國 立 政 治 大 學 應 用 數 學 系

數 學 教 學 碩 士 在 職 專 班

碩士學位論文

DIFFY PENTAGON

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中華民國　101　年　8　月　24　日

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Diffy Pentagon

Abstract

In Diffy box, we write down numbers on the four vertices of square, and then on the midpoint of each side write the difference between the two numbers at its endpoints. It is known that the numbers on the four vertices of a square will converge to zero finally. In this article, we use the same operations as Diffy box to discuss pentagons which we call" Diffy pentagon ". We find it will converge, too.

Keywords: Diffy pentagon, Strong induction

迪菲五邊形

中文摘要

在迪菲方塊中，我們將正方形的四個頂點皆填入數值，再利用相鄰兩頂點相減，再取絕對值的方式觀察其數列行爲，發現四個頂點的數字最後皆會收斂至 ***0***。在本文中，我們將之推廣至五邊形，我們稱它爲迪菲五邊形。我們套用同樣的運算模式後，發現亦有特殊的收斂行爲。

關鍵字：　迪菲五邊形、強勢數學歸納法

Chapter 1 Introduction

Diffy box, also called difference box, is a simple method that provide us subtraction practice. The idea's original author is unknow, we could only trace back to Professor Juanita Copley of the University of Houston who has introduced it as a problem-solving activity in professional development sessions.

To create a Diffy box is as follows:

1. Draw a square, and label each vertex with a (rational) number.
2. On the midpoint of each side write the difference between the two numbers at its endpoints.
3. Inscribe a new square in the old one, using these new numbers to label the vertices.
4. Repeat this process, and continue inscribing new boxes until reaching a square that has all four vertices labeled *0*.

In Diffy box, we only consider squares. And here, we use the same operations as Diffy box to discuss pentagons which we call" Diffy pentagon." We try to find its convergence laws.

Chapter 2 The Description of the Convergence Properties

2.1 Definitions and Theorems

Definition 2.1 *Let a,b,c,d,e be nonnegative integers, we define an ordered set* $(a\ b\ c\ d\ e)$ *be the five vertices of the regular-pentagon, and another ordered set* $)a\ b\ c\ d\ e($ *be the five vertices of the anti-pentagon.*

Note that we often use the first element of the ordered set $(a\ b\ c\ d\ e)$ to be the upper left of the regular-pentagon, and in clockwise order. Also, we use the first element of the ordered set $)a\ b\ c\ d\ e($ to be the upper left of the anti-pentagon, and in clockwise order.

Example 2.1 Let *a,b,c,d,e* be nonnegative integers, we take an ordered set $(a\ b\ c\ d\ e)$ on the regular-pentagon as in Figure 2.1. And an ordered set $)a\ b\ c\ d\ e($ on the anti-pentagon as in Figure 2.2.

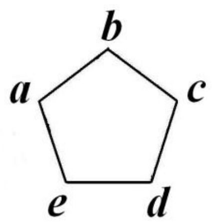


Figure 2.1: Regular-pentagon

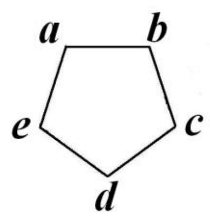


Figure 2.2: Anti-pentagon

Definition 2.2 *We define the child of an ordered set* $(a\ b\ c\ d\ e)$ *to be the ordered set* $)|a-b|\ |b-c|\ |c-d|\ |d-e|\ |e-a|($ *and* $(a\ b\ c\ d\ e)$ *a parent of* $)|a-b|\ |b-c|\ |c-d|\ |d-e|\ |e-a|($ *(see Figure 2.3).*

Also, we define the child of an ordered set)a b c d e(to be the ordered set $(|e-a|\ |a-b|\ |b-c|\ |c-d|\ |d-e|)$ *and)a b c d e(a parent of* $(|e-a|\ |a-b|\ |b-c|\ |c-d|\ |d-e|)$ *(see Figure 2.4).*

And write their relation as parent $\Rightarrow$ *child.*

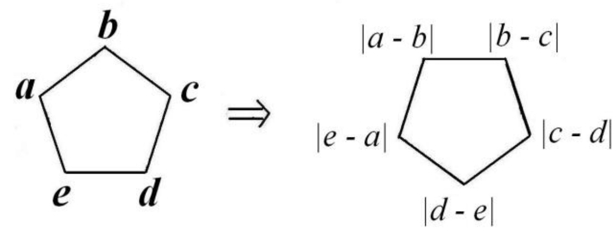


Figure 2.3: The first pentagon is the parent, and the second pentagon is the child.

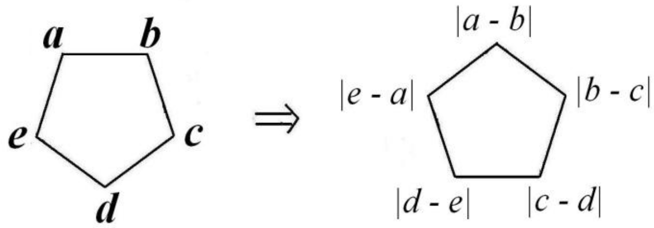


Figure 2.4: The first pentagon is the parent, and the second pentagon is the child.

It is easy to see if we rotate both anti-pentagon and its child (see Figure 2.4) counterclockwise 36^0 through the center, we obtain the regular-pentagon and its child (see Figure 2.5), and vice versa.

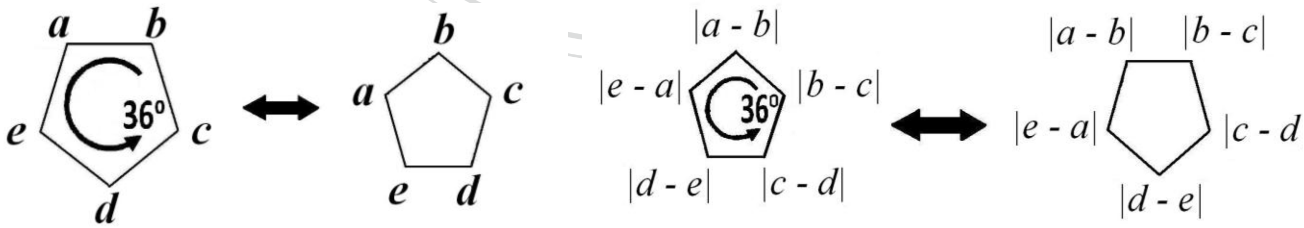


Figure 2.5:

Since *(a b c d e)* and *)a b c d e(* differ by a rotation of 36^0, sometimes, we don't want to distinguish between them. For convenience, we define an ordered set *[a b c d e]* to be *(a b c d e)* or *)a b c d e(* .

Definition 2.3 *For any* $[a_1\ b_1\ c_1\ d_1\ e_1] \Rightarrow [a_2\ b_2\ c_2\ d_2\ e_2] \Rightarrow [a_3\ b_3\ c_3\ d_3\ e_3] \Rightarrow \ldots \Rightarrow [a_t\ b_t\ c_t\ d_t\ e_t] \Rightarrow \ldots \Rightarrow [a_k\ b_k\ c_k\ d_k\ e_k] \ldots$ *If there exists one* $[a_k\ b_k\ c_k\ d_k\ e_k] = [a_t\ b_t\ c_t\ d_t\ e_t]$*, for the smallest t, k, and k > t, then we call* $[a_t\ b_t\ c_t\ d_t\ e_t] \Rightarrow \ldots \Rightarrow [a_{k-1}\ b_{k-1}\ c_{k-1}\ d_{k-1}\ e_{k-1}] \Rightarrow [a_k\ b_k\ c_k\ d_k\ e_k]$ *the cycle convergence of* $[a_1\ b_1\ c_1\ d_1\ e_1]$

Example 2.2 Take a pentagon with *[3 1 2 4 1]*, find its cycle convergence.

Solution :

$$\text{Since } [3\ 1\ 2\ 4\ 1] \tag{1}$$

$$\Rightarrow [2\ 1\ 2\ 3\ 2] \tag{2}$$

$$\Rightarrow [0\ 1\ 1\ 1\ 1] \tag{3}$$

$$\Rightarrow [1\ 0\ 0\ 0\ 1] \tag{4}$$

$$\Rightarrow [0\ 1\ 0\ 0\ 1] \tag{5}$$

$$\Rightarrow [1\ 1\ 0\ 1\ 1] \tag{6}$$

Since the ordered set (3) is equal to (6), by Definition 2.3, we take $t = 3$ and $k = 6$, and then call $[0\ 1\ 1\ 1\ 1] \Rightarrow [1\ 0\ 0\ 0\ 1] \Rightarrow [0\ 1\ 0\ 0\ 1] \Rightarrow [1\ 1\ 0\ 1\ 1] = [0\ 1\ 1\ 1\ 1]$ the cycle convergence of *[3 1 2 4 1]*.

Definition 2.4 *[a b c d e] is isomorphic to [a' b' c' d' e'] and denoted by* $[a\ b\ c\ d\ e] \approx [a'\ b'\ c'\ d'\ e']$ *if they have the same child.*

Definition 2.5 *[a b c d e] and [a' b' c' d' e'] are similar and denoted by* $[a\ b\ c\ d\ e] \sim [a'\ b'\ c'\ d'\ e']$ *if* $i' = ki$*, where* $i \in \{a, b, c, d, e\}$*, k is a nonzero integer.*

Example 2.3 Two pentagons : *[3 1 2 4 1]* and *[3k k 2k 4k k]*, *k* be any positive integer. Find their cycle convergences.

Solution :

Since *[3 1 2 4 1]*	Since *[3k k 2k 4k k]*
$\Rightarrow$ *[2 1 2 3 2]*	$\Rightarrow$ *[2k k 2k 3k 2k]*
$\Rightarrow$ *[0 1 1 1 1]*	$\Rightarrow$ *[0 k k k k]*
$\Rightarrow$ *[1 0 0 0 1]*	$\Rightarrow$ *[k 0 0 0 k]*
$\Rightarrow$ *[0 1 0 0 1]*	$\Rightarrow$ *[0 k 0 0 k]*
$\Rightarrow$ *[1 1 0 1 1]*	$\Rightarrow$ *[k k 0 k k]*

Since the ordered set *[0 1 1 1 1]* is equal to *[1 1 0 1 1]*,

[0 1 1 1 1] $\Rightarrow$ *[1 0 0 0 1]* $\Rightarrow$ *[0 1 0 0 1]* $\Rightarrow$ *[1 1 0 1 1] = [0 1 1 1 1]* is the cycle convergence of *[3 1 2 4 1]*.

Since the formula *[0 k k k k]* is equal to *[k k 0 k k]*,

[0 k k k k] $\Rightarrow$ *[k 0 0 0 k]* $\Rightarrow$ *[0 k 0 0 k]* $\Rightarrow$ *[k k 0 k k] = [0 k k k k]* is the cycle convergence of *[3k k 2k 4k k]*.

Note that they have the same kind of cycle convergences which just need to multiply each component number by *k* .

Theorem 2.1 *For any [a b c d e], where a, b, c, d, e are nonnegative integers with a = b = c = d = e, then the child of the pentagon is [0 0 0 0 0].*

Proof :

Since $a = b = c = d = e$,

$[a\ b\ c\ d\ e] \Rightarrow [|a-b|\ |b-c|\ |c-d|\ |d-e|\ |e-a|] = [0\ 0\ 0\ 0\ 0]$

And the proof is complete.

Theorem 2.2 *Let a, b, c, d, e be any nonnegative integers, and k be any integer, then [a b c d e] and [a+k b+k c+k d+k e+k] are isomorphic.*

Proof :

It is easy to see $[a\ b\ c\ d\ e] \Rightarrow [|a-b|\ |b-c|\ |c-d|\ |d-e|\ |e-a|]$

and $[a+k\ b+k\ c+k\ d+k\ e+k] \Rightarrow [|a-b|\ |b-c|\ |c-d|\ |d-e|\ |e-a|]$

Since they have the same child

$[a\ b\ c\ d\ e] \approx [a+k\ b+k\ c+k\ d+k\ e+k]$.

Theorem 2.3 *For any [a b c d e], a, b, c, d, e be any nonnegative integers, and a, b, c, d, e are not all equal, if one of its child is as [0 0 0 n n] or [0 0 n 0 n] or [n 0 n n n] for some positive n, then it must have a cycle convergence constituted by [0 0 0 n n], [0 0 n 0 n], and [n 0 n n n], in cyclic order.*

Proof :

First, we make a relation with *[0 0 0 n n]*, *[0 0 n 0 n]*, and *[n 0 n n n]*.

Let *[0 0 0 n n]* be a regular-pentagon,

then its child is *[0 0 n 0 n]*,

and then its child is *[n 0 n n n]*,

and then its child is *[n n 0 0 0]*. (see Figure 2.6)

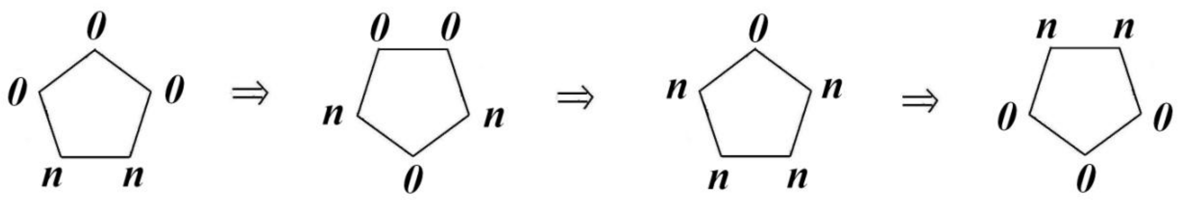


Figure 2.6: The first pentagon is *[0 0 0 n n]*, and then *[0 0 n 0 n]*, and then *[n 0 n n n]*, and then *[n n 0 0 0]*.

Since *[a b c d e]* is a cycle order pentagon, it is easy to see *[n n 0 0 0] = [0 0 0 n n]* by a cyclic rotation, and similarly for *[0 0 n 0 n]* and *[n 0 n n n]*, thus the theorem hold for regular-pentagons.

Next, let *[0 0 0 n n]* be an anti-pentagon.

Then its child is *[n 0 0 n 0]*,

and then its child is *[n 0 n n n]*,

and then its child is *[0 n n 0 0]*. (see Figure 2.7)

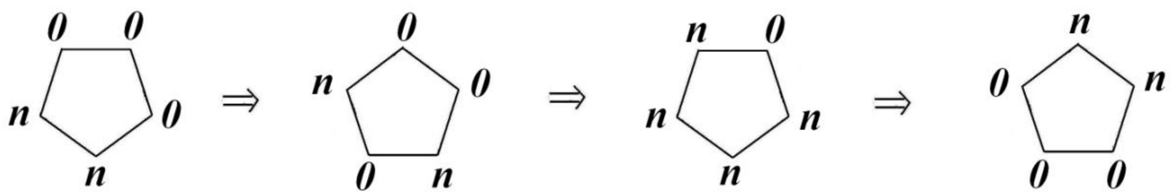


Figure 2.7: The first pentagon is *[0 0 0 n n]*, and then *[n 0 0 n 0]*, and then *[n 0 n n n]*, and then *[0 n n 0 0]*.

Since *[a b c d e]* is a cycle order pentagon, it is easy to see *[0 n n 0 0] = [0 0 0 n n]* by a cyclic rotation, and similarly for *[0 0 n 0 n]* and *[n 0 n n n]*, thus the theorem holds in anti-pentagons. Therefore, if one child of any cycle order pentagon as *[0 0 0 n n]*, *[0 0 n 0 n]*, or *[n 0 n n n]*, then it must have a cycle convergence which is constituted by *[0 0 0 n n]*, *[0 0 n 0 n]*, and *[n 0 n n n]*, in cyclic order.

Remark 2.1 *For any [a b c d e] and [a' b' c' d' e'], where a, b, c, d, e, a', b', c', d', e' are nonnegative integers, and a, b, c, d, e are not all equal, and a', b', c', d', e' are not all equal, either. If [a b c d e]* $\approx$ *[a' b' c' d' e'], and one of them has a cycle convergence, then the other must has the same cycle convergence.*

Theorem 2.4 *If* $0 \leqq a < M$ *and* $0 \leqq b < M$*, then* $0 \leqq |a-b| < M$*.*

Proof :

Since $0 \leqq a < M$ and $0 \leqq b < M$

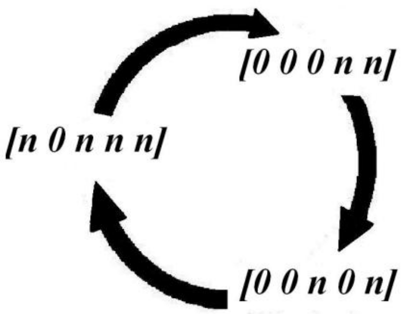
$\Rightarrow \quad 0 - M < a - b < M - 0$

$\Rightarrow \quad -M < a - b < M$

$\Rightarrow \quad 0 \leqq |a-b| < M$

2.2 Description of Feature

In our study, we find that for any *[a b c d e]*, where *a, b, c, d, e* are all nonnegative integers and not all equal, then its child must make a cycle convergence as the following figure, where n is any positive integer.



Next, we will take an example to show this feature.

Example 2.4 Take a pentagon with *[5 3 2 1 3]*, please find its cycle convergence.

Solution :

Since *[5 3 2 1 3]*

$\Rightarrow$ *[2 1 1 2 2]*

$\Rightarrow$ *[0 1 0 1 0]*

$\Rightarrow$ *[1 1 1 1 0]*

$\Rightarrow$ *[1 0 0 0 1]*

$\Rightarrow$ *[1 0 0 1 0] = [0 1 0 1 0]*

By this feature, we say

[0 1 0 1 0] $\Rightarrow$ *[1 1 1 1 0]* $\Rightarrow$ *[1 0 0 0 1]* $\Rightarrow$ *[1 0 0 1 0] = [0 1 0 1 0]*

is the cycle convergence of the pentagon with *[5 3 2 1 3]*.

Note that if we rotation with these three pentagons and let $n = 1$, we will get *[0 0 n 0 n] = [0 1 0 1 0]*, *[n 0 n n n] = [1 1 1 1 0]*, and *[0 0 0 n n] = [1 0 0 0 1]*.

At next section, we prove for any *[a b c d e]* of pentagon, *a, b, c, d, e* be any nonnegative integer, and *a, b, c, d, e* are not all equal, then *[a b c d e]* must have a cycle convergence which constituted by *[0 0 0 n n]*, *[0 0 n 0 n]*, and *[n 0 n n n]* for some positive natural number *n*.

For convenience, we call:

[0 0 0 n n] is type Ⅰ

[0 0 n 0 n] is type Ⅱ

[n 0 n n n] is type Ⅲ

And by Theorem 2.3, we get for any pentagon, if one of its child is as like as type Ⅰ or type Ⅱ or typeⅢ, then it must have a cycle convergence as:

$$\text{type I} \Rightarrow \text{type II} \Rightarrow \text{type III} \Rightarrow \text{type I}$$
$$\text{or type II} \Rightarrow \text{type III} \Rightarrow \text{type I} \Rightarrow \text{type II}$$
$$\text{or type III} \Rightarrow \text{type I} \Rightarrow \text{type II} \Rightarrow \text{type III}$$

And all cycle convergences just have three ordered sets.

Chapter 3　Pentagon with Cycle Convergence

3.1　Introduction

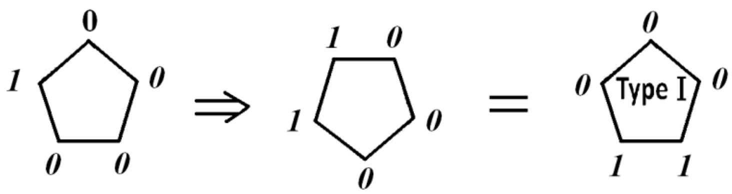
Let $[a\ b\ c\ d\ e]$ be a cycle order pentagon, a, b, c, d, e are all nonnegative integers and not all equal, and let $m = min\{a, b, c, d, e\}$, then at least one element of $[a-m, b-m, c-m, d-m, e-m]$ must be zero.

Since $[a-m, b-m, c-m, d-m, e-m]$ and $[a\ b\ c\ d\ e]$ have the same child, by Theorem 2.2, $[a-m, b-m, c-m, d-m, e-m] \approx [a\ b\ c\ d\ e]$, and it is easy to see if one of them has a cycle convergence, the other must have the same cycle convergence, too.　So we just need to prove $[a-m, b-m, c-m, d-m, e-m]$ has a cycle convergence.

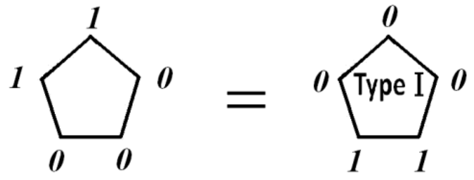
3.2　The Proof with Strong Induction

We let $[a\ b\ c\ d\ e]$ be a cycle order pentagon again, and assume at least one element of a, b, c, d, e be zero.　By strong induction to prove cycle convergence, we check the following pentagons first:

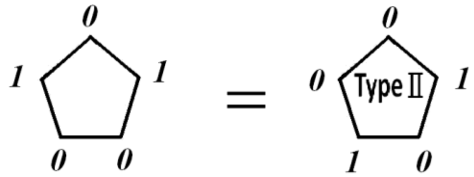
1. $[1\ 0\ 0\ 0\ 0] \Rightarrow [1\ 0\ 0\ 0\ 1] = [0\ 0\ 0\ 1\ 1]$ (type Ⅰ)



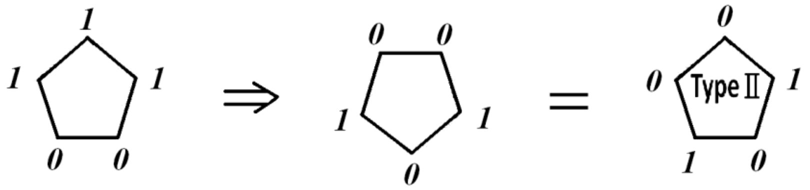
2. $[1\ 1\ 0\ 0\ 0] = [0\ 0\ 0\ 1\ 1]$ (type Ⅰ)



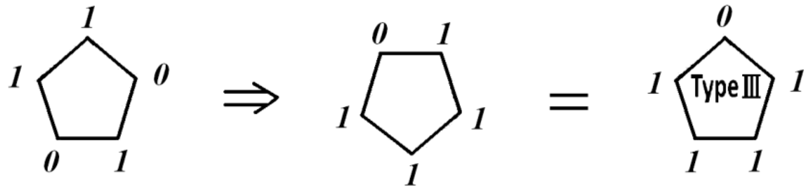
3. *[1 0 1 0 0] = [0 0 1 0 1]* (type Ⅱ)



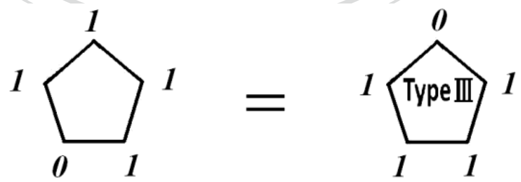
4. *[1 1 1 0 0]* $\Rightarrow$ *[0 0 1 0 1]* (type Ⅱ)



5. *[1 1 0 1 0]* $\Rightarrow$ *[0 1 1 1 1] = [1 0 1 1 1]* (type Ⅲ)



6. *[1 1 1 1 0] = [1 0 1 1 1]* (type Ⅲ)



And then, we assume for any pentagon must have cycle convergence if the value of *max{a, b, c, d, e}* equal $1,2,3,\cdots,M-1$.

Finally, we just need check if the value of *max{a, b, c, d, e}* equal M, then the pentagon still has cycle convergence, too.

To check this, assume that n is the number of M's and t is the number of 0's in a, b, c, d, e. We let the remaining elements of pentagon, we call x, y, z , and let $0 < x, y, z < M$. And then, classify with the numbers of elements of " M " and " 0 " as the following cases:

■ Case 1.

Let $n = 1$, and $t = 1$,

(1): $[M\ 0\ x\ y\ z]$

$[M\ 0\ x\ y\ z] \Rightarrow [M\ x\ |x-y|\ |y-z|\ M-z]$

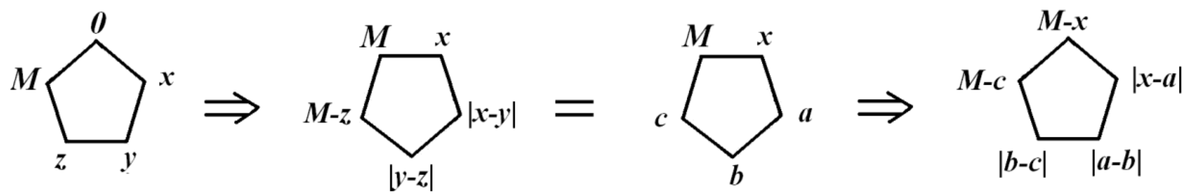
Let $a = |x-y|$, $b = |y-z|$, $c = M-z$,

Since $0 < x, y, z < M$, and $0 < c < M$, by Theorem 2.4, $0 \leqq a, b < M$.

And then $[M\ x\ |x-y|\ |y-z|\ M-z] = [M\ x\ a\ b\ c] \Rightarrow [M-c\ M-x\ |x-a|\ |a-b|\ |b-c|]$

By Theorem 2.4, $0 \leqq |x-a|, |a-b|, |b-c| < M$ and since $0 \leqq M-x, M-c < M$, $[M-c\ M-x\ |x-a|\ |a-b|\ |b-c|]$ must have cycle convergence.

And then, $[M\ 0\ x\ y\ z]$ has cycle convergence, too.



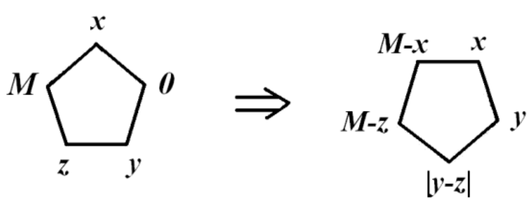
(2): $[M\ x\ 0\ y\ z]$

$[M\ x\ 0\ y\ z] \Rightarrow [M-x\ x\ y\ |y-z|\ M-z]$

Since $0 < y, z < M$, by Theorem 2.4, $0 \leqq |y-z| < M$.

Since $0 < M-x, x, y, M-z < M$, $[M-x\ x\ y\ |y-z|\ M-z]$ must have cycle convergence.

And then, $[M\ x\ 0\ y\ z]$ has cycle convergence, too.



■ Case 2.

Let $n = 1$, and $t = 2$,

(1): $[M\ 0\ 0\ x\ y]$

$[M\ 0\ 0\ x\ y] \Rightarrow [M\ 0\ x\ |x-y|\ M-y]$

(a) If $x \neq y$: By case 1 (1), $[M\ 0\ x\ |x-y|\ M-y]$ has cycle convergence.

And then, $[M\ 0\ 0\ x\ y]$ has cycle convergence, too.

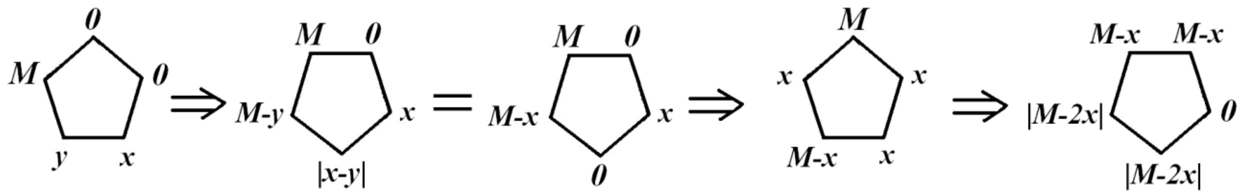
(b) If $x = y$: $[M\ 0\ x\ |x-y|\ M-y] = [M\ 0\ x\ 0\ M-x] \Rightarrow [x\ M\ x\ x\ M-x]$

$\Rightarrow [M-x\ M-x\ 0\ |M-2x|\ |M-2x|]$

Since $0 < M-x, x < M$, by Theorem 2.4, $0 \leqq |M-2x| < M$.

Therefore, $[M-x\ M-x\ 0\ |M-2x|\ |M-2x|]$ must have cycle convergence.

And then, $[M\ 0\ 0\ x\ y]$ has cycle convergence, too.



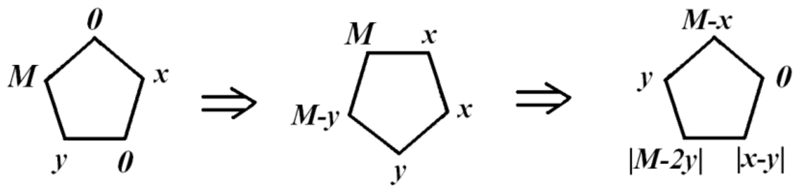
(2): $[M\ 0\ x\ 0\ y]$

$[M\ 0\ x\ 0\ y] \Rightarrow [M\ x\ x\ y\ M-y] \Rightarrow [y\ M-x\ 0\ |x-y|\ |M-2y|]$

Since $0 < M-y, x, y < M$, by Theorem 2.4, $0 \leqq |M-2y|, |x-y| < M$.

Since $0 < M-x < M$, $[y\ M-x\ 0\ |x-y|\ |M-2y|]$ must have cycle convergence.

And then, $[M\ 0\ x\ 0\ y]$ has cycle convergence, too.



(3): $[M\ 0\ x\ y\ 0]$

$[M\ 0\ x\ y\ 0] \Rightarrow [M\ x\ |x-y|\ y\ M]$

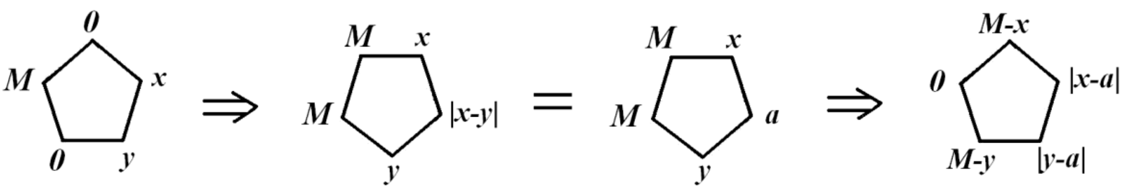
Let $a = |x-y|$, since $0 < x, y < M$, by Theorem 2.4, $0 \leqq a < M$.

So $[M\ x\ |x-y|\ y\ M] = [M\ x\ a\ y\ M] \Rightarrow [0\ M-x\ |x-a|\ |y-a|\ M-y]$

Since $0 \leqq a < M$ and $0 < x, y < M$, by Theorem 2.4, $0 \leqq |x-a|, |y-a| < M$

Since $0 < M-x, M-y < M$, $[0\ M-x\ |x-a|\ |y-a|\ M-y]$ must have cycle convergence.

And then, $[M\ 0\ x\ y\ 0]$ has cycle convergence, too.

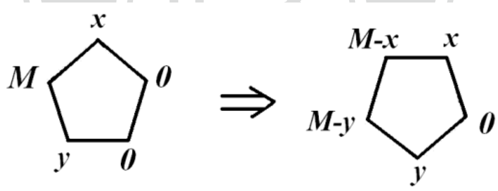


(4): $[M\ x\ 0\ 0\ y]$

$[M\ x\ 0\ 0\ y] \Rightarrow [M-x\ x\ 0\ y\ M-y]$

Since $0 < M-x, M-y, x, y < M$, $[M-x\ x\ 0\ y\ M-y]$ must have cycle convergence.

And then, $[M\ x\ 0\ 0\ y]$ has cycle convergence, too.



■ Case 3.

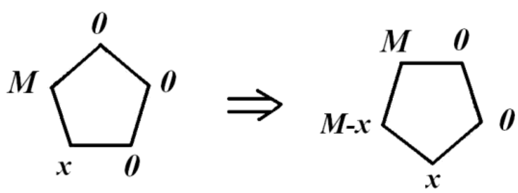
Let $n = 1$, and $t = 3$,

(1):$[M\ 0\ 0\ 0\ x]$

$[M\ 0\ 0\ 0\ x] \Rightarrow [M\ 0\ 0\ x\ M-x]$

Since $0 < M-x < M$, by case 2 (1), $[M\ 0\ 0\ x\ M-x]$ has cycle convergence.

And then, $[M\ 0\ 0\ 0\ x]$ has cycle convergence, too.

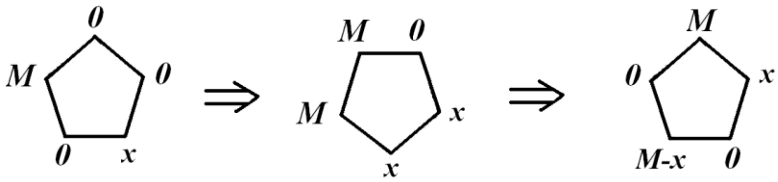


(2): $[M\ 0\ 0\ x\ 0]$

$[M\ 0\ 0\ x\ 0] \Rightarrow [M\ 0\ x\ x\ M] \Rightarrow [0\ M\ x\ 0\ M-x]$

Since $0 < x < M \Rightarrow 0 < M-x < M$, by case 2 (2), $[0\ M\ x\ 0\ M-x]$ has cycle convergence.

And then, $[M\ 0\ 0\ x\ 0]$ has cycle convergence, too.



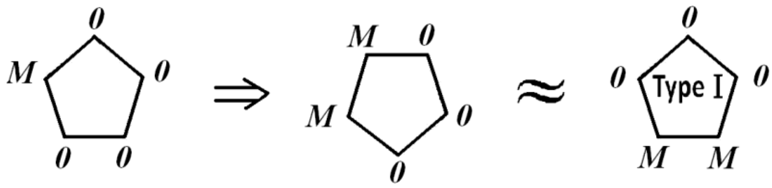
■ Case 4.

Let $n = 1$, and $t = 4$,

(1):$[M\ 0\ 0\ 0\ 0]$

$[M\ 0\ 0\ 0\ 0] \Rightarrow [M\ 0\ 0\ 0\ M] \approx$ type I

So $[M\ 0\ 0\ 0\ 0]$ has cycle onvergene.



■ Case 5.

Let $n = 2$, and $t = 1$,

(1):$[M\ M\ 0\ x\ y]$

$[M\ M\ 0\ x\ y] \Rightarrow [0\ M\ x\ |x-y|\ M-y]$

(a) If $x \neq y$: Since $0 < x, y < M$, by theorem2.4, $0 < |x-y| < M$.

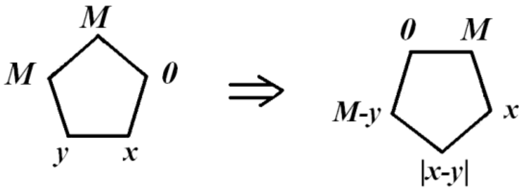
By case 1 (1), $[0\ M\ x\ |x-y|\ M-y]$ has cycle convergence.

And then, $[M\ M\ 0\ x\ y]$ has cycle convergence, too.

(b) If $x = y$: $[0\ M\ x\ |x-y|\ M-y] = [0\ M\ x\ 0\ M-x]$:

Since $0 < M-x < M$, by case 2 (2), $[0\ M\ x\ 0\ M-x]$ has cycle convergence.

And then, $[M\ M\ 0\ x\ y]$ has cycle convergence, too.

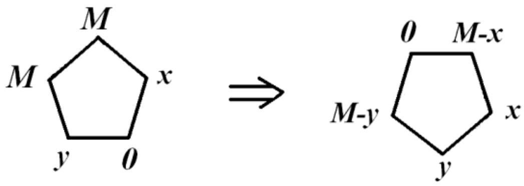


(2):$[M\ M\ x\ 0\ y]$

$[M\ M\ x\ 0\ y] \Rightarrow [0\ M-x\ x\ y\ M-y]$

Since $0 < M-x, M-y, x, y < M$, $[0\ M-x\ x\ y\ M-y]$ has cycle convergence.

And then, $[M\ M\ 0\ x\ y]$ has cycle convergence, too.



(3): $[M\ 0\ M\ x\ y]$

$[M\ 0\ M\ x\ y] \Rightarrow [M\ M\ M-x\ |x-y|\ M-y]$

(a) If $x = y$: $[M\ M\ M-x\ |x-y|\ M-y] = [M\ M\ M-x\ 0\ M-x]$

Since $0 < M-x < M$, by case 5 (2), $[M\ M\ M-x\ 0\ M-x]$ has cycle convergence.

And then, $[M\ 0\ M\ x\ y]$ must have cycle convergence, too.

(b) If $x \neq y$: Let $k = min\{\ M-x, |x-y|, M-y\ \}$

Then $[M\ M\ M-x\ |x-y|\ M-y] \approx [\ M-k\ M-k\ M-x-k\ |x-y|-k\ M-y-k]$

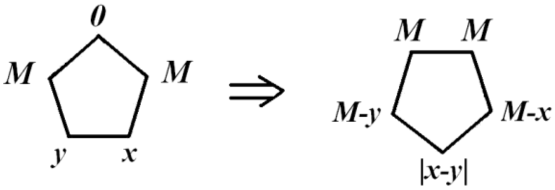
Since at least one element of $M-x-k$, $|x-y|-k$, $M-y-k$ is zero and all elements

of $M-k$,

$M-x-k$, $|x-y|-k$, $M-y-k$ are smaller then M.

It means $[M-k\ M-k\ M-x-k\ |x-y|-k\ M-y-k]$ has cycle convergence.

And then, $[M\ 0\ M\ x\ y]$ has cycle convergence, too.



(4): *[M x M 0 y]*

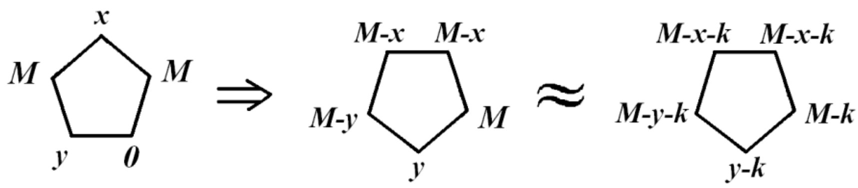
$[M\ x\ M\ 0\ y] \Rightarrow [M-x\ M-x\ M\ y\ M-y]$

Since $0 < x, y < M$, we get $0 < M-x, M-y < M$

Let $k = min\{ M-x, M-y, y\}$

Then $[M-x\ M-x\ M\ y\ M-y] \approx [M-x-k\ M-x-k\ M-k\ y-k\ M-y-k]$

Since at least one element of $M-x-k, y-k, M-y-k$ is zero and all elements of $M-x-k$, $M-k, y-k, M-y-k$ are smaller then M, it means $[M-x-k\ M-x-k\ M-k\ y-k\ M-y-k]$ has cycle convergence. And then, *[M x M 0 y]* has cycle convergence, too.



■ Case 6.

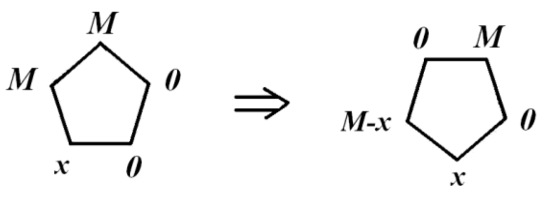
Let $n = 2$, and $t = 2$,

(1): *[M M 0 0 x]*

$[M\ M\ 0\ 0\ x] \Rightarrow [0\ M\ 0\ x\ M-x]$

By case 2 (3), $[0\ M\ 0\ x\ M-x]$ has cycle convergence.

And then, *[M M 0 0 x]* has cycle convergence, too.

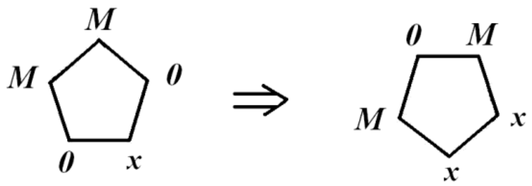


(2): *[M M 0 x 0]*

$[M\ M\ 0\ x\ 0] \Rightarrow [0\ M\ x\ x\ M]$

By case 5 (3) : *[0 M x x M]* has cycle convergence.

And then, *[M M 0 x 0]* has cycle convergence, too.



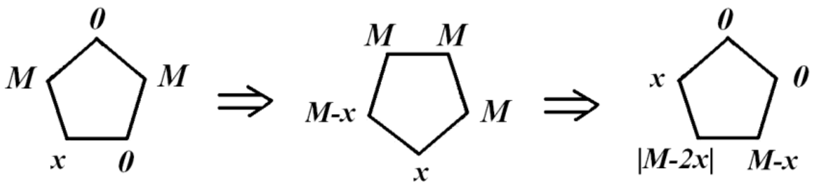
(3): *[M 0 M 0 x]*

$[M\ 0\ M\ 0\ x] \Rightarrow [M\ M\ M\ x\ M-x] \Rightarrow [x\ 0\ 0\ M-x\ |M-2x|]$

Since $0 < x, M-x < M$, by Theorem 2.4, $0 \leqq |M-2x| < M$.

And since $0 \leqq x, M-x, |M-2x| < M$, $[x\ 0\ 0\ M-x\ |M-2x|]$ has cycle convergence.

And then, *[M 0 M 0 x]* has cycle convergence, too.

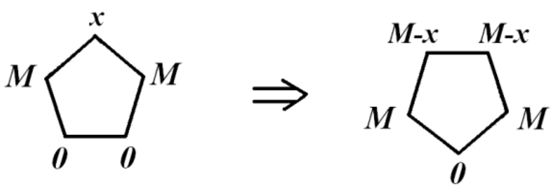


(4): *[M x M 0 0]*

$[M\ x\ M\ 0\ 0] \Rightarrow [M-x\ M-x\ M\ 0\ M]$

Since $0 < M-x < M$, by case 5 (3), $[M-x\ M-x\ M\ 0\ M]$ has cycle convergence.

And then, *[M x M 0 0]* has cycle convergence, too.



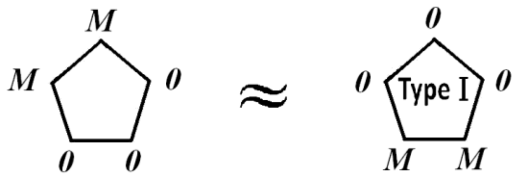
■ Case 7.

Let $n = 2$, and $t = 3$,

(1):*[M M 0 0 0]*

[M M 0 0 0] ≈ type Ⅰ

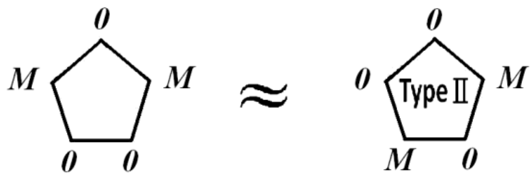
So *[M M 0 0 0]* has cycle convergence.



(2):*[M 0 M 0 0]*

[M 0 M 0 0] ≈ typeⅡ

So *[M 0 M 0 0]* has cycle convergence.



■ Case 8.

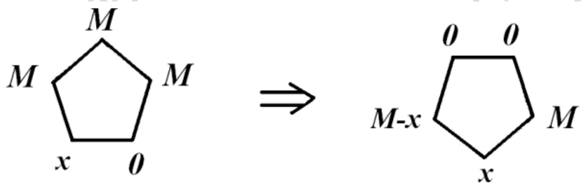
Let $n = 3$, and $t = 1$,

(1):*[M M M 0 x]*

[M M M 0 x] ⇒ *[0 0 M x M − x]*

Since $0 < M - x < M$, by case 2 (1), *[0 0 M x M − x]* has cycle convergence.

And then, *[M M M 0 x]* has cycle convergence, too.

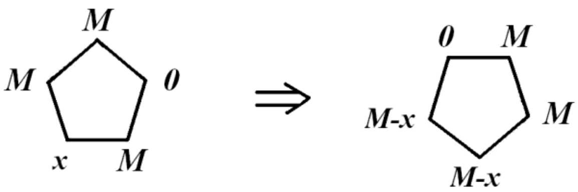


(2):*[M M 0 M x]*

[M M 0 M x] ⇒ *[0 M M M − x M − x]*

Since $0 < M - x < M$, by case 5 (1), *[0 M M M − x M − x]* has cycle convergence.

And then, *[M M 0 M x]* has cycle convergence, too.



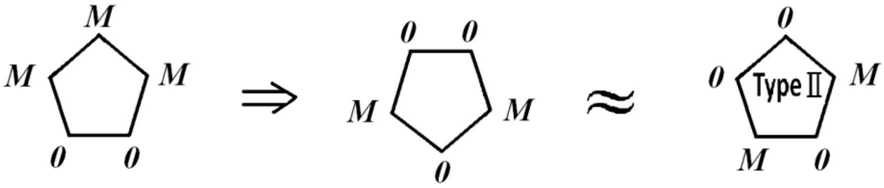
■ Case 9.

Let $n = 3$, and $t = 2$,

(1):*[M M M 0 0]*

[M M M 0 0] ⇒ *[0 0 M 0 M]* ≈ type Ⅱ

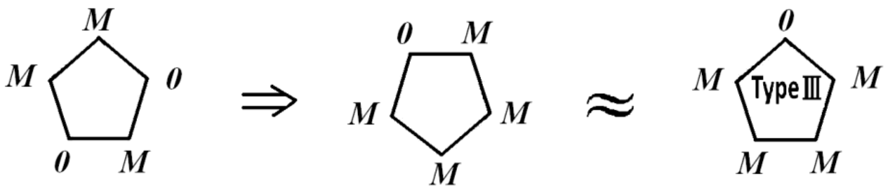
So *[M M M 0 0]* has cycle convergence.



(2):*[M M 0 M 0]*

[M M 0 M 0] = *[M 0 M M M]* ≈ type Ⅲ

So *[M M 0 M 0]* has cycle convergence.



■ Case 10.

Let $n = 4$, and $t = 1$,

(1): *[M M M M 0]*

[M M M M 0] = *[M 0 M M M]* ≈ type Ⅲ

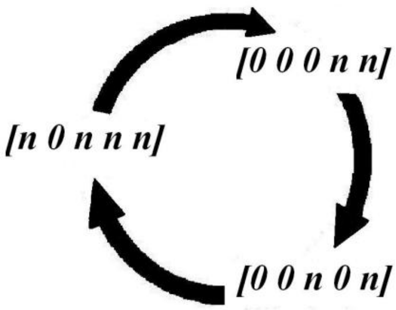
So *[M M M M 0]* has cycle convergence.

Since all the pentagons have cycle convergences with the largest value of *a, b, c, d, e* is equal *M*, by strong induction, all the pentagons must have cycle convergences if all the elements of pentagons are nonnegative integers and not all equal.

Chapter 4　Conclusion and Promotion

In our study, we find all the Diffy pentagons must have the following features if their elements are all nonnegative integers and not all equal:

1. All the Diffy pentagons must have cycle convergences.
2. All the cycle convergences must have the same type as the following figure for some integer *n*:



3. If all the values of vertices of pentagon are equal, this Diffy pentagon must converges to *[0 0 0 0 0]*

Next, if we take *[a b c d e]* to be a cycle order pentagon and assume at least one element of *a, b, c, d, e* is negative, we can subjoin one value which equal the absolute of the *min{ a, b, c, d, e }*.　Then the new pentagon must be isomorphic to the first pentagon.

Also, if at least one element is rational, we can multiply each element by the least common multiple of denomicators such that all numbers of this pentagon become integers, then we can use the method above such that all numbers of this pentagon become nonnegative integers.

By these two ways, we can easily promote our conclusion to rational number.

During our study, we still find there are some rules of cycle convergences with hexagon, heptagon, and so on.　I hope this article can help us to continue to study Diffy pentagon even further.

Appendix

Strong induction

The principle of mathematical induction asserts that P(k) being true implies P($k+1$) is true, but sometimes is not enough. Strong induction is a variant on proof by induction. It comprises of the following steps:

1. Set up a statement P(n), $n \in$ N
2. Confirm $n=1$ is true.
3. Assume P(1), P(2), P(3), ……, P(k) are true.
4. Show P($k+1$) is true.

And then, we can say P(n) is true for all $n \in$ N.

Example 1 Let P(n) be n is the product of primes, where n is integer and $n > 1$.

Proof:

It is easy to check P(2) is ok.

Assume P(2), P(3), P(4), ……, P(k) are true.

Then we just need prove $k+1$ is the product of primes.

First, if $k+1$ is a prime:

Then P($k+1$) = ($k+1$), it means P($k+1$) is true.

Second, if $k+1$ is not a prime:

Then there must has some i and $j \in$ N, such that $k+1 = i \times j$,where $1 < i, j < k$.

Since P(i), P(j) are true for all $1 < i, j < k$,

we have P($k+1$)=P(i) $\times$ P(j) is also true and the proof is completed.

Example 2 Every positive integer n can be expressed as $n = a_r 2^r + a_{r-1} 2^{r-1} + \ldots + a_2 2^2 + a_1 2 + a_0$, where a_i is 0 or 1 and r is some nonegative integer.

Proof:

For $n = 1$: Let $a_0 = 1$, ok.

Assume n is ok for all $1 \leqq n \leqq k-1$.

We just need to show $n = k$ is ok, too.

Let $n = k$ be even:

Then $\frac{k}{2}$ is an integer and $\frac{k}{2} \leqq k-1$

Let $\frac{k}{2} = a_r 2^r + a_{r-1} 2^{r-1} + \ldots + a_2 2^2 + a_1 2 + a_0$

Then $n = k = a_r 2^{r+1} + a_{r-1} 2^r + \ldots + a_2 2^3 + a_1 2^2 + a_0 2$

Let $n = k$ be odd:

Then $\frac{k-1}{2}$ is an integer and $\frac{k-1}{2} \leqq k-1$

Let $\frac{k-1}{2} = a_r 2^r + a_{r-1} 2^{r-1} + \ldots + a_2 2^2 + a_1 2 + a_0$

Then $n = k = a_r 2^{r+1} + a_{r-1} 2^r + \ldots + a_2 2^3 + a_1 2^2 + a_0 2 + 1$ and the proof is complete.

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