satisfy the definition of opposition graphs, then T is not an opposition graph.

3.3 There Are Two Vertices u,v in $R(T)$

If there are exactly two vertices u and v in $R(T)$, then T must be the tree shown as Figure 13, we call it *wing graph.* We will discuss whether T is an opposition graph.

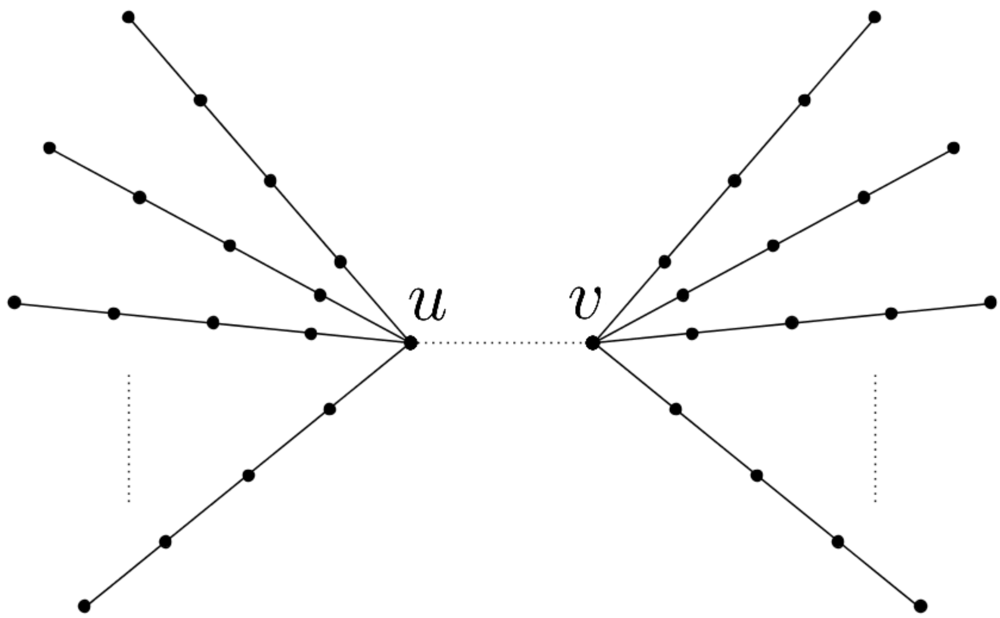

Figure 13: T is a wing graph.

Now, if we delete all the vertices between u and v, then we can get two subtrees containing u and v, we call them T_1 and T_2. Observably, the degrees of u and v are greater than or equal to 2. The trees T_1 and T_2 are paths or sunshine graphs because the degrees of every vertices are less than 3 except u and v.

Theorem 3.8. *Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If at least one of T_1 and T_2 does not contain P_4, then T is an opposition graph.*

Proof. Suppose T_2 does not contain P_4 and v is in T_2. Let u be the root of the tree T. We can give an orientation of edges of T as follows :

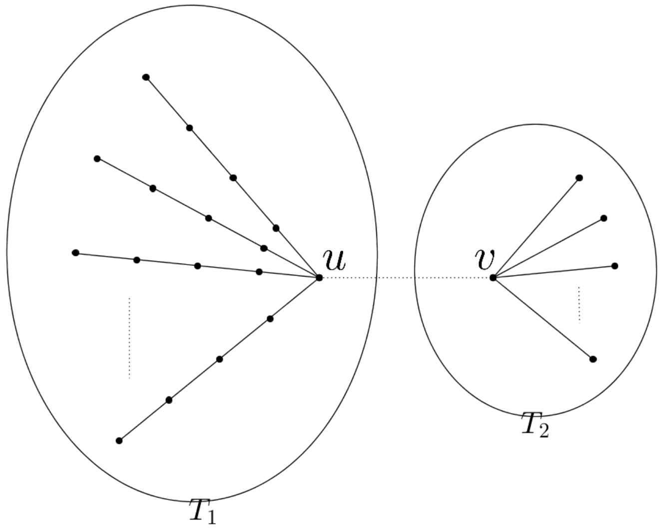

Figure 14: The graph of Theorem 3.8.

Level $i \to$ level $i+1$ for all $i = 4k, 4k+1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Level $i+1 \to$ level i for all $i = 4k+2, 4k+3$, $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Then T is an opposition graph shown as Figure 15. □

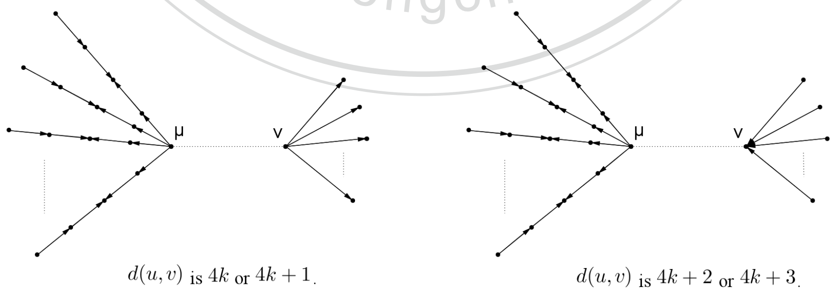

Figure 15: The orientation of Theorem 3.8.

Theorem 3.9. *Let T be a tree and $v \in R(T)$. There are n paths Q_1, Q_2,...,Q_n with endpoint v. Let $v_{11} \in Q_1$, $v_{12} \in Q_2$,...,$v_{1n} \in Q_n$ be the vertices whose distance from v is 1. Let $v_{21} \in Q_1$, $v_{22} \in Q_2$,...,$v_{2n} \in Q_n$ be some vertices whose distance from v is 2. If T is an opposition graph, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ must be as follows:*

Case 1 *The directions are $v \to v_{1i}$ for all $i = 1, ..., n$ and $v_{1i} \to v_{2i}$ for all $i = 1, ..., n$.*

Case 2 *The directions are $v_{1i} \to v$ for all $i = 1, ..., n$ and $v_{2i} \to v_{1i}$ for all $i = 1, ..., n$.*

Proof. T is a tree. Let $u \in R(T)$ be the root of T. Suppose the path Q_1 is between u and v. By Theorem 3.7, we give an orientation for T, there are two cases in the edge between v_{11} and v_{21}:

Case 1 If we give the direction $v_{11} \to v_{21}$, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ is $v \to v_{1i}$ for all $i = 2, ..., n$, $v_{1i} \to v_{2i}$ for some $i = 2, ..., n$, and $v \to v_{11}$.

Case 2 If we give the direction $v_{21} \to v_{11}$, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ is $v_{1i} \to v$ for all $i = 2, ..., n$, $v_{2i} \to v_{1i}$ for some $i = 2, ..., n$, and $v_{11} \to v$.

So there are only two cases for the directions of the edges uv_{1i} and $v_{1i}v_{2i}$. □

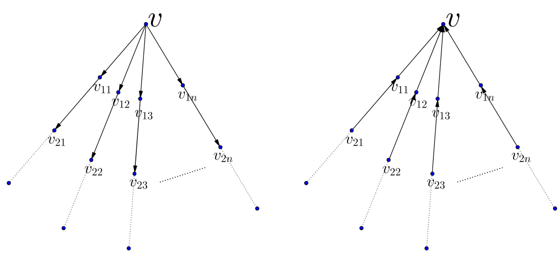

Figure 16: The orientation of Theorem 3.9.

Theorem 3.9 can give us a way to determine if T is an opposition graph. For a tree T, by Theorem 3.7, we can give an orientation, then the orientation of every vertex u in $R(T)$ must satisfy Theorem 3.9. If the orientation of any vertex u in $R(T)$ doesn't satisfy Theorem 3.9, then T is not an opposition graph.

Then we will discuss that both T_1 and T_2 contain P_4. We have the following two cases :

Case 1 If dist(u,v) is odd.

Case 2 If dist(u,v) is even.

Theorem 3.10. *Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If both T_1 and T_2 contain P_4 and dist(u,v) is odd, then T is not an opposition graph.*

Proof. Suppose u is in T_1 and v is in T_2. Let u be the root of the tree T. We can give an orientation of edges of T by Corollary 3.7. Then the orientation of T is shown as Figure 17. The orientation of T_2 doesn't satisfy Theorem 3.9, so T is not an opposition graph. □

Theorem 3.11. *Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If both T_1 and T_2 contain P_4 and dist(u,v) is even, then T is an opposition graph.*

Proof. Let u be the root of the tree T. We can give an orientation of edges of T by Corollary 3.7. Then the orientation of T is shown as Figure 18, so T is an opposition graph. □

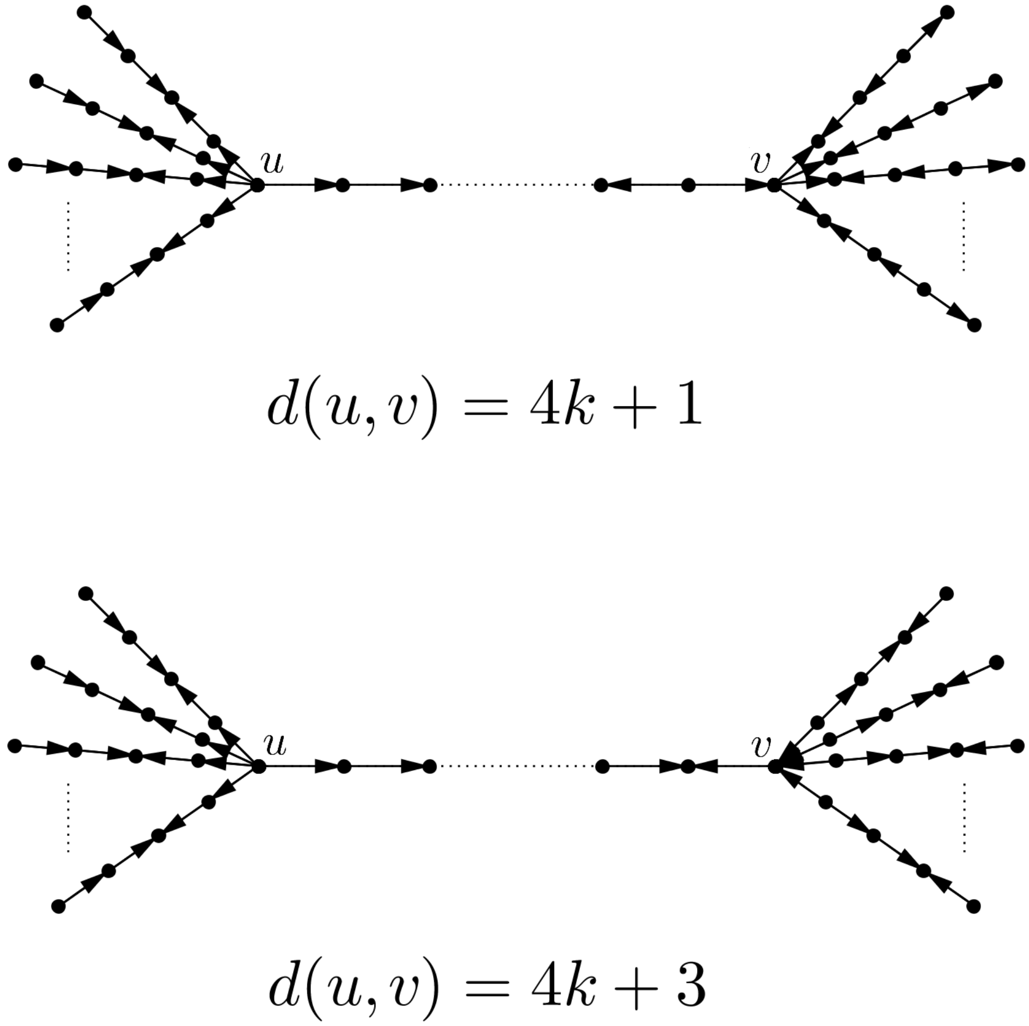

Figure 17: The orientation of Theorem 3.10.

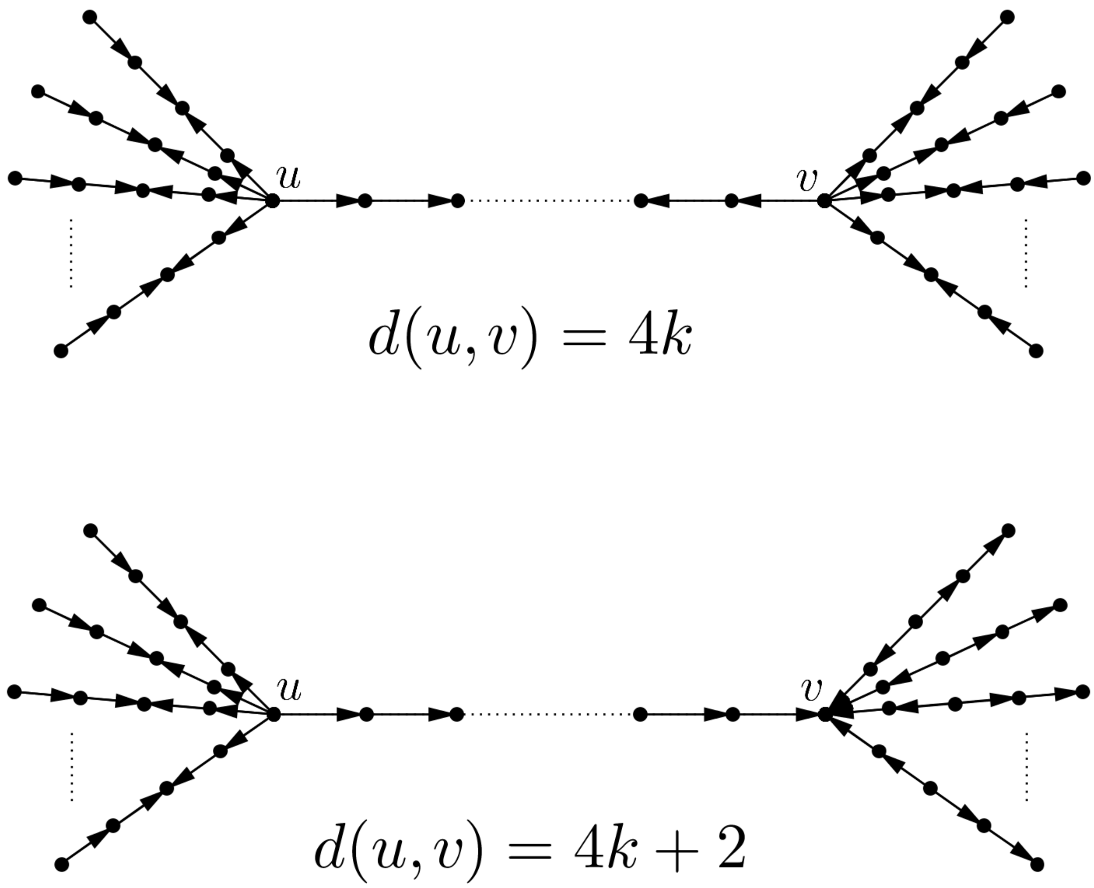

Figure 18: The orientation of Theorem 3.11.

3.4 There Are More Than Two Vertices in R

Theorem 3.12. *Let T be a tree. Let $R(T) = \{v_1, v_2, ..., v_n\}$ be the set of vertices in T whose degree is greater than or equal to 3. If $d(v_i, v_{i+1})$ is even for all $i = 1, ..., n$, then T is an opposition graph.*

Proof. We use the induction on $R(T)$ to prove the statement. Let T be a tree and $R(T) = \{v_1, v_2 ... v_n\}$ be the set of vertices in T which degree is greater than or equal to 3.

Basic step Suppose n=2. By Theorem 3.11, T is an opposition graph.

Induction step Suppose $n > 2$. Let v_1 be the root of the tree T. Suppose $dist(v(i, v_1) \leq dist(v_j, v_1)$ for all $i < j$. Let T_n be the subtree of T whose vertex set $V(T_n)$ are v_n and all of its descendant. Let T' be the subtree of T whose vertex set $V(T')$ are $\{v_n\} \cup V(T) - V(T_n)$. Now, $|R(T')| = n - 1$, so T' is an opposition graph by induction hypothesis.

Let v_1 be the root of T'. We can give an orientation to T':

Level $i \to$ level $i + 1$ for all $i = 4k$, $4k + 1$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T'.

Level $i + 1 \to$ level i for all $i = 4k + 2$, $4k + 3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T'.

Then we give the orientation for T_n and add T_n to T'. Let v_n be the root of T_n. There are two cases in T_n:

case 1 If $d(v_1, v_n) = 4k$, then level $i \to$ level $i + 1$ for all $i = 4k$, $4k + 1$ and level $i + 1 \to$ level i for all $i = 4k + 2$, $4k + 3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

case 2 If $d(v_1, v_n) = 4k + 2$, then level $i + 1 \to$ level i for all $i = 4k$, $4k + 1$ and level $i \to$ level $i + 1$ for all $i = 4k + 2$, $4k + 3$, where $k \in \mathbb{N}$ and $i < l$, l is the height of T.

Hence, T is an opposition graph for $n > 2$.

□

Definition 3.13. Let the path $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$ be two P_4. We add an odd path between u_2 and v_2, the graph is called *H graph* shown as Figure 19.

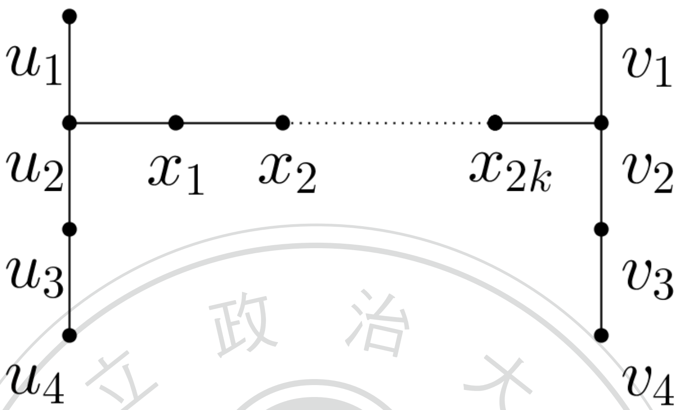

Figure 19: H graph.

Theorem 3.14. *If T be an H graph, then T is a minimal obstruction for the class of opposition graphs.*

Proof. If we remove u_1, then there is only one vertex v_2 which degree is greater than or equal to 3, by Theorem 3.4, T is an opposition graph. If we remove u_4, the path $u_1u_2u_3$ is a P_3, then by Theorem 3.8, T is an opposition graph. Similar for the vertices v_1 and v_4. □

4 Some Families of Opposition Graphs

Theorem 4.1. *If P is an induced P_4 in G, then $\overline{P}$ is an induced P_4 in $\overline{G}$.*

Proof. If the path $abcd$ is an induced P_4 in G, then $cadb$ is an induced P_4 in $\overline{G}$ □

Corollary 4.2. *$\overline{P}$ is an induced P_4 in $\overline{G}$ if and only if P is an induced P_4 in G.*

Proof. P is an induced P_4 in G, by Theorem 4.1, $\overline{P}$ is an induced P_4 in $\overline{G}$. $\overline{P}$ is an induced P_4 in $\overline{G}$, by Lemma 4.1, P is an induced P_4 in G □

If the path $v_i v_{i+1} v_{i+2} v_{i+3}$ is an induced P_4 in P_n, then the path $v_{i+2} v_i v_{i+3} v_{i+1}$ is an induced P_4 in $\overline{P}_n$

Figure 20:

Theorem 4.3. *$\overline{P_n}$ is an opposition graph.*

Proof. Let P_n be v_1 , v_2 , v_3,...,v_n. We give an orientation for $\overline{P_n}$ as following :

$v_{k+2} \to v_k$ for all k is even.

$v_i \to v_j$ for all $i < j$.

Then we can check the orientation for all induced P_4 in $\overline{P_n}$:

Case 1 If n $<$ 4, then $\overline{P_n}$ has no P_4, so $\overline{P_n}$ is an opposition graph.

Case 2 If n $\geq$ 4, by Corollary 4.2, $\overline{P}$ is a P_4 in $\overline{P_n}$ if and only if P is a P_4 in P_n. Suppose the path $v_i v_{i+1} v_{i+2} v_{i+3}$ is an induced P_4 in P_n, then the path $v_{i+2} v_i v_{i+3} v_{i+1}$ is an induced P_4 in $\overline{P}_n$, the orientation is as follows :

Figure 21:

If i is odd, then the orientation is $v_{i+2}\leftarrow v_i \rightarrow v_{i+3} \rightarrow v_{i+1}$, shown as Figure 21 (a).

If i is even, then the orientation is $v_{i+2}\rightarrow v_i \rightarrow v_{i+3} \leftarrow v_{i+1}$, shown as Figure 21 (b).

So $\overline{P_n}$ is an opposition graph.

□

Theorem 4.4. *$\overline{T_2}$ is not an opposition graph.*

Proof. T_2 is expressed in Figure 22, there are six P_4 in T_2 : $a_2a_1ob_1$, $a_2a_1oc_1$, $b_2b_1oa_1$, $b_2b_1oc_1$, $c_2c_1oa_1$, $c_2c_1ob_1$. By Theorem 4.2, there are six P_4 in $\overline{T}_2$: $a_1b_1a_2o$, $a_1c_1a_2o$, $b_1a_1b_2o$, $b_1c_1b_2o$, $c_1a_1c_2o$, $c_1b_1c_2o$. We can suppose the direction of the edge a_1b_1 is $a_1 \rightarrow b_1$, then we have the following direction : $o \rightarrow a_2$, $a_1 \rightarrow c_1$, $c_2 \rightarrow o$, $b_1 \rightarrow c_1$, $b_2 \rightarrow o$, then the P_4 $b_1a_1b_2o$ gives us a contradictory. Similar for the direction $b_1 \rightarrow a_1$. So $\overline{T_2}$ is not an opposition graph. □

Corollary 4.5. *$\overline{T_n}$ is not an opposition graph for all $n > 1$.*

Proof. Because $T_i \subseteq T_j$ for all $i < j$, so $\overline{T}_i \subseteq \overline{T}_j$ for all $i < j$. By Theorem 4.4, $\overline{T_2}$ is not an opposition graph, so $\overline{T_n}$ is not an opposition graph for all $n > 1$. □

Theorem 4.6. *The graphs C_n is an opposition graph if and only if $n = 4k$ or $n = 3$.*

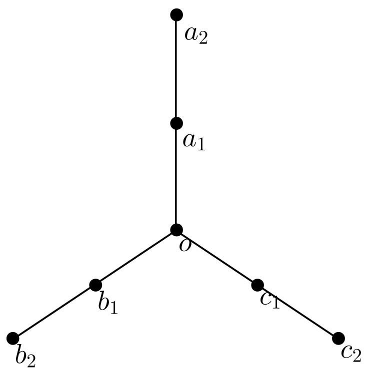

Figure 22: T_2

Proof. The graphs C_3 and C_4 don't have an induced P_4, so C_3 and C_4 are opposition graphs. Let $v_1, v_2, v_3, ..., v_n$ be the vertices of C_n. Deleting the edge v_nv_1, the graph is a path P_n. By Theorem 3.3, we can give an orientation as follows :

$v_i \to v_{i+1}$ for all $i = 4k,\ 4k+1$, where $k \in \mathbb{N}$ and $i < n$.

$v_{i+1} \to v_i$ for all $i = 4k+2,\ 4k+3$, where $k \in \mathbb{N}$ and $i < n$.

case 1 If $n = 4k$ where $k \in \mathbb{N}$, then the orientation of the edge v_nv_1 is $v_1 \to v_n$. Hence, C_n is an opposition graph.

case 2 If $n = 4k+1$ where $k \in \mathbb{N}$, then the path $v_{4k-1}v_{4k}v_{4k+1}v_1$ is an induced P_4, the orientation of the edge $v_{4k+1}v_1$ is $v_{4k+1} \to v_1$. The induced P_4 $v_{4k+1}v_1v_2v_3$ gives us a contradictory, so C_n is not an opposition graph.

case 3 If $n = 4k+2$ where $k \in \mathbb{N}$, then the path $v_{4k}v_{4k+1}v_{4k+2}v_1$ is an induced P_4, the orientation of the edge $v_{4k+2}v_1$ is $v_{4k+2} \to v_1$. The induced P_4 $v_{4k+2}v_1v_2v_3$ gives us a contradictory, so C_n is not an opposition graph.

case 4 If $n = 4k+3$ where $k \in \mathbb{N}$, then the path $v_{4k+1}v_{4k+2}v_{4k+3}v_1$ is an induced P_4, the orientation of the edge $v_{4k+3}v_1$ is $v_1 \to v_{4k+3}$. The induced P_4 $v_{4k+2}v_{4k+3}v_1v_2$ gives us a contradictory, so C_n is not an opposition graph.

Hence, C_n is an opposition graph if and only if $n = 4k$ or $n = 3$. $\square$

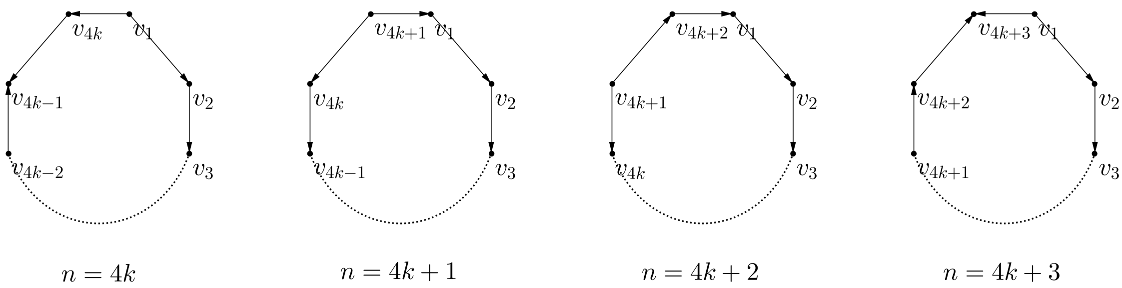

Figure 23:

5 Open Problems and Further Directions of Studies

In this article, we prove some graphs are opposition graphs. In a tree T, R is the set of vertices with degree greater than or equal to 3, if every distance of any two vertices is even, then T is an opposition graph. In a cycle C_n, if $n = 4k$ for all k is integer, then C_n is an opposition graph. There are still some open problems for future studies:

1. In Chapter 3, we have known some classes of trees are opposition graphs. Furthermore,

  a. We would like to find out the necessary and sufficient conditions of trees being opposition graphs.

  b. We would like to find out the necessary and sufficient conditions of any graph being an opposition graph.

2. In the Figure 1, we know that P_4 is an opposition graph but not a threshold graph; C_6 is a perfect graph but not an opposition graph. Furthermore,

  a. We would like to find out the relation between opposition graphs and perfect graphs.

  b. We would like to find out the relation between opposition graphs and the other graphs.

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