國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

A Classification Scheme for Nonoscillatory Solutions of Two-Dimensional Nonlinear Dynamical Systems

二維非線性動態系統之非振盪解的分類法

碩士班學生: 黃雅雯 撰

指導教授: 符聖珍 博士

中華民國一百零一年十一月二十九日

謝辭

在政大好多年，時間過得好快，我終於即將要畢業了。想想剛進研究所時，真的是一個一心只是想著玩想著打工賺錢的學生，就這個樣子混混噩噩浪費了一年的時間。

首先，最要感謝我的指導教授符聖診老師，在我碩士一年級學期末結束前，提醒了我，我當學生的首要任務就是應該好好的把書念好，其餘的事應該都是排在念書之後，讓我驚覺應該好好讀書。由於，自己的數學底子及英文並不是很好，因此非常感謝符老師在這兩、三年的細心及耐心指導，並在課業及課業之外，提供了許許多多的意見，讓我在求學之路上所遇到的困難都能一一解決。並在寫論文時期，提供我很多方向、教導我如何抒寫一篇好的文章及遇到問題時的解決方式。真的很感謝符老師在如此忙碌的時間裡還能耐心的指導我。

很感謝陳天進老師，經過陳老師的實變課程訓練，雖然很累，但確實是一門很札實的課程更從中學習到，上一門數學課應該如何上課做筆記及應該如何思考數學，並常常在課後時，關心我的論文及念書進度。陳老師真的是一位亦師亦友的好老師。

另外，很感謝張宜武老師、蔡炎龍老師及余屹正老師平日的鼓勵，也謝謝蔡老師的關心，以及余老師在LaTex編寫上的教導。也很謝謝兩位美麗的助教曾琬婷、李偉慈在校務行政方面給予很大的幫助。謝謝在政大這麼多年，給予我很多幫助的老師及助教們。

================同學們========================

在政大研究室的這幾年，認識好棒的好同學。一起遊玩、一起歡笑的日子想起來真的很快樂。謝謝靜儒、丞偉、博翔、治陞及盈穎，在我的學生生涯裡給了我許許多多的幫助，並且我們一起走過考實變前一星期在研究室像壓力鍋要爆炸的感覺，仔細想想很有趣。好在我們也都一起走過來了。

謝謝宥柔、裕哲、增堂、偉哲及沛承這幾位學弟、妹總是能夠在研究

室帶來無限的歡笑。讓緊張的情緒總是能輕鬆不少。我想離開學校之後，一定也還很想念和你們一起玩樂、讀書的日子。謝謝大澤佑在我論文的最後階段時，適時的提供了很多的幫助，不論是tex或者是分析上的問題。

================家人、朋友===========================

回想求學這幾年，眞的很感謝我的父、母親讓我能夠毫無經濟上的煩惱供我讀書，專心做一個學生，只是很可惜，我自己有好一些年並不夠珍惜這些父母的苦心。對於我的父母親我的太多感謝之意無法用言語好好表現，眞的深深感謝他們，我想我的畢業就是給予他們的最好的禮物，也好讓他終於放下心中的一塊大石。再一次感謝父母親這些年的培育。

很感謝我的姐妹淘們，雖然你們早已踏入社會，但，你們仍願意聽我在學校發生的任何事情，當我的垃圾桶。聽我分享我所有的喜怒哀樂。

在研究室的這段時間，非常感謝，我的好友海豹，你幫了我許多的忙，在論文英文上做生活鎖事，總是能在背後提供我好的意見及方向，對你有千言萬語感謝，因爲有你，總是讓我在我猶豫不決之時，你總是能夠給予我最棒的意見及想法，讓我不會迷失方向。

謝謝我的微積分啓蒙老師，吳老師及師母；讓我的微積分基礎穩得多及生活上許多的照顧。謝謝我的男朋友在我情緒不是很好的時候，總是給予我很多的鼓勵以及默默地讓我發脾氣、拿衛生紙幫我擦乾眼淚。謝謝你這三年的陪伴。在我壓力很大的時候，帶我出去走走散散心，到處吃吃喝喝。

眞的非常感謝，我這求學路上，一切給予我幫助的所有人。這一切我都會好好放在心上，再一次感激大家。

此篇論文謹獻給我親愛的家人、師長和朋友們。

黃雅雯　謹誌于

國立政治大學應用數學系

中華民國一百零一年九月

Abstract

In this thesis, we provide a classification scheme for nonoscillatory solutions of a class of two-dimensional dynamical systems in terms of their asymptotic values. In addition, we find the sufficient and necessary conditions for the existence of these solutions.

中文摘要

在此篇論文中，我們提供二維非線性動態系統之非振盪解的一個分類法，此分類法是依據解的漸近值作分類，同時我們也得到具有此漸近值之非振盪解的存在性的充分必要條件。

Contents

1 Introduction

The study of dynamic equations on time scales has received a lot of attentions since it not only can unify the calculation of difference and differential equations but also has various applications. Recently, some researchers have focused on the establishment of the oscillation and nonoscillation criteria for two-dimensional dynamic systems; see, for example, [2], [3], [6].

Based on the above works, in this thesis, we will provide classification schemes for nonosicllatory solutions of the two-dimensional nonlinear dynamic system

$$x^{\Delta}(t) = p(t)f(y(t)), \forall t \in \mathbb{T}, \tag{1.1a}$$

$$y^{\Delta}(t) = -q(t)g(x(t)), \forall t \in \mathbb{T}, \tag{1.1b}$$

where $\mathbb{T}$ is an arbitrary time scale (i.e., nonempty closed subset of $\mathbb{R}$) which is unbounded above. Here we assume that $p, q : \mathbb{T} \to \mathbb{R}$ are right-dense continuous with $p > 0$ and $q \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$, and q is not eventually zero. Moreover, we assume that $f, g : \mathbb{T} \to \mathbb{R}$ are nondecreasing continuous functions satisfying

$$sf(s) > 0, \; sg(s) > 0, \; \text{for } s \neq 0. \tag{1.2}$$

This thesis is organized as follows. Firstly, we review some basic definitions and theorems on the theory of time scales in Section 2. Then, in Section 3, we present several useful lemmas. Finally, we provide classification schemes for nonoscillatory solutions of (1.1) in Section 4.

2 The Fundamental Theory of Time Scales

For completeness, we state some fundamental definitions and results concerning dynamic equations on time scales that will be used in the sequel. More details can be found in [1]. Throughout this thesis, we assume that $t_0, t_1 \in \mathbb{T}$ and $t_0 < t_1$. For convenience, we define the time-scale interval $[t_0, t_1]_{\mathbb{T}} := \{t \in \mathbb{T} : t_0 \leq t \leq t_1\}$.

Definition 2.1 *Let* $\mathbb{T}$ *be a time scale. For* $t \in \mathbb{T}$*, we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$*, then* t *is called right-dense, and if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$*, then* t *is called left-dense. If* $\sigma(t) > t$*, we say that* t *is right-scattered, while if* $\rho(t) < t$ *we say that* t *is left-scattered.*

Definition 2.2 *We define the set* $\mathbb{T}^\kappa$ *which is derived from the time scale* $\mathbb{T}$ *as follows: If* $\mathbb{T}$ *has a left-scattered maximum* m*, then* $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$*. Otherwise,* $\mathbb{T}^\kappa = \mathbb{T}$*. In summary,*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \diagdown (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \;\; \sup \mathbb{T} < \infty \\ \mathbb{T} & if \;\; \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.3 *Assume that* $h : \mathbb{T} \to \mathbb{R}$ *is a function and* $t \in \mathbb{T}^\kappa$*, then we define* $h^\Delta(t)$ *to be a number (provided it exists) with the property that given any* $\varepsilon > 0$*, there exists a neighborhood* U *of* t *(i.e.* $U = (t - \delta, t + \delta)_{\mathbb{T}}$ *for some* $\delta > 0$*), such that*

$$|[h(\sigma(t)) - h(s)] - h^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, \textit{ for all } s \in U.$$

We call $h^{\Delta}(t)$ the delta (or Hilger) derivative of h at t.

Moreover, we say that h is delta (or Hilger) differentiable on $\mathbb{T}^{\kappa}$ provided $h^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $h^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called the (delta) derivative of h on $\mathbb{T}^{\kappa}$.

Theorem 2.4 *Assume $h : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$, then we have the following:*

(i) If h is differentiable at t, then h is continuous at t.

(ii) If h is continuous and t is right-scattered, then h is differentiable at t with

$$h^{\Delta}(t) = \frac{h(\sigma(t)) - h(t)}{\mu(t)}.$$

(iii) If t is right-dense, then h is differentiable at t if and only if

$$\lim_{s \to t} \frac{h(t) - h(s)}{t - s}$$

exists. In this case,

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s}.$$

(iv) If h is differentiable at t, then $h(\sigma(t)) = h(t) + \mu(t)h^{\Delta}(t)$, where $\mu(t) = \sigma(t) - t$ is the forward graininess function.

Definition 2.5 *A function $h : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of all right-dense continuous functions from $\mathbb{T}$ to $\mathbb{R}$ will be denote by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.6 *A function $H : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of a function $h : \mathbb{T} \to \mathbb{R}$ if $H^{\Delta}(t) = h(t)$, and we define $\int_r^s h(t)\Delta t = H(s) - H(r)$. Note that every right-dense continuous function has an antiderivative.*

Theorem 2.7 *If $h \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} h(r)\Delta r = \mu(t)h(t).$$

Theorem 2.8 *If $h \in C_{rd}$ and $h^{\Delta} \geq 0$, then h is nondecreasing.*

Theorem 2.9 *If $h(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b h(t)\Delta t \geq 0$.*

Theorem 2.10 *Let $a,\ b \in \mathbb{T}$ and $h, h^* \in C_{rd}$. If $|h(t)| \leq h^*(t)$ on $[a, b)$, then*

$$\left|\int_a^b h(t)\Delta t\right| \leq \int_a^b h^*(t)\Delta t.$$

Definition 2.11 *If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and h is right-dense continuous on $[a, \infty)$, then we define the improper integral by*

$$\int_a^{\infty} h(t)\Delta t := \lim_{b \to \infty} \int_a^b h(t)\Delta t. \tag{2.1}$$

If the limit on the right-hand side of (2.1) exists, then we say that the improper integral $\int_a^{\infty} h(t)\Delta t$ converges. Otherwise, we say that it diverges.

3 Preparatory Lemmas

In this section, we will introduce some useful lemmas for proving the main theorems.

Definition 3.1

(i) A solution $(x(t), y(t))$ of (1.1) is said to be oscillatory if both component functions $x(t)$ and $y(t)$ are oscillatory (i.e., neither eventually positive nor eventually negative); otherwise it is called nonoscillatory.

(ii) We say that the nonlinear system (1.1) is oscillatory if all its solutions are oscillatory.

Lemma 3.2 *The component functions $x(t)$ and $y(t)$ of a nonoscillatory solution $(x(t), y(t))$ of (1.1) are nonoscillatory.*

Proof. For contradiction, we assume that $x(t)$ is oscillatory but $y(t)$ is eventually positive. Then, we have $x^{\Delta}(t) = p(t)f(y(t)) > 0$ eventually, which holds according to $p(t) > 0$ and (1.2). Hence $x(t) > 0$ or $x(t) < 0$ for all large $t \in \mathbb{T}$, which is a contradiction. The proof for the other case where $y(t)$ is eventually negative is similar. Likewise, we assume that $y(t)$ is oscillatory while $x(t)$ is eventually positive or eventually negative leads to comparable contradictions. ❑

Lemma 3.3 *Suppose that $\int_{t_0}^{\infty} p(r)\Delta r < \infty$ and $(x(t), y(t))$ is a nonoscillatory solution of (1.1), then $\lim_{t\to\infty} x(t)$ exists.*

Proof. Applying Lemma 3.2, we know that $x(t)$ is nonoscillatory. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$. Using (1.2) and assumption $q \geq 0$, we deduce from (1.1b) that $y^{\Delta}(t) = -q(t)g(x(t)) \leq 0$ on $[t_0, \infty)_{\mathbb{T}}$, which implies that $y(t)$ is nonincreasing for $t \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 3.2, we know that $y(t)$ is nonoscillatory, which implies that $y(t)$ is eventually one sign. Hence, there exists $t_1 \geq t_0$ such that

$$y(t) < 0 \text{ on } [t_1, \infty)_{\mathbb{T}} \tag{3.1}$$

or

$$y(t) > 0 \text{ on } [t_1, \infty)_{\mathbb{T}}. \tag{3.2}$$

Suppose that (3.1) holds. Using (3.1) and (1.2), we have

$$f(y(t)) < 0 \text{ on } [t_1, \infty)_{\mathbb{T}}, \tag{3.3}$$

which, together with the assumption $p > 0$, gives that

$$x^{\Delta}(t) = p(t)f(y(t)) \leq 0 \text{ on } [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

This implies $x(t)$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. Since $x(t)$ is nonincreasing and $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, it follows that $\lim_{t\to\infty} x(t)$ exists.

Suppose that (3.2) holds. Using (1.2), (3.2), and assumption $p > 0$, we get

$$x^{\Delta}(t) = p(t)f(y(t)) > 0 \text{ on } [t_1, \infty)_{\mathbb{T}}. \tag{3.5}$$

On the other hand, since $y(t)$ is nonincreasing, it follows that

$$0 < y(t) \leq y(t_1) \text{ on } [t_1, \infty)_{\mathbb{T}}. \tag{3.6}$$

Then, using (1.1a), (3.6) and the assumption that f is nondecreasing, we obtain

$$x^{\Delta}(t) = p(t)f(y(t)) \leq p(t)f(y(t_1)). \tag{3.7}$$

Integrating (3.7) from t_1 to t, we get

$$\begin{aligned} x(t) &\leq x(t_1) + f(y(t_1)) \int_{t_1}^{t} p(r)\Delta r \\ &\leq x(t_1) + f(y(t_1)) \int_{t_1}^{\infty} p(r)\Delta r \\ &< \infty, \end{aligned} \tag{3.8}$$

where we have used $f(y(t_1)) > 0$ and the assumption $\int_{t_1}^{\infty} p(r)\Delta r < \infty$. Therefore, using (3.5) and (3.8), we obtain $\lim_{t\to\infty} x(t)$ exists.

❑

Theorem 3.4 *If* $\int_{t_0}^{\infty} p(r)\Delta r = \infty$ *and* $\int_{t_0}^{\infty} q(r)\Delta r = \infty$*, then each solution of (1.1) is oscillatory.*

Proof. For contradiction, we assume that $(x(t), y(t))$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ on $[t^*, \infty)_{\mathbb{T}}$, for some $t^* \geq t_0$. Then, using (1.2) and the assumption that $q \geq 0$, we have

$$y^{\Delta}(t) = -q(t)g(x(t)) \leq 0, \ \forall t \in [t^*, \infty)_{\mathbb{T}},$$

which implies that y is nonincreasing on $[t^*, \infty)_{\mathbb{T}}$. By lemma 3.2, we know that $y(t)$ is nonoscillatory. This implies that there exists $t_1 \geq t^*$ such that

$$y(t) < 0 \text{ on } [t_1, \infty)_{\mathbb{T}} \tag{3.9}$$

or

$$y(t) > 0 \text{ on } [t_1, \infty)_{\mathbb{T}}. \tag{3.10}$$

Suppose that (3.9) holds. Since y is nonincreasing, it follows that $y(t) \leq y(t_1)$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then the monotonicity of f gives that

$$f(y(t)) \leq f(y(t_1)) < 0, \ \forall t \in [t, \infty)_{\mathbb{T}} \tag{3.11}$$

where we used (1.2). Together with the assumption that $p > 0$, we get

$$x^{\Delta}(t) = p(t)f(y(t)) < 0, \tag{3.12}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Integrating (3.12) from t_1 to t, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} p(s)f(y(s))\Delta s. \tag{3.13}$$

Taking limits on both sides of (3.13), using (3.11), and the assumption that $\int_{t_0}^{\infty} p(r)\Delta r = \infty$, we obtain that

$$\begin{aligned}\lim_{t\to\infty} x(t) &= x(t_1) + \int_{t_1}^{\infty} p(s)f(y(s))\Delta s \\ &\le x(t_1) + f(y(t_1)) \int_{t_1}^{\infty} p(s)\Delta s \\ &= -\infty,\end{aligned}$$

which contradicts the fact that $x(t) > 0$.

For the case (3.10), since $y(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, (1.2) implies that $f(y(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Together with the assumption that $p(t) > 0$, we deduce from (1.1a) that $x^{\Delta}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, which implies that $x(t)$ is increasing. Hence $x(t) > x(t_1) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Together with (1.2) and the fact that g is nondecreasing, we have

$$g(x(t)) \ge g(x(t_1)) > 0, \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.14}$$

Integrating (1.1b) from t_1 to t and rearranging the resulting equation, we obtain

$$\begin{aligned}y(t_1) &= y(t) + \int_{t_1}^{t} q(s)g(x(s))\Delta s \\ &\ge y(t) + g(x(t_1)) \int_{t_1}^{t} q(s)\Delta s \\ &\ge g(x(t_1)) \int_{t_1}^{t} q(s)\Delta s\end{aligned} \tag{3.15}$$

where we have used (3.14) and $y(t) > 0$. Taking limits on both sides of (3.15) and noting that $g(x(t_1)) > 0$, we get

$$g(x(t_1)) \int_{t_1}^{\infty} q(s)\Delta s \le y(t_1) < \infty,$$

which contradicts the assumption $\int_{t_1}^{\infty} q(s)\Delta s = \infty$. ❑

4 The Main Results

For $a, b \in [-\infty, \infty]$, we denote the collection of all nonoscillatory solution $(x(t), y(t))$ of (1.1) such that

$$\lim_{t\to\infty} x(t) = a \text{ and } \lim_{t\to\infty} y(t) = b$$

by $C^*(a, b)$.

Later, we will apply the Knaster's fixed-point theorem [4] to prove our main result. For readers' convenience, we state this theorem in the following.

Lemma 4.1 (Knaster's fixed-point theorem) *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum and supermum of M belong to X, as well as every nonempty subset of M has the infimum and supermum which belong to M. Let $T : M \to M$ be an increasing mapping. i.e., $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in M.*

4.1 The Case $\int_{t_0}^{\infty} p(r)\Delta r = \infty$

Theorem 4.2 *Suppose that*

$$\int_{t_0}^{\infty} p(r)\Delta r = \infty \text{ and } \int_{t_0}^{\infty} q(r)\Delta r < \infty.$$

Then (1.1) has a nonoscillatory solution $(x(t), y(t))$ which belongs to $C^*(\infty, m)$ for some $m > 0$ if and only if

$$\int_{t_0}^{\infty} q(s) g\left(c \int_{t_0}^{s} p(r)\Delta r\right)\Delta s < \infty,$$

for some $c > 0$.

Proof. Suppose $(x(t), y(t))$ is a nonoscillatory solution of (1.1) such that

$$\lim_{t\to\infty} x(t) = \infty \text{ and } \lim_{t\to\infty} y(t) = m > 0.$$

Then there exist a positive constant c_1 and $t_1 \geq t_0$ such that

$$x(t) \geq c_1 \text{ on } [t_1, \infty)_{\mathbb{T}}, \tag{4.1}$$

which, together with (1.2) and the assumption that f is nondecreasing, follows that

$$f(x(t)) \geq f(c_1) > 0, \ \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.2}$$

Integrating (1.1a) from t_1 to t and using (4.2), we have

$$x(t) = x(t_1) + \int_{t_1}^{t} p(s)f(x(s))\Delta s \geq x(t_1) + \int_{t_1}^{t} p(s)f(c_1)\Delta s, \ \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.3}$$

On the other hand, integrating the equation (1.1b) from t_1 to ∞, and using (4.2), (4.3), $y(\infty) = m$, and the assumption that g is nondecreasing, we obtain

$$\begin{aligned} y(t_1) - m &= \int_{t_1}^{\infty} q(s)g(x(s))\Delta s \\ &\geq \int_{t_1}^{\infty} q(s)g\left(x(t_1) + \int_{t_1}^{s} p(r)f(c_1)\Delta r\right)\Delta s \\ &\geq \int_{t_1}^{\infty} q(s)g\left(c\int_{t_1}^{s} p(r)\Delta r\right)\Delta s, \end{aligned}$$

where $c = f(c_1)$. Hence,

$$\int_{t_0}^{\infty} q(s)g\left(c\int_{t_0}^{s} p(r)\Delta r\right)\Delta s < \infty.$$

Conversely, suppose

$$\int_{t_0}^{\infty} q(s)g\left(c\int_{t_0}^{s} p(r)\Delta r\right)\Delta s < \infty, \tag{4.4}$$

for some $c > 0$. Pick $t_1 \geq t_0$ such that

$$\int_{t}^{\infty} q(s)g\left(c\int_{t_1}^{s} p(r)\Delta r\right)\Delta s < m^*, \ \forall t \geq t_1, \tag{4.5}$$

where $m^* = f^{-1}(c)/2$. Let $B = C_{rd}(\mathbb{T}, \mathbb{R})$ be a Banach space of all right-dense continuous functions on $\mathbb{T}$ with the norm

$$\|x\| = \sup_{t \geq t_1} |x(t)|$$

and the usual pointwise ordering "$\leq$". Define a subset Ω of B as follows:

$$\Omega = \{x \in B \mid f(m^*) \int_{t_1}^{t} p(r)\Delta r \leq x(t) \leq f(2m^*) \int_{t_1}^{t} p(r)\Delta r\,,\ \forall t \in [t_1, \infty)_{\mathbb{T}}\}.$$

It is easy to see that $\inf \Omega \in B$ and $\sup \Omega \in B$. Moreover, for any subset Q of Ω, we have $\inf Q \in \Omega$ and $\sup Q \in \Omega$. Define an operator $L : \Omega \to B$ by

$$(Lx)(t) = \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g(x(r))\Delta r\right)\Delta s,\ \forall t \in [t_1, \infty)_{\mathbb{T}}.$$

We claim that $L\Omega \subseteq \Omega$. Let $x \in \Omega$. Since f, g are nondecreasing and $p > 0$, $q \geq 0$, we have

$$\begin{aligned}(Lx)(t) &= \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g(x(r))\Delta r\right)\Delta s \\ &\geq \int_{t_1}^{t} p(s) f(m^*)\Delta s \\ &= f(m^*) \int_{t_1}^{t} p(s)\Delta s\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Here we have used (1.2). On the other hand, using (4.5), we obtain

$$\begin{aligned}(Lx)(t) &= \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g(x(r))\Delta r\right)\Delta s \\ &\leq \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g\left(f(2m^*) \int_{t_1}^{r} p(u)\Delta u\right)\Delta r\right)\Delta s \\ &= \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g\left(c \int_{t_1}^{r} p(u)\Delta u\right)\Delta r\right)\Delta s \\ &\leq \int_{t_1}^{t} p(s) f(2m^*)\Delta s \\ &= f(2m^*) \int_{t_1}^{t} p(s)\Delta s,\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Hence $L\Omega \subseteq \Omega$.

Furthermore, L is increasing since if $x, \widetilde{x} \in \Omega$ with $\widetilde{x} \geq x$, then

$$\begin{aligned}(Lx)(t) &= \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g(x(r))\Delta r\right)\Delta s \\ &\leq \int_{t_1}^{t} p(s) f\left(m^* + \int_{s}^{\infty} q(r) g(\widetilde{x}(r))\Delta r\right)\Delta s \\ &= (L\widetilde{x})(t).\end{aligned}$$

Here we have used the assumption that f and g are nondecreasing. By Lemme 4.1, we can conclude that there exists $\hat{x} \in \Omega$ such that $\hat{x} = L\hat{x}$.

Now we set

$$\hat{y}(t) = m^* + \int_t^\infty q(r)g(\hat{x}(r))\Delta r, \tag{4.6}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$, then we have

$$\hat{y}^\Delta(t) = -q(t)g(\hat{x}(t))$$

and

$$\hat{x}^\Delta(t) = (L\hat{x})^\Delta(t) = p(t)f\left(m^* + \int_t^\infty q(r)g(\hat{x}(r))\Delta r\right) = p(t)f(\hat{y}(t)).$$

Taking limits on both sides of the equation $\hat{x} = L\hat{x}$, we get

$$\begin{aligned}\lim_{t\to\infty} \hat{x}(t) &= \lim_{t\to\infty} \int_{t_1}^t p(r)f\left(m^* + \int_s^\infty q(r)g(\hat{x}(r))\Delta r\right)\Delta s \\ &= \int_{t_1}^\infty p(r)f\left(m^* + \int_s^\infty q(r)g(\hat{x}(r))\Delta r\right)\Delta s \\ &\geq f(m^*)\int_{t_1}^\infty p(r)\Delta r \\ &= \infty,\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$, where we have used the assumption $\int_{t_0}^\infty p(r)\Delta r = \infty$. Therefore, $\lim_{t\to\infty} \hat{x}(t) = \infty$. Since $\hat{x} \in \Omega$, it follows that

$$\hat{x}(t) \leq c\int_{t_1}^t p(r)\Delta r, \ \forall t \in [t_1, \infty)_{\mathbb{T}}.$$

Together with the monotonicity of g and the assumption (4.4), we get

$$\int_t^\infty q(r)g(\hat{x}(r))\Delta r \leq \int_t^\infty q(r)g\left(c\int_{t_1}^r p(s)\Delta s\right)\Delta r < \infty, \ \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.7}$$

Taking limits on both sides of (4.6) and using (4.7), we get

$$\lim_{t\to\infty} \hat{y}(t) = \lim_{t\to\infty}\left(m^* + \int_t^\infty q(r)g(\hat{x}(r))\Delta r\right) = m^*.$$

Hence $(\hat{x}(t), \hat{y}(t))$ is a nonoscillatory solution of (1.1) which belongs to $C^*(\infty, m^*)$.
❑

4.2 The Case $\int_{t_0}^{\infty} p(r)\Delta r < \infty$

Theorem 4.3 *(1.1) has a nonoscillatory solution* $(x(t), y(t))$ *which belongs to* $C^*(\ell, k)$ *for some* $0 < \ell, k < \infty$ *if and only if*

$$\int_{t_0}^{\infty} p(r)\Delta r < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} q(r)\Delta r < \infty. \tag{4.8}$$

Proof. Suppose $(x(t), y(t))$ is a nonoscillatory solution of (1.1) such that

$$\lim_{t\to\infty} x(t) = \ell \text{ and } \lim_{t\to\infty} y(t) = k, \text{ for some } 0 < \ell, k < \infty.$$

Then there exist four positive constants $\mu_1, \mu_2, \nu_1, \nu_2$, and $t_1 \geq t_0$ such that

$$\mu_1 \leq x(t) \leq \mu_2 \tag{4.9}$$

and

$$\nu_1 \leq y(t) \leq \nu_2 \tag{4.10}$$

for all $t \geq t_1$.

Integrating (1.1a) from t_1 to ∞, using (4.10), (1.2), $x(\infty) = \ell$ and the monotonicity of f, we have

$$\ell = x(t_1) + \int_{t_1}^{\infty} p(r)f(y(r))\Delta r \geq x(t_1) + f(\nu_1)\int_{t_1}^{\infty} p(r)\Delta r,$$

which implies that $\int_{t_1}^{\infty} p(r)\Delta r < \infty$. Similarly, integrating (1.1b) from t_1 to ∞, using (4.9), (1.2), $y(\infty) = k$ and the monotonicity of g and rearranging the resulting equation, we obtain

$$y(t_1) = k + \int_{t_1}^{\infty} q(r)g(x(r))\Delta r \geq k + g(\mu_1)\int_{t_1}^{\infty} q(r)\Delta r,$$

which follows that $\int_{t_1}^{\infty} q(r)\Delta r < \infty$. Hence (4.8) holds.

Conversely, suppose (4.8) holds. Then, for given positive constants ℓ^*, k^*, we set

$$M = f\left(k^* + g(2\ell^*)\int_{t_0}^{\infty} q(s)\Delta s\right),$$

which is a finite number since $\int_{t_0}^{\infty} q(r)\Delta r < \infty$. Note that $\int_{t_0}^{\infty} p(r)\Delta r < \infty$. It follows that there exists $t_1 \geq t_0$ such that $\int_{t}^{\infty} p(r)\Delta r < \ell^*/M$, $\forall t \geq t_1$, which together with the monotonicity of f, gives that

$$\begin{aligned}
&\int_t^{\infty} p(r) f\left(k^* + g(2\ell^*)\int_r^{\infty} q(s)\Delta s\right)\Delta r \\
&\leq \int_t^{\infty} p(r) f\left(k^* + g(2\ell^*)\int_{t_0}^{\infty} q(s)\Delta s\right)\Delta r \\
&= M\int_t^{\infty} p(r)\Delta r \\
&< \ell^*,
\end{aligned} \tag{4.11}$$

for all $t \geq t_1$. Let B be given as in the proof of Theorem 4.2. Define a subset Ω of B as follows:

$$\Omega = \{\, x \in B \mid \ell^* \leq x(t) \leq 2\ell^* \,,\ \forall t \in [t_1, \infty)_{\mathbb{T}} \,\}.$$

It is easy to see that $\inf \Omega \in B$ and $\sup \Omega \in B$. Moreover, for any subset Q of Ω, we have $\inf Q \in \Omega$ and $\sup Q \in \Omega$. Define an operator $L : \Omega \to B$ given by

$$(Lx)(t) = \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_r^{\infty} q(s) g(x(s))\Delta s\right)\Delta r,\ \forall t \in [t_1, \infty)_{\mathbb{T}}.$$

We claim that $L\Omega \subseteq \Omega$. Let $x \in \Omega$. Since f, g are nondecreasing and $p > 0$, $q \geq 0$, we have

$$(Lx)(t) = \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_r^{\infty} q(s) g(x(s))\Delta s\right)\Delta r \geq \ell^*$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. On the other hand, using (4.11), we obtain

$$\begin{aligned}
(Lx)(t) &= \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_r^{\infty} q(s) g(x(s))\Delta s\right)\Delta r \\
&\leq \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_r^{\infty} q(s) g(2\ell^*)\Delta s\right)\Delta r \\
&= \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + g(2\ell^*)\int_r^{\infty} q(s)\Delta s\right)\Delta r \\
&\leq \ell^* + \int_{t_1}^{\infty} p(r) f\left(k^* + g(2\ell^*)\int_r^{\infty} q(s)\Delta s\right)\Delta r \\
&\leq \ell^* + \ell^* = 2\ell^*,
\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Hence $L\Omega \subseteq \Omega$

Furthermore, L is increasing since if $x, \widetilde{x} \in \Omega$ with $\widetilde{x} \geq x$, then

$$\begin{aligned}(Lx)(t) &= \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_{r}^{\infty} q(s) g(x(s)) \Delta s\right) \Delta r \\ &\leq \ell^* + \int_{t_1}^{t} p(r) f\left(k^* + \int_{r}^{\infty} q(s) g(\widetilde{x}(s)) \Delta s\right) \Delta r \\ &= (L\widetilde{x})(t).\end{aligned}$$

Here we have used the assumption g is nondecreasing. By Lemme 4.1, we can conclude that there exists an $\hat{x} \in \Omega$ such that $\hat{x} = L\hat{x}$. Now we set

$$\hat{y}(t) = k^* + \int_{t}^{\infty} q(s) g(\hat{x}(s)) \Delta s, \tag{4.12}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$, then we have

$$\hat{y}^{\Delta}(t) = -q(t) g(\hat{x}(t))$$

and

$$\hat{x}^{\Delta}(t) = (L\hat{x})^{\Delta}(t) = p(t) f(\hat{y}(t)).$$

Since $\hat{x} \in \Omega$, it follows that $\hat{x} \leq 2\ell^*$. Together with the monotonicity of g and the assumption $\int_{t_0}^{\infty} q(r)\Delta r < \infty$, we get

$$\int_{t}^{\infty} q(s) g(\hat{x}(s)) \Delta s \leq g(2\ell^*) \int_{t}^{\infty} q(s) \Delta s < \infty. \tag{4.13}$$

Taking limits on both sides of (4.12) and using (4.13), we get

$$\lim_{t \to \infty} \hat{y}(t) = \lim_{t \to \infty} \left(k^* + \int_{t}^{\infty} q(s) g(\hat{x}(s)) \Delta s\right) = k^*.$$

In addition, since $\int_{t_0}^{\infty} p(r)\Delta r < \infty$, Lemme 3.3 asserts that $\ell := \lim_{t \to \infty} \hat{x}(t)$ exists and $\ell \geq \ell^* > 0$. Hence $(\hat{x}(t), \hat{y}(t))$ is a nonoscillatory solution of (1.1) which belongs to $C^*(\ell, k^*)$.

❑

Theorem 4.4 *Suppose that*

$$\int_{t_0}^{\infty} p(r)\Delta r < \infty \text{ and } \int_{t_0}^{\infty} q(r)\Delta r < \infty.$$

Then (1.1) has a nonoscillatory solution $(x(t), y(t))$ *which belongs to* $C^*(\xi, 0)$ *for some* $\xi > 0$ *if and only if*

$$\int_{t_0}^{\infty} p(s) f\left(\int_s^{\infty} q(r) g(\xi^*)\Delta r\right)\Delta s < \infty \tag{4.14}$$

for some $\xi^* > 0$.

Proof. Suppose $(x(t), y(t))$ is a nonoscillatory solution of (1.1) such that

$$\lim_{t\to\infty} x(t) = \xi \text{ and } \lim_{t\to\infty} y(t) = 0, \quad \text{for some } \xi > 0.$$

Then there exist a positive constant σ_1 and $t_1 \geq t_0$ such that

$$x(t) \geq \sigma_1 \text{ on } [t_1, \infty)_{\mathbb{T}},$$

which, together with the monotonicity of g, gives that

$$g(x(t)) \geq g(\sigma_1), \ \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.15}$$

Integrating the equation (1.1b) from t to ∞, and rearranging the resulting equation, we have

$$y(t) = \int_t^{\infty} q(s) g(x(s))\Delta s \geq \int_t^{\infty} q(s) g(\sigma_1)\Delta s, \ \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.16}$$

where we have used (4.15) and the assumption that $y(\infty) = 0$. Since f is nondecreasing, (4.16) implies that

$$f(y(t)) \geq f\left(\int_t^{\infty} q(s) g(\sigma_1)\Delta s\right), \ \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.17}$$

Proceeding to integrate the equation (1.1a) from t_1 to ∞, and using (4.17) , we get

$$\begin{aligned}\xi &= x(t_1) + \int_{t_1}^{\infty} p(s) f(y(s))\Delta s \\ &\geq x(t_1) + \int_{t_1}^{\infty} p(s) f\left(\int_s^{\infty} q(r) g(\sigma_1)\Delta r\right)\Delta s.\end{aligned}$$

Hence (4.14) holds with $\xi^* = \sigma_1$.

Conversely, Suppose

$$\int_{t_0}^{\infty} p(s) f\left(\int_s^{\infty} q(r) g(\xi^*) \Delta r\right) \Delta s < \infty,$$

for some $\xi^* > 0$. Then there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(s) f\left(\int_s^{\infty} q(r) g(\xi^*) \Delta r\right) \Delta s < \tau, \tag{4.18}$$

where $\tau = \xi^*/2 > 0$. Let B be given as in the proof of Theorem 4.2. Define a subset Ω of B as follows:

$$\Omega = \{\ x \in B \mid \tau \leq x(t) \leq 2\tau,\ \forall t \in [t_1, \infty)_{\mathbb{T}}\ \}.$$

It is easy to see that $\inf \Omega \in B$ and $\sup \Omega \in B$. Moreover, for any subset Q of Ω, we have $\inf Q \in \Omega$ and $\sup Q \in \Omega$. Define an operator $L : \Omega \to B$ by

$$(Lx)(t) = \tau + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} q(r) g(x(r)) \Delta r\right) \Delta s,\ \forall\ t \in [t_1, \infty)_{\mathbb{T}}.$$

we claim that $L\Omega \subseteq \Omega$. Let $x \in \Omega$. Since f, g are nondecreasing and $p > 0$, $q \geq 0$, we have

$$(Lx)(t) = \tau + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} q(r) g(x(r)) \Delta r\right) \Delta s \geq \tau,$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. On the other hand, using (4.18), we obtain

$$\begin{aligned}
(Lx)(t) &= \tau + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} q(r) g(x(r)) \Delta r\right) \Delta s \\
&\leq \tau + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} q(r) g(2\tau) \Delta r\right) \Delta s \\
&= \tau + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} q(r) g(\xi^*) \Delta r\right) \Delta s \\
&\leq \tau + \tau = 2\tau,
\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Hence $L\Omega \subseteq \Omega$.

Furthermore, L is increasing since if $x, \widetilde{x} \in \Omega$ with $\widetilde{x} > x$, then

$$\begin{aligned}(Lx)(t) &= \tau + \int_{t_1}^{r} p(s) f\left(\int_s^\infty q(r) g(x(r))\Delta r\right)\Delta s \\ &\le \tau + \int_{t_1}^{r} p(s) f\left(\int_s^\infty q(r) g(\widetilde{x}(r))\Delta r\right)\Delta s \\ &= (L\widetilde{x})(t).\end{aligned}$$

Here we have used the assumption g is nondecreasing. By Lemma 4.1, we can conclude that there exists $\hat{x} \in \Omega$ such that $\hat{x} = L\hat{x}$. Now we set

$$\hat{y}(t) = \int_t^\infty q(r) g(\hat{x}(r))\Delta r, \tag{4.19}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$, then we have

$$\hat{y}^{\Delta}(t) = -q(t) g(\hat{x}(t)),$$

and

$$\hat{x}^{\Delta}(t) = (L\hat{x})^{\Delta}(t) = p(t) f\left(\int_t^\infty q(r) g(\hat{x}(r))\Delta r\right) = p(t) f(\hat{y}(t)).$$

Since $\hat{x} \in \Omega$, it follows that $\hat{x} \le 2\tau$. Together with the monotonicity of g and the assumption $\int_{t_0}^\infty q(r)\Delta r < \infty$, we get

$$\int_t^\infty q(s) g(\hat{x}(s))\Delta s \le \int_t^\infty q(s) g(2\tau)\Delta s < \infty. \tag{4.20}$$

Taking limits on both sides of (4.19) and using (4.20), we get

$$\lim_{t\to\infty} \hat{y}(t) = \lim_{t\to\infty}\left(\int_t^\infty q(s) g(\hat{x}(s))\Delta s\right) = 0.$$

In addition, since $\int_{t_0}^\infty p(r)\Delta r < \infty$, Lemma 3.3 asserts that $\xi := \lim_{t\to\infty} \hat{x}(t)$ exists and $\xi \ge \tau > 0$. Hence $(\hat{x}(t), \hat{y}(t)$ is a nonoscillatory solution of (1.1) which belongs to $C^*(\xi, 0)$. ❑

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