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SEMILINEAR WAVE EQUATIONS

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Stability of positive solutions for some semilinear wave equations under nonlinear perturbation near blow-up solutions in 1-space dimension

$$\Box\ \mathrm{u}\ -\ \mathrm{u}^p+\lambda \mathrm{u}^q\ =\ 0\quad \text{(I)}$$

Abstract

In this research we treat the stability of positive solutions of some particular semilinear wave equations under nonlinear perturbation in bounded domain near blow-up solutions in 1-space dimension.

1.Introduction

In this paper we want to study the stability of positive solutions for the semilinear wave equation

$$\Box\, u = u^p + \lambda u^q \text{ in } [0,T)\times(R_1,R_2) \tag{0.1}$$

with boundary value null and initial values $u(0,x) = u_0(x) \in H^2(R_1,R_2)\cap H_0^1(R_1,R_2)$ and $\dot{u}(0,x) = u_1(x) \in H_0^1(R_1,R_2)$, where $p,q\in(1,\infty)$ and $(R_1,R_2)\subset\mathbb{R}$.

We will use the following notations:

$$\cdot := \frac{\partial}{\partial t}, Du := \left(\dot{u}, \frac{\partial u}{\partial x}\right), \Box u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2},$$

$$a(t) := \int_{R_1}^{R_2} u^2(t,x)\,dx, E_\lambda(t) := \int_{R_1}^{R_2}\left(|Du|^2 - \frac{2}{p+1}u^{p+1} - \frac{2\lambda}{q+1}u^{q+1}\right)(t,x)\,dx.$$

For a Banach space X and $0<T\le\infty$ we set

$$C^k(0,T,X) = \text{Space of } C^k - \text{ functions} : [0,T)\to X,$$
$$H1 := C^1\left(0,T,H_0^1(R_1,R_2)\right)\cap C^2\left(0,T,L^2(R_1,R_2)\right).$$

The existence result to the equation (0.1) is proved $[Li\ 3]$ and the positive solution blows-up in finite time if $\lambda \geq 0$ $[Li\ 2]$, this means that the positive solutions for the semilinear wave equation

$$\begin{aligned} \Box\, u &= u^p \text{ in } [0,T) \times (R_1, R_2)\,, \\ u(0,x) &= u_0(x) \in H^2(R_1,R_2) \cap H_0^1(R_1,R_2)\,, \\ \dot{u}(0,x) &= u_1(x) \in H_0^1(R_1,R_2)\,, \end{aligned} \tag{0.2}$$

is stable under nonlinear perturbation λu^q providing $p > 1, q > 1,\ \lambda > 0$; but it is not clearly whether it is also true for any $p > 1, q > 1,\ \lambda < 0$? If so, we would want to estimate the blow-up time and the blow-up rate under such a situation.

It is also important to study the asymptotic behavior of the solution u_λ; the velocity and the rate of the approximation for λ approaches to zero.

Such questions are also not easy to answer even under the case for the ordinary differential equation

$$\begin{aligned} u'' &= u^p\left(c + \lambda u'(t)^q\right), \\ u(0) &= u_0, u'(0) = u_1, \\ p &> 1, q > 1, c > 0, \lambda > 0. \end{aligned} \tag{0.3}$$

We have studied the blow-up behavior of the solution for problem (0.3) and got some estimates on blow-up time and blow-up rate [Li4] but it is difficult to find the real blow-up time (life-span). Further literature could be fund in [S], [R], [W1] and [W2].

In this study we hope that our ideals used in [Li 4], [Li 5] can do help us dealing such problem (0.1) on our topics.

1. **Definition and Fundamental Lemma**

There are many definitions of the weak solutions of the initial-boundary problems of the wave equation, we use here as following.

Definition 1.1: *For* $p > 1$, $u \in H1$ *is called a positive weakly solution of equation* (0.1)*, if*

$$\int_0^t \int_{R_1}^{R_2} \left[\begin{array}{c} \dot{u}(r,x)\,\dot{\varphi}(r,x) - \frac{\partial}{\partial x}u(r,x)\,\frac{\partial}{\partial x}\varphi(r,x) \\ +\left(u^p + \lambda u^q\right)(r,x)\,\varphi(r,x) \end{array} \right] dxdr = 0 \ \forall \varphi \in H1$$

and

$$\int_0^t \int_{R_1}^{R_2} u(r,x)\,\psi(r,x)\,dxdr \geq 0$$

for each positive $\psi \in C_0^\infty([0,T) \times (R_1,R_2))$.

Remark 1.2:
1) Our definition 1.1 is resulted from the multiplication with φ to the equation (0.1) and integration in (R_1,R_2) from 0 to t.
2) From the local Lipschitz functions $u^p + \lambda u^q, p > 1, q > 1$ the initial-boundary value problem (0.1) possesses a unique solution in $H1$ [Li1]. Hereafter we use the notations:

$$\frac{1}{C_{(R_1,R_2)}} := \eta_1 = \sup\left\{ \|u\|_2 / \left\| \frac{\partial u}{\partial x} \right\|_2 : u \in H_0^1(R_1,R_2) \right\},$$

$$\lambda_q \ = \sup\left\{ \|u\|_q / \left\| \frac{\partial u}{\partial x} \right\|_2 : u \in H_0^1(\Omega) \cap L_q(\Omega) \right\}, q > 1.$$

In this study we need the following lemmas

Lemma 1.3: *Suppose that* $u \in H1$ *is a weakly positive solution of* (SL) with $E_\lambda(0) = 0$ for $p > 1, q > 1$, *then for* $a(0) > 0$ *we have*:

(i) $a \in C^2(\mathbb{R}^+)$ *and* $E_\lambda(t) = E_\lambda(0) \quad \forall t \in [0,T)$.

(ii) $a'(t) > 0 \quad \forall t \in [0,T)$, *provided* $a'(0) > 0$.

(iii) $a'(t) > 0 \quad \forall t \in (0,T)$, *if* $a'(0) = 0$.

(iv) *For* $a'(0) < 0$, *there exists a constant* $t_0 > 0$ *with*

$$a'(t) > 0 \quad \forall t > t_0$$

and $a'(t) = 0$.

Lemma 1.4: *Suppose that* u *is a positive weakly solution in* $H1$ *of equation (0.1) with* $u(0,\cdot) = 0 = \dot{u}(0,\cdot)$ in $L^2(R_1,R_2)$*. For* $p > 1, q > 1, \lambda > 0$*, we have* $u \equiv 0$ *in* $H1$.

According to Lemma 1.4, we discuss the following theme

(3) $E_\lambda(0) = 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

(4) $E_\lambda(0) < 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

We should study the problem (0.1) under the following situations:

2. Estimates for the Life-Span of the Solutions of (0.1) under Null-Energy

We study the case that $E_\lambda(0) = 0,\ \ p > 1, q > 1, \lambda > 0$ and divide it into two parts

(i) $a(0) > 0, a'(0) \geq 0$
and
(ii) $a(0) > 0, a'(0) < 0$.

Remark 2.1. 1) The local existence and uniqueness of solutions of equation (0.1) in $H1$ are known [Li2].

2) For higher space dimensional special cases under general bounded domain $\Omega \subset \mathbb{R}^n,\ \lambda = 0$:

i) If $n = 2$, $p > 1$ and $E_\lambda(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded by

$$T \leq \alpha_1 := k_2^{-1} \sin^{-1}\left(\frac{k_2}{k_1 a^{\frac{p-1}{4}}(0)}\right)$$

with

$$k_1 := \frac{p-1}{4} \cdot a^{-\frac{p-1}{4}}(0)\sqrt{a'(0)\, a^{-2}(0) + 4C_\Omega^2},\ k_2 := \frac{p-1}{2} C_\Omega,$$
$$\frac{1}{C_\Omega} := \eta_1 = \sup\left\{\|u\|_2 / \|Du\|_2 : u \in H_0^1(\Omega)\right\}.$$

ii) For $n = 3$, $p = 2$ and $E_\lambda(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded

$$T \leq \alpha_2 := 2\eta_1 \sin^{-1}\left[2C_\Omega\left(a'(0)^2 a(0)^{-2} + 4C_\Omega^2\right)^{-\frac{1}{2}}\right]$$

for some constant C_Ω. If $T = \alpha_2$, then $a(t) \to \infty, t \to T$.

iii) For $n = 3$, $p = 3, E_\lambda(0) = 0$ the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded

$$T \leq \alpha_3 := \eta_1 \sin^{-1}\left[2C_\Omega\left(a'(0)^2 a(0)^{-2} + 4C_\Omega^2\right)^{-\frac{1}{2}}\right].$$

If $T = \alpha_3$, then $a(t) \to \infty, t \to T$.

iv) For $a'(0) = 0$, we have

$$\alpha_1 = \frac{\pi}{p-1} C_\Omega.$$

v) For $|\Omega| \to \infty$, we have also $\alpha_1 \to \dfrac{1}{p-1}\dfrac{a(0)}{a'(0)}$.

As $|\Omega| \to 0$, then $\alpha_1 \to \dfrac{2}{p-1}\sin^{-1}\left(\dfrac{1}{4}C_\Omega^{-1}\right)$.

3. **Estimates for the Life-Span of the Solutions of equation (0.1) under Negativ-Energy**

We use the following result and those argumentations of proof are not true for positive energy, so under positive energy we need another method to show the similar results.

Lemma 3: *Suppose that $u \in H1$ is a positive weakly solution of equation (0.1) with $a(0) > 0$ and $E_\lambda(0) < 0$ for $\lambda = 0$. Then*

(i) for $a'(0) \geq 0$, we have

$$a'(t) > 0 \ \ \forall t > 0.$$

(ii) for $a'(0) < 0$, there exists a constant $t_5 > 0$ with

$$a'(t) > 0 \ \ \forall t > t_5, \ a'(t_5) = 0$$

and

$$t_5 \leq t_6 := \frac{-a'(0)}{(p-1)(\delta^2 - E_\lambda(0))},$$

where δ is the positive root of the equation

$$\frac{2}{p+1}\lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E_\lambda(0) = 0.$$

4. **Stability of positive solutions of equation (0.1) near blow-up solutions under Negativ-Energy**

In this study we use our ideals used in $[Li4], [Li5]$ and $[Li\ 7]$to deal such problem (0.1) on our topics under negative energy and obtain the following results:

Theorem 4.1: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (SL) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1,$ *then for* $a(0) > 0$ *we have*:

The equation (0.1) *is stable for* $\lambda \to 0^+$; *this means that weakly positive solution* u_λ *of* (SL) *blows up in finite time for* $\lambda \to 0^+$.

Theorem 4.2: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (SL) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1,$ *then for* $a(0) > 0$ *we have*:

The equation (0.1) *is stable for* $p > q$, $\lambda \to 0^-$; *this means that weakly positive solution* u_λ *of* (SL) *blows up in finite time for* $p > q$, $\lambda \to 0^-$.

Theorem 4.3: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (SL) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1,$ *then for* $a(0) > 0$ *we have*:

The equation (0.1) *is unstable for* $p < q$, $\lambda \to 0^-$; *this means that some weakly positive solution* u_λ *of* (SL) *blow up in finite time for* $p < q$, $\lambda \to 0^-$; *but also there were some global weakly positive solution* u_λ *of* (SL) *for* $p < q$, $\lambda \to 0^-$.

Remarks:

The decade rate of the difference of life-spans T_λ of u_λ and T of u, can not be estimated very well for $\lambda \to 0$; thus it will be a good topic on asymptotic behavior near the blow-up solutions.

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