行政院國家科學委員會專題研究計畫 成果報告

半純函數共值與共少函數之研究
研究成果報告(精簡版)

計 畫 類 別 ：個別型
計 畫 編 號 ：NSC 98-2115-M-004-003-
執 行 期 間 ：98 年 08 月 01 日至 99 年 07 月 31 日
執 行 單 位 ：國立政治大學應用數學學系

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處 理 方 式 ：本計畫可公開查詢

中 華 民 國 99 年 10 月 19 日

ON THE SHARING VALUES AND SMALL FUNCTIONS OF MEROMORPHIC FUNCTIONS

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1. Introduction

A well-known result by Picard [**?**] says that any non-constant entire function $f(z)$ can omit at most one finite complex value, which we call a Picard exceptional value of $f(z)$. Nevanlinna generalized the idea of omitting values, and define now called the Nevanlinna deficiency $\delta(a, f)$ to measure the degree of a meromorphic function $f(z)$ "misses" the value a. We say that an extended complex value a is a deficient value of $f(z)$ if $\delta(a, f) > 0$. Under this terminology, if a is a Picard exceptional value of $f(z)$, then $\delta(a, f) = 1$.

Yang [**?**, **?**] proved that any non-constant rational function $f(z)$ has exactly one deficient value a. Also, we can easily calculate the Nevanlinna deficiency $\delta(a, f)$ for the corresponding deficient value a. For completeness, we will state Yang's results in Section 2.

To construct a meromorphic function with two deficient values, our approach is as follows. First, we consider a meromorphic function $g(z)$ with two Piacrd exceptional values a and b. Then, take a polynomial $P(z)$, and consider the meromorphic function $f(z) = P(g(z))$. We will show that $f(z)$ has at most two deficient values, and the only possible deficient values are $P(a)$ and $P(b)$. If $g(z)$ is of finite order, both $P(a)$ and $P(b)$ are deficient values of $f(z)$, and the corresponding deficiencies can be computed explicitly. While a polynomial $P(z)$ is fixed, we define $\nu(\alpha)$ to be the multiplicity of the zero of $P(z) - P(\alpha)$ at $z = \alpha$ if α is a finite complex number, and $\nu(\infty)$ to be the degree of $P(z)$.

Now, given a non-constant meromorphic function $g(z)$ with two Picard exceptional values 0 and ∞, then it is well-known that $g(z) = e^{h(z)}$, where $h(z)$ is an entire function. Moreover, if $g(z)$ is of finite order, then $h(z)$ must be a non-constant polynomial [**?**]. In this case, we have the following theorem:

Theorem A. *Let $h(z)$ be a non-constant polynomial and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have*

(i) *If $k \geq 1$, then 0 and ∞ are the only two deficient values of $f(z)$. Moreover, $\delta(0, f) = \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*
(ii) *If $k = 0$, then a_0 and ∞ are the only two deficient values of $f(z)$. Moreover, $\delta(a_0, f) = \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*

In general, we have the following result.

Theorem B. *Let $g(z)$ be a non-constant meromorphic function of finite order, such that $g(z)$ has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have*

(i) *If $P(a) = P(b)$, then $P(a)$ is the only two deficient value of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)+\nu(b)}{n}$.*
(ii) *If $P(a) \neq P(b)$ and a, b are finite, then $P(a)$ and $P(b)$ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(P(b), P(g)) = \frac{\nu(b)}{n}$.*
(iii) *If a is finite and $b = \infty$, then $P(a)$ and ∞ are the only deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(\infty, P(g)) = \frac{\nu(\infty)}{n} = 1$.*

When $g(z)$ is of infinite order, we can get similar but somewhat weaker results as Theorem A and B, which will be stated in Section 3 and 4.

The proofs of Theorem A and B, given in Section 3 and 4, are based on the theory of value distribution. We will assume the reader is familiar with the basic notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [**?**]. In particular, we will denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set E of finite linear measure.

2. The deficient values of rational functions

Let $f(z)$ be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. First, we define the Nevanlinna deficiency $\delta(a, f)$ to measure the degree of a meromorphic function $f(z)$ "misses" the value a.

Definition. *Let $f(z)$ be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. The deficiency of a with respect to $f(z)$ is defined to be*

$$\delta(a, f) = \varliminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

If $\delta(a, f) > 0$, then a is called a deficient value of $f(z)$.

Clearly, by the definition, $0 \leq \delta(a, f) \leq 1$. If $\delta(a, f)$ is much more close to 1, this means $N(r, \frac{1}{f-a})$ much smaller than $T(r, f)$. In other words, the lack of $f(z)$ at a is much more acuter. In general, it is quite difficult to find the deficient values of an arbitrary meromorphic function. However, for rational function, C. C. Yang [**?**] proved the following.

Theorem 1. *Let $f(z)$ be a non-constant rational function defined by*

$$f(z) = \frac{a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0}{b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0},$$

where $a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0$ and $b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0$ are relatively prime. Then

(i) $N(r, f) = q \log r$ *and* $N(r, \frac{1}{f}) = p \log r$.
(ii) $m(r, f) = \begin{cases} (p-q)\log r + O(1) & \text{if } p > q \\ O(1) & \text{if } p \leq q \end{cases}$

(iii) $N(r, \frac{1}{f-a}) = \begin{cases} \max\{p,q\}\log r & \text{if } p \neq q \\ p\log r & \text{if } p = q \text{ and } a_p \neq ab_q \\ k\log r & \text{if } p = q \text{ and } a_p = ab_q \text{ for some } 0 \leq k \leq p-1, \end{cases}$
where a is a non-zero complex number.

(iv) $T(r, f) = \max\{p, q\}\log r$.

It follows from Theorem **??**, we can completely classify all deficient values and their corresponding deficiency for rational functions as follows:

Corollary 1. *If $f(z)$ is a non-constant rational function, then $f(z)$ has only one deficient value $f(\infty)$. More precisely, we have the following cases:*

(i) *If $p > q$, then ∞ is the only deficient value of $f(z)$ and $\delta(\infty, f) = 1 - \frac{q}{p}$.*
(ii) *If $p < q$, then 0 is the only deficient value of $f(z)$ and $\delta(0, f) = 1 - \frac{p}{q}$.*
(iii) *If $p = q$, then $\frac{a_p}{b_q}$ is the only deficient value of $f(z)$ and $\delta(\frac{a_p}{b_q}, f) = 1 - \frac{k}{p}$, where k is the largest nonnegative integer j such that $a_j \neq ab_j$.*

3. The Proof of Theorem A

Let $g(z)$ be a non-constant meromorphic function with two Picard exceptional values 0 and ∞, so $g(z) = e^{h(z)}$, where $h(z)$ is an entire function. In this section, we study the deficient values and deficiencies of $P(g)$, where $P(z)$ is a non-constant polynomial. First, we establish some lemmas.

Lemma 1. *Let $h(z)$ be a non-constant entire function and $f(z) = a + be^{h(z)}$, where a and b are non-zero complex numbers. Then*

$$m(r, \frac{1}{f}) = S(r, e^h).$$

Proof. By the Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r, \tfrac{1}{f}) &= T(r, f) + O(1) \\ &\leq \overline{N}(r, \tfrac{1}{f}) + \overline{N}(r, \tfrac{1}{f-a}) + \overline{N}(r, f) + S(r, f) \\ &\leq \overline{N}(r, \tfrac{1}{f}) + \overline{N}(r, \tfrac{1}{be^h}) + S(r, f) \\ &\leq N(r, \tfrac{1}{f}) + S(r, f). \end{aligned}$$

Hence,

$$m(r, \frac{1}{f}) = T(r, \frac{1}{f}) - N(r, \frac{1}{f}) = S(r, f) = S(r, e^h).$$

❑

Lemma 2. *Let $h(z)$ be a non-constant entire function and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial, where a_0 and a_n are non-zero complex numbers. If $f(z) = P(e^{h(z)})$, then*

$$m(r, \frac{1}{f}) = S(r, e^h).$$

Proof. Write $P(z) = c\prod_{j=1}^{n}(z - \alpha_j)$. Clearly, $\alpha_j \neq 0$ for all $1 \leq j \leq n$. By Lemma **??**, we have

$$\begin{aligned} m(r, \tfrac{1}{f}) &= m(r, \tfrac{1}{c\prod_{j=1}^{n}(e^h - \alpha_j)}) \\ &\leq \textstyle\sum_{j=1}^{n} m(r, \tfrac{1}{e^h - \alpha_j}) + O(1) \\ &= S(r, e^h). \end{aligned}$$

❑

In order to find $m(r, \frac{1}{P(e^h)})$, we need the following fact [**?**] about the characteristic function of polynomial in a meromorphic function.

Theorem 2. *Let $g(z)$ be a non-constant meromorphic function and $P(z) = a_n z^n + \cdots + a_0$, where $a_0, \ldots, a_n$ are small functions of $g(z)$. Then*

$$T(r, P(g)) = nT(r, g) + S(r, g).$$

In particular, if $g(z)$ is of finite order, so is $P(g)$.

Now, we can express $m(r, \frac{1}{P(e^h)})$ in terms of $m(r, \frac{1}{e^h})$, which is fundamental to the proofs of Theorem A and B.

Theorem 3. *Let $h(z)$ be a non-constant entire function and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. If $f(z) = P(e^{h(z)})$, then*

$$m(r, \frac{1}{f}) = k\, m(r, \frac{1}{e^h}) + S(r, e^h).$$

Proof. Write $P(z) = z^k Q(z)$ and $Q(z) = c \prod_{j=1}^{n-k} (z - \alpha_j)$, where $\alpha_j \neq 0$ for all $1 \leq j \leq n-k$. Then, by Lemma **??** and Theorem **??**,

$$\begin{aligned}
T(r, P(e^h)) &= T(r, \tfrac{1}{P(e^h)}) + O(1) \\
&= N(r, \tfrac{1}{P(e^h)}) + m(r, \tfrac{1}{P(e^h)}) + O(1) \\
&= N(r, \tfrac{1}{Q(e^h)}) + m(r, \tfrac{1}{P(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \frac{1}{e^h - \alpha_j}) + m(r, \frac{1}{P(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \frac{1}{e^h - \alpha_j}) + k\, m(r, \frac{1}{e^h}) + m(r, \frac{1}{Q(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \frac{1}{e^h - \alpha_j}) + k\, m(r, \frac{1}{e^h}) + S(r, e^h) \\
&\leq \textstyle\sum_{j=1}^{n-k} T(r, \frac{1}{e^h - \alpha_j}) + k\, T(r, \frac{1}{e^h}) + S(r, e^h) \\
&= nT(r, e^h) + S(r, e^h) \\
&= T(r, P(e^h)) + S(r, e^h).
\end{aligned}$$

Therefore, we have equality everywhere. In particular,

$$m(r, \frac{1}{f}) = k\, m(r, \frac{1}{e^h}) + S(r, e^h).$$

❑

Now, we are ready to prove Theorem A.

Theorem A. *Let $h(z)$ be a non-constant polynomial and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have*

(i) *If $k \geq 1$, then 0 and ∞ are the only two deficient values of $f(z)$. Moreover, $\delta(0, f) = \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*

(ii) *If $k = 0$, then a_0 and ∞ are the only two deficient values of $f(z)$. Moreover, $\delta(a_0, f) = \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*

Proof. Note that $h(z)$ is a polynomial, e^h is of finite order. We have $S(r, e^h) = o(T(r, e^h))$ as $r \to \infty$. Clearly, in any case, ∞ is a deficient value of $f(z)$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.

For $k \geq 1$, we have $\nu(0) = k$ and, by Theorem **??**,

$$\begin{aligned}\delta(0, f) &= \lim_{r\to\infty} \frac{m(r, \frac{1}{f})}{T(r, f)} \\ &= \lim_{r\to\infty} \frac{km(r, \frac{1}{e^h}) + S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\ &= \frac{k}{n}.\end{aligned}$$

On the other hand, for any $a \neq 0$, by Lemma **??**, we have

$$\begin{aligned}\delta(a, f) &= \lim_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \\ &= \lim_{r\to\infty} \frac{S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\ &= 0.\end{aligned}$$

Hence, 0 and ∞ are the only two deficient values of $f(z)$ and $\delta(0, f) = \frac{k}{n}$, $\delta(\infty, f) = 1$. This proves (i).

For $k = 0$, we can write $P(z) - a_0 = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_l z^l$, where $a_l \neq 0$ and $\nu(0) = l$. As above, we have

$$\begin{aligned}\delta(a_0, f) &= \lim_{r\to\infty} \frac{m(r, \frac{1}{f-a_0})}{T(r, f)} \\ &= \lim_{r\to\infty} \frac{lm(r, \frac{1}{e^h}) + S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\ &= \frac{l}{n}.\end{aligned}$$

Moreover, for any $a \neq a_0$, by Lemma **??**, we have

$$\begin{aligned}\delta(a, f) &= \lim_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \\ &= \lim_{r\to\infty} \frac{S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\ &= 0.\end{aligned}$$

Therefore, a_0 and ∞ are the only two deficient values of $f(z)$ and $\delta(a_0, f) = \frac{l}{n}$, $\delta(\infty, f) = 1$, which proves (ii). ❑

For general transcendental entire function $h(z)$, due to the fact that $S(r, f) = o(T(r, f))$ as $r \to \infty$ and $r \notin E$, where E is a set of finite linear measure, we cannot get Theorem A. However, as in the proof of Theorem A, we have the following.

Theorem A′. *Let $h(z)$ be a transcendental entire function of infinite order and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have*

(i) *If $k \geq 1$, then $\delta(0, f) \leq \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$. In particular, 0 and ∞ are the only possible deficient values of $f(z)$.*

(ii) *If $k = 0$, then $\delta(a_0, f) \leq \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$. In particular, a_0 and ∞ are the only possible deficient values of $f(z)$.*

4. The Proof of Theorem B

Since 0 and ∞ are the Picard exceptional values of e^h, Theorem A says that $P(0)$ and $P(\infty)$ are the only deficient values of $f(z) = P(e^{h(z)})$. Hence, it is reasonable to conjecture that if e^h is replaced by any meromorphic function $g(z)$ with two Picard exceptional values a and b, then $P(a)$ and $P(b)$ are the only deficient values of $P(g)$. Indeed, it is true. First, we need some lemmas.

Lemma 3. *Let g be a non-constant meromorphic function with two Picard exceptional values a and b. Then*

$$m(r, \frac{1}{g-\alpha}) = S(r, g)$$

for any $\alpha \in \mathbb{C}_\infty \setminus \{a, b\}$.

Proof. Given $\alpha \in \mathbb{C}_\infty \setminus \{a, b\}$. We may assume that α, a and b are finite. By the Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r, \tfrac{1}{g-\alpha}) &= T(r, g) + O(1) \\ &\leq \overline{N}(r, \tfrac{1}{g-\alpha}) + \overline{N}(r, \tfrac{1}{g-a}) + \overline{N}(r, \tfrac{1}{g-b}) + S(r, g) \\ &\leq \overline{N}(r, \tfrac{1}{g-\alpha}) + S(r, g) \\ &\leq N(r, \tfrac{1}{g-\alpha}) + S(r, g). \end{aligned}$$

Hence,

$$m(r, \frac{1}{g-\alpha}) = T(r, \frac{1}{g-\alpha}) - N(r, \frac{1}{g-\alpha}) = S(r, g).$$

❑

The following theorem is fundamental in finding the deficiency of $P(g)$.

Theorem 4. *Let $g(z)$ be a non-constant meromorphic function with two finite Picard exceptional values a, b and let $P(z)$ be a non-constant polynomial of degree n. We have*

(i) *If $P(a) \neq P(b)$, then*

$$m(r, \frac{1}{P(g) - P(a)}) = \nu(a) m(r, \frac{1}{g-a}) + S(r, g), \ \text{and}$$

$$m(r, \frac{1}{P(g) - P(b)}) = \nu(b) m(r, \frac{1}{g-b}) + S(r, g).$$

(ii) *If $P(a) = P(b)$, then*

$$m(r, \frac{1}{P(g) - P(a)}) = (\nu(a) + \nu(b)) m(r, \frac{1}{g-a}) + S(r, g).$$

Proof. Denote $k_1 = \nu(a)$ and $k_2 = \nu(b)$. Then we can write

$$P(z) - P(a) = c(z-a)^{k_1} \prod_{i=1}^{n-k_1} (z - \alpha_i)$$

and

$$P(z) - P(b) = c(z-b)^{k_2} \prod_{j=1}^{n-k_2} (z - \beta_j),$$

where $\alpha_i \neq a$ for all $1 \leq i \leq n - k_1$, and $\beta_j \neq b$ for all $1 \leq j \leq n - k_2$.

Note that if $P(a) \neq P(b)$, then $\alpha_i \neq a, b$ for all $1 \leq i \leq n-k_1$ and $\beta_j \neq a, b$ for all $1 \leq j \leq n-k_2$. By Lemma **??** and Thoerem **??**, we have

$$\begin{aligned}
T(r, P(g)) &= T(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&= N(r, \tfrac{1}{P(g)-P(a)}) + m(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} N(r, \tfrac{1}{g-\alpha_i}) + m(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} N(r, \tfrac{1}{g-\alpha_i}) + k_1\, m(r, \tfrac{1}{g-a}) + \sum_{i=1}^{n-k_1} m(r, \tfrac{1}{g-\alpha_i}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} T(r, \tfrac{1}{g-\alpha_i}) + k_1\, T(r, \tfrac{1}{g-a}) + S(r, g) \\
&= nT(r, g) + S(r, g) \\
&= T(r, P(g)) + S(r, g).
\end{aligned}$$

Hence,

$$m(r, \frac{1}{P(g)-P(a)}) = k_1\, m(r, \frac{1}{g-a}) + S(r, g).$$

Similarly, we have

$$m(r, \frac{1}{P(g)-P(b)}) = k_2\, m(r, \frac{1}{g-b}) + S(r, g).$$

This proves (i).

If $P(a) = P(b)$, then we can write

$$P(z) - P(a) = c(z-a)^{k_1}(z-b)^{k_2} \prod_{j=1}^{n-k_1-k_2} (z - \gamma_j),$$

where $\gamma_j \neq a, b$ for all $1 \leq j \leq n - k_1 - k_2$. As in the proof of (i), we still get

$$m(r, \frac{1}{P(g)-P(a)}) = (k_1 + k_2) m(r, \frac{1}{g-a}) + S(r, g),$$

which proves (ii). ❑

In Theorem **??**, we assume that both a and b are finite values. If one of a and b is ∞, say $b = \infty$, then $P(a) \neq P(b)$ and $P(g)$ is entire. So, as in the proof of Theorem **??**, we have

Theorem 4′. *Let g be a non-constant meromorphic function with two Picard exceptional values a and ∞ and let $P(z)$ be a non-constant polynomial of degree n. Then we have*

$$m(r, \frac{1}{P(g)-P(a)}) = \nu(a) m(r, \frac{1}{g-a}) + S(r, g)$$

and

$$m(r, P(g)) = \nu(\infty) m(r, g) + S(r, g) = T(r, P(g)).$$

Now, we are in the position to prove Theorem B.

Theorem B. *Let $g(z)$ be a non-constant meromorphic function of finite order, such that $g(z)$ has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have*

- (i) *If $P(a) = P(b)$, then $P(a)$ is the only deficient value of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)+\nu(b)}{n}$.*
- (ii) *If $P(a) \neq P(b)$ and a, b are finite, then $P(a)$ and $P(b)$ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(P(b), P(g)) = \frac{\nu(b)}{n}$.*

(iii) *If a is finite and $b=\infty$, then $P(a)$ and ∞ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a),P(g))=\frac{\nu(a)}{n}$ and $\delta(\infty,P(g))=\frac{\nu(\infty)}{n}=1$.*

Proof**.** Note that $g(z)$ is of finite order, so is $P(g)$ by Theorem **??**. We have $S(r,g)=o(T(r,g))$ as $r\to\infty$.

If $P(a)=P(b)$, then a and b must be finite values. By Theorem **??**, we get

$$\delta(P(a),P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-P(a)})}{T(r,P(g))}=\frac{\nu(a)+\nu(b)}{n}.$$

On the other hand, for any $\alpha\neq P(a)$, we can write $P(z)-\alpha=c\prod_{j=1}^{n}(z-\alpha_j)$, where $\alpha_j\neq a,b$ for all $1\le j\le n$. Then, by Lemma **??**, we have

$$\delta(\alpha,P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-\alpha})}{T(r,P(g))}=\varliminf_{r\to\infty}\frac{S(r,g)}{T(r,P(g))}=0.$$

Therefore, $P(a)$ is the only deficient value of $P(g)$ and $\delta(P(a),P(g))=\frac{\nu(a)+\nu(b)}{n}$. This proves (i).

If $P(a)\neq P(b)$ and a, b are finite, then, by Theorem **??**, we have

$$\delta(P(a),P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-P(a)})}{T(r,P(g))}=\frac{\nu(a)}{n}$$

and

$$\delta(P(b),P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-P(b)})}{T(r,P(g))}=\frac{\nu(b)}{n}.$$

Moreover, as in the proof of (i), for any $\alpha\neq P(a),P(b)$, we have

$$\delta(\alpha,P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-\alpha})}{T(r,P(g))}=\varliminf_{r\to\infty}\frac{S(r,g)}{T(r,P(g))}=0.$$

Therefore, $P(a)$ and $P(b)$ are the only deficient values of $P(g)$ and $\delta(P(a),P(g))=\frac{\nu(a)}{n}$, $\delta(P(b),P(g))=\frac{\nu(b)}{n}$. This proves (ii).

Finally, if a is finite and $b=\infty$, then, by Theorem 4′, we have

$$\delta(P(a),P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-P(a)})}{T(r,P(g))}=\frac{\nu(a)}{n}$$

and

$$\delta(\infty,P(g))=\varliminf_{r\to\infty}\frac{m(r,P(g))}{T(r,P(g))}=\frac{\nu(\infty)}{n}=1.$$

Moreover, as in the proof of (i), for any $\alpha\neq P(a),P(b)$, we have

$$\delta(\alpha,P(g))=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{P(g)-\alpha})}{T(r,P(g))}=\varliminf_{r\to\infty}\frac{S(r,g)}{T(r,P(g))}=0.$$

Therefore, $P(a)$ and ∞ are the only deficient values of $P(g)$ and $\delta(P(a),P(g))=\frac{\nu(a)}{n}$, $\delta(P(a),P(g))=1$, which proves (iii). ❑

For arbitrary meromorphic function $g(z)$ with two Picard exceptional values, as the reasoning in the end of Section 3, we have the following result.

Theorem B′. *Let $g(z)$ be a non-constant meromorphic function of infinite order, such that $g(z)$ has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have*

(i) *If* $P(a) = P(b)$*, then* $\delta(P(a), P(g)) \leq \frac{\nu(a)+\nu(b)}{n}$*. In particular,* $P(a)$ *is the only possible deficient value of* $P(g)$*.*

(ii) *If* $P(a) \neq P(b)$ *and* a, b *are finite, then* $\delta(P(a), P(g)) \leq \frac{\nu(a)}{n}$ *and* $\delta(P(b), P(g)) \leq \frac{\nu(b)}{n}$*. In particular,* $P(a)$ *and* $P(b)$ *are the only possible deficient values of* $P(g)$*.*

(iii) *If* a *is finite and* $b = \infty$*, then* $\delta(P(a), P(g)) \leq \frac{\nu(a)}{n}$ *and* $\delta(\infty, P(g)) = \frac{\nu(\infty)}{n} = 1$*. In particular,* $P(a)$ *and* ∞ *are the only possible deficient values of* $P(g)$*.*

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無研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：陳天進	計畫編號：98-2115-M-004-003-
計畫名稱：半純函數共值與共少函數之研究	

成果項目			量化：實際已達成數（被接受或已發表）	量化：預期總達成數(含實際已達成數)	量化：本計畫實際貢獻百分比	單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
國內	論文著作	期刊論文	1	1	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
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		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
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其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以 100 字為限）

   　　□實驗失敗

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   　　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：■已發表 □未發表之文稿 □撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   We study the deficient values and the Nevanlinna deficiencies of meromorphic function f(z) of the form f(z)=P(g(z)), where P(z) is a polynomial and g(z) is a meromorphic function with two Picard exceptional values. In fact, we prove that all deficient values of f(z) are of the form $(a), where a is a Picard exceptional value of g(z). Furthermore, if g(z) is of finite order, we compute explicitly the Nevanlinna deficiencies for f(z). We also give an upper bound estimation for the deficiencies of f(z) while g(z) is of infinite order.