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Ferguson-Dirichlet 過程在 n 維空間上的隨機函數分配
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行政院國家科學委員會補助專題研究計畫 ■ **成 果 報 告**
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On the distribution of the random functional of a Ferguson-Dirichlet process over the n-dimensional space

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中文摘要

自從Ferguson於1973年提出 Ferguson-Dirichlet 過程後，就一直有很多的研究者探討它的隨機函數，但是大部份的研究者都專注於低維度問題的探討。本研究中，我們探討並且提出用於得到 Ferguson-Dirichlet 過程在任何n維球體表面的隨機函數之機率密度函數的一套理論，這對目前僅有低維度的結果而言，是一個重要的擴充。

關鍵詞: Ferguson-Dirichlet過程；對稱分配；隨機函數；c-特徵函數

Abstract

Ferguson-Dirichlet process was first introduced by Ferguson (1973). Since then, many researchers have studied its random functional. However, most of them focus on the low dimensions. In this research, we study and give a unified theory to find the probability density functions of the random functional of the Ferguson-Dirichlet process over the spherical surface of any n-dimensional ball. This would be a very important generalization of the current low dimensional results.

Keywords: Ferguson-Dirichlet process, symmetrical distribution, random functional, c-characteristic function.

1 Introduction

The study of distribution of random functional of the Ferguson-Dirichlet process has been in active research in the recent decades, e.g., Hannum, Hollander, and Langberg (1981), Yamato (1984), Jiang (1988), Diaconis and Kemperman (1996), Muliere and Tardella (1998), Nomachi and Yamato (1999), and Jiang, Dickey, and Kuo (2004). Most of these papers study the cases on real line. Although Jiang, Dickey, and Kuo (2004) and Jiang and Kuo (2008a) study the cases in higher dimensions, the results are still restricted to the cases in two or three dimensions only. We give a unify theory and many interesting results in the spherical surface of any n-dimensional ball in section 2. We then give conclusions in section 3.

2 Random function of a Ferguson-Dirichlet process with parameter measure over the surface of a n-dimensional ball.

Let μ be a finite non-null measure on$(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of Euclidean space Ω, and let U be a stochastic process indexed by elements of $\mathcal{B}$. We say U is a Ferguson-Dirichlet process with parameter μ, denoted by $U \sim D(\mu)$ on Ω, if for every finite measurable partition $B_1, ..., B_m$ of Ω (i.e., the $B_i's$ are measurable, disjoint, and $\bigcup_{i=1}^{m} B_i = \Omega$), the random vector $(U(B_1), \ldots, U(B_m))$ has a Dirichlet distribution with parameter vector$(\mu(B_1), ..., \mu(B_m))$, where$\sum_{i=1}^{m} U(B_i) = 1$.

Let $\boldsymbol{X}$ be a N-dimensional random vector defined as

$$\boldsymbol{X} = \int_{\Omega} \boldsymbol{\ell}(\boldsymbol{y}) dU(\boldsymbol{y}) = (\int_{\Omega} \ell_1(\boldsymbol{y}) dU(\boldsymbol{y}), ..., \int_{\Omega} \ell_N(\boldsymbol{y}) dU(\boldsymbol{y}))', \tag{2.1}$$

where $U \sim D(\mu)$ on Ω, $\mu(\Omega) = c$, $\boldsymbol{y} = (\boldsymbol{y}_1, ..., \boldsymbol{y}_L)'$, $\boldsymbol{\ell}(\boldsymbol{y}) = (\ell_1(\boldsymbol{y}), ..., \ell_N(\boldsymbol{y}))'$, and the $\ell_n(\boldsymbol{y})'s$ are bounded measurable real-valued functions defined on Ω.

The following lemma gives the c-characteristic function expression for $\mathbf{X}$.

Lemma 2.1. *The c-characteristic function of $\boldsymbol{X}$, as in Eq.(2.1), can be expressed by*

$$g(\boldsymbol{t}; \boldsymbol{X}, c) = exp[-\int_{\Omega} \ln(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}(\boldsymbol{y})) d\mu(\boldsymbol{y})],$$

where $\boldsymbol{t} = (t_1, \ldots, t_N)'$.

The following corollary can be obtained by applying the above Lemma 2.1 and Jiang,Dickey,and Kuo(2004).

Corollary 2.2. *Let $\boldsymbol{X} = (X_1, \ldots, X_N)'$ and $\boldsymbol{\ell}(\boldsymbol{y}) = (\ell_1(\boldsymbol{y}), ..., \ell_N(\boldsymbol{y}))'$, which are defined in*

Eq.(2.1). Then

$$E(X_n) = \frac{1}{c}\int_{\Omega} \ell_n(\boldsymbol{y})d\mu(\boldsymbol{y}) \quad for \quad 1 \leqslant n \leqslant N$$

and

$$Cov(X_n, X_m) = \frac{1}{c+1}[\frac{1}{c}\int_{\Omega} \ell_n(\boldsymbol{y})\ell_m(\boldsymbol{y})d\mu(\boldsymbol{y}) - (\frac{1}{c}\int_{\Omega} \ell_n(\boldsymbol{y})d\mu(\boldsymbol{y}))(\frac{1}{c}\int_{\Omega} \ell_m(\boldsymbol{y})d\mu(\boldsymbol{y}))] \quad for \quad 1 \leqslant n \leqslant N$$

The following corollary is obtained immediately from Corollary 2.2.

Corollary 2.3. *Let $\boldsymbol{Y}$ be a random vector associated with the probability measure μ/c.Then the mean vector and the variance-covariance matrix of $\boldsymbol{X}$, Eq.(2.1), can be expressed as*

$$E(\boldsymbol{X}) = E(\boldsymbol{\ell}(\boldsymbol{Y})) \quad and \quad Cov(\boldsymbol{X}) = \frac{1}{1+c}Cov(\boldsymbol{\ell}(\boldsymbol{Y}))$$

respectively.

In the following, we want to study the distributions of the random functionals,

$$\widetilde{\boldsymbol{S}}_{n,r} = \int_{S_{n,r}} \boldsymbol{y} dU_{n,r}(\boldsymbol{y}) \qquad U_{n,r} \sim D(\mu_n, r) \tag{2.2}$$

where n is any positive integer, $S_{n,r} = \{y \in \mathbb{R}^n ||y| = r\}$, and $r > 0$, is the spherical surface of the n-dimensional ball, and where $\mu_{n,r}$ is the usual Lebesgue measure(i.e., usual rotation-invariant measure) on $S_{n,r}$ with total measure c.

First, we give the c-characteristic function expression for $\widetilde{\boldsymbol{S}}_{n,r}$ in the next theorem.

Theorem 2.4. *The c-characteristic functions of $\widetilde{\boldsymbol{S}}_{n,r}$ in Eq.(2.2) can be expressed as*

$$g(t; \widetilde{\boldsymbol{S}}_{n,r}, c) = exp\{\sum_{k=1}^{\infty} \frac{c(1/2,k)}{2k(n/2,k)}[-r^2(t_1^2 + ... + t_n^2)]^k\}$$

where $t = (t_1, \ldots, t_n)'$.

Theorem 2.4 shows that the c-characteristic function of $\widetilde{\boldsymbol{S}}_{n,r}$ is a function of $|t|$ and c, only. It, then, can be shown that $\widetilde{\boldsymbol{S}}_{n,r}$ has a spherical distribution. Moreover, the c-characteristic function of any one-dimensional margin of $\widetilde{\boldsymbol{S}}_{n,r}$, say $\widetilde{S}_{n,r}$, is expressed as

$$g(t;\widetilde{S}_{n,r},c) = exp\{\sum_{k=1}^{\infty}\frac{c(1/2,k)}{2k(n/2,k)}(-r^2t^2)^k\} \tag{2.3}$$

Our aim now is to find the probability density function of $\widetilde{\boldsymbol{S}}_{\boldsymbol{n,r}}$. Lord (1954) demonstrated that a spherical symmetric distribution can be determined by its marginal distribution. Therefore, if the probability density function of the marginal random variable $\widetilde{S}_{n,r}$ is known, then we can obtain the probability density function of the random vector $\widetilde{\boldsymbol{S}}_{n,r}$. Before giving the probability density function of $\widetilde{S}_{n,r}$, we show that an interesting random mean of a Ferguson-Dirichlet process and another related random mean have the same c-characteristic function, and hence the same distribution, as the one-dimensional marginal $\widetilde{S}_{n,r}$.

Lemma 2.5. *Let n be any positive integer and $c > 0$. Suppose that $\widetilde{R}_{n,r} = \int_{-r}^{r} ydW_{n,r}(y)$ where $W_{n,r} \sim D(\alpha_{n,r})$ and let $\alpha_{n,r}$ be the measure on $\{-r,r\}$ with $\alpha_{1,r}(\{-r\}) = \alpha_{1,r}(\{r\}) = c/2$ if $n =$ 1, and be the measure on $(-r,r)$ with density*

$$d\alpha_{n,r}(y) = \frac{c(r^2-y^2)^{(n-3)/2}}{B(1/2,(n-1)/2)r^{n-2}}dy$$

if $n > 1$. Then the c-characteristic function of $\widetilde{R}_{n,r}$ can be expressed as

$$g(t;\widetilde{R}_{n,r},c) = exp\{\sum_{k=1}^{\infty}\frac{c(1/2,k)}{2k(n/2,k)}(-r^2t^2)^k\}$$

In particular, $g(t;\widetilde{R}_{1,r},c) = (1+r^2t^2)^{-c/2}$

Corollary 2.6. *Let $\widetilde{S}_{n,r}$ be any one-dimensional margin of $\widetilde{S}_{n,r}$ in Theorem 2.4. Then $\widetilde{S}_{n,r}$ has the same distribution as $\widetilde{R}_{n,r}$ in Lemma 2.5.*

Next, we provide the probability density functions of $\widetilde{R}_{n,r}$, which can be derived by using the inversion formula of the c-characteristic function (Jiang and Kuo(2008b)), when c = 1.

Lemma 2.7. *Let f(x; n, r) be the probability density function of $\widetilde{R}_{n,r}$.*

Then, for $-r < x < r$,

(*a*) $f(x;1,r) = \dfrac{1}{\pi\sqrt{r^2-x^2}}$,

that is, $\widetilde{R}_{1,r}$ *is distributed as* $-ru_1 + ru_2$ *where* (u_1, u_2) *has a Dirichlet distribution with parameter vector* $(1/2, 1/2)$*,and* $\widetilde{R^2}_{1,r}/r^2$ *is distributed as* $Beta(1/2, 1/2)$*;*

(*b*) $f(x;2,r) = \dfrac{2\sqrt{r^2-x^2}}{r^2\pi}$, *that is,* $\widetilde{R^2}_{2,r}/r^2$ *is distributed as* $Beta(1/2, 3/2)$*;*

(*c*) $f(x;3,r) = \dfrac{e}{\pi}(r+x)^{-(r+x)/(2r)}(r-x)^{-(r-x)/(2r)}\cos\dfrac{\pi x}{2r}$*;*

(*d*) $f(x;4,r) = \dfrac{2}{r\pi}e^{1/2-x^2/r^2}\sin(\dfrac{x\sqrt{r^2-x^2}}{r^2} + 2\arcsin\sqrt{\dfrac{r+x}{2r}})$*;*

More generally, we have

(*e*) *when n is an odd integer greater than 3,*

$$\begin{aligned} f(x;n,r) =& \frac{1}{r\pi}exp\Bigg\{\frac{-1}{B(1/2,(n-1)/2)}\sum_{k=0}^{(n-3)/2}\begin{pmatrix}\frac{n-3}{2}\\ k\end{pmatrix}\frac{(-1)^k}{2k+1} \\ &\times\Bigg[\left(1+\frac{x^{2k+1}}{r^{2k+1}}\right)\ln\left(1+\frac{x}{r}\right)+\left(1-\frac{x^{2k+1}}{r^{2k+1}}\right)\ln\left(1-\frac{x}{r}\right) \\ &-2\sum_{m=0}^{k}\frac{x^{2k-2m}}{(2m+1)r^{2k-2m}}\Bigg]\Bigg\}\sin\left(\int_{-r}^{x}\pi d\alpha_{n,r}(y)\right), \quad -r<x<r; \end{aligned}$$

(*f*) *when n is an even integer greater than 4,*

$$\begin{aligned} f(x;n,r) =& \frac{2}{r\pi}exp\Bigg\{\frac{2^{3-n}\pi}{B(1/2,(n-1)/2)}\sum_{k=0}^{(n-4)/2}\begin{pmatrix}n-2\\ k\end{pmatrix}\frac{\cos[(n-2-2k)\arcsin(x/r)]}{n-2-2k}\Bigg\} \\ &\times\sin\left(\int_{-r}^{x}\pi d\alpha_{n,r}(y)\right), \quad -r<x<r. \end{aligned}$$

By Lemma 2.7 and Lord (1954), we have the following theorem.

Theorem 2.8. *Let* $h_{\widetilde{\boldsymbol{S}}}(\boldsymbol{x};n,r)$ *be the probability density functions of* $\widetilde{\boldsymbol{S}}_{n,r} = \int_{S_n}\boldsymbol{y}dU_{n,r}(\boldsymbol{y})$ *defined in Eq.(2.2) where* $\boldsymbol{x} = (x_1, ..., x_n)'$*,* $n \geq 2$*, and* $\mu_{n,r}(S_{n,r}) = 1$*.*

(*a*) *When n is odd,*

$$h_{\tilde{\mathbf{S}}}(x;n,r) = \left(\frac{-1}{2\pi t}\frac{d}{dt} \right)^m f(t;n,r) \qquad |x| < r$$

with $t = \sqrt{x_1^2 + ... + x_n^2}$ *and* $m = (n-1)/2$

(*b*) *When n is even,*

$$h_{\tilde{\mathbf{S}}}(x;n,r) = \left(\frac{-1}{2\pi s}\frac{d}{ds} \right)^m \left(\frac{-1}{\pi} \int_s^r \frac{f'(t;n,r)}{\sqrt{t^2 - s^2}} \right) \qquad |\boldsymbol{x}| < r$$

with $s = \sqrt{x_1^2 + ... + x_n^2}$ *and* $m = (n-1)/2$,

where f(t; n, r) is given in Lemma 2.7 and

$$\left(\frac{-1}{2\pi t}\frac{d}{dt} \right)^m f(t;n,r) = \frac{-1}{2\pi t}\frac{d}{dt}\left[\left(\frac{-1}{2\pi t}\frac{d}{dt} \right)^{m-1} f(t;n,r) \right]$$

for $m \geq 1$,*where* $\left(\frac{-1}{2\pi t}\frac{d}{dt} \right)^0 \equiv 1$.

3 Conclusion

Through the c-characteristic function, we have given a new approach for studying spherical distributions. The c-characteristic function expression of any functional of any Ferguson-Dirichlet process has also been given. In addition, we have given the exact distribution in n dimensions of the random mean of a symmetric Ferguson-Dirichlet process with parameter measure over any spherical surface.

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Some new application results of the c-characteristic function

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1 Introduction

The c-characteristic function, which is in a special form of the generalized Stieltjes transformation, has been shown to be a useful alternative tool for cases that the traditional characteristic function is hard or too complicated to use. First, we give its definition and properties. Then, we use it to give new results on random moments of Ferguson-Dirichlet processes with some interesting parameter measures.

2 Definition and properties of c-characteristic function

Definition 1. *The c-characteristic function of a bounded random variable X (with a support in $[-a, a]$, $a > 0$) is defined as*

$$g(t; X, c) = E(1 - itX)^{-c},$$

where $|t| < a^{-1}$ and $c > 0$.

Definition 2. *A random vector $\boldsymbol{u} = (u_1, \ldots, u_n)'$ is said to have a Dirichlet distribution with parameter $\boldsymbol{b} = (b_1, \ldots, b_n)'$, denoted by $\boldsymbol{u} \sim Dir(\boldsymbol{b})$, if its PDF has the form*

$$f(\boldsymbol{u}; \boldsymbol{b}) = \frac{1}{B(\boldsymbol{b})} \prod_{j=1}^{n} u_j^{b_j - 1},$$

for all $\boldsymbol{u}$ in the probability simplex $\{\boldsymbol{u} \mid$ each $u_j \geq 0,\ u_+ = 1\}$, where each $b_j > 0$, $u_+ = \sum_{j=1}^{n} u_j$, and $B(\boldsymbol{b}) = \prod_{j=1}^{n} \Gamma(b_j)/\Gamma(b_+)$.

Properties:

(a) If $\boldsymbol{u} \sim Dir(\boldsymbol{b})$, then $g(t; \boldsymbol{a}'\boldsymbol{u}, b_+) = \prod_{j=1}^{n}(1 - ita_j)^{-b_j}$, where $\boldsymbol{a}' = (a_1, \ldots, a_n)$.

(b) For any c, there is a one-to-one correspondence between $g(t; X, c)$ and X.

(c) For any c, if $\lim_{n\to\infty} g(t; X_n, c) = g(t; X, c)$ for all $|t| < a^{-1}$, then $X_n \to X$ in distribution.

(d) Inversion formulas.
Let X be a random variable on (a, b). Suppose that $g(t; X, c)$ is the univariate c-characteristic function of X. Set $G(z) = z^{-c} g(i/z; X, c)$. Then the related PDF and CDF can be expressed as

(i) $$\frac{f_X(x^+) + f_X(x^-)}{2} = \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i} \int_{D_{\epsilon,x}} (z+x)^{c-1} G'(z)\, dz, \text{ and}$$

(ii) $$\frac{F_X(x^+) + F_X(x^-)}{2} = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{D_{\epsilon,x}} (z+x)^{c-1} G(z)\, dz,$$

where $D_{\epsilon,x}$ is the contour which starts at the point $-x - i\epsilon$, proceeds along the straight line $\operatorname{Im} z = -\epsilon$ to the point $-a - i\epsilon$, then along the semi-circle $|z + a| = \epsilon$, $\operatorname{Re} z \geq -a$, to the point $-a + i\epsilon$, and finally along the line $\operatorname{Im} z = \epsilon$ to the point $-x + i\epsilon$. In particular, when $c = 1$, (i) can be simplied as

(iii) $$\frac{f_X(x^+) + f_X(x^-)}{2} = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \left[G(-x - i\epsilon) - G(-x + i\epsilon)\right].$$

3 Random moments of a Ferguson-Dirichlet process

Definition 3. *Let μ be a finite non-null measure on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is the σ-field of Borel subsets of Euclidean space Ω, and let U be a stochastic process indexed by elements of $\mathcal{A}$. We say that U is a Ferguson-Dirichlet process with parameter μ, denoted by $U \sim D(\mu)$ on Ω, if for every finite measurable partition $\{A_1, \ldots, A_m\}$ of Ω, the random vector $(U(A_1), \ldots, U(A_m))$ has a Dirichlet distribution with parameter $(\mu(A_1), \ldots, \mu(A_m))$.*

Define $\xi_\mu(h) = \int_\Omega h(y)\, dU(y)$, where $U \sim D(\mu)$ on Ω and $h(y)$ is a bounded measurable function on Ω.

This random functional can be applied in many areas such as
nonparametric density estimation: see, e.g. Lo (1984) and Dickey, Garthwaite, and Bian (1995)
smoothness prior distribution: see, e.g. Dickey and Jiang (1998)
quality control problems: see, e.g. Epifani, Guglielmi, and Melilli (2006)

One possible derivation of the random functional $\xi_\mu(h)$ is through the limit of $X_n = \sum_{j=1}^n d_{nj} u_j$ as n approaches ∞, where $d_{nj} = h(a_{nj})$, $(u_1, \ldots, u_n) \sim Dir(b_1, \ldots, b_n)$, $a_{nj} \in A_{nj}$, $\{A_{n1}, \ldots, A_{nn}\}$ is a partition of Ω, $b_{nj} = \mu(A_{nj})$, and $\max_{1\leq j\leq n} \mu(A_{nj}) \to 0$ as n approaches ∞. The traditional characteristic function (the Fourier transform) of X_n is

$$\phi(t; X_n) = e^{itd_{nn}} \sum_{\substack{m_j=0 \\ 1\leq j\leq n-1}}^{\infty} \frac{(b_{n1}, m_1)\cdots(b_{n,n-1}, m_{n-1})}{(b_{n,+}, m_1 + \cdots + m_{n-1}) m_1! \cdots m_{n-1}!} \times (i(d_{n1} - d_{nn})t)^{m_1} \cdots (i(d_{n,n-1} - d_{nn})t)^{m_{n-1}},$$

where $(b_{n1}, m_1) = b_{n1}(b_{n1}+1)\cdots(b_{n1}+m_1-1)$ and $b_{n,+} = \sum_{j=1}^{n} b_{nj}$. (See also Exton (1976, p. 233).) However, the $c(= b_{n,+})$-characteristic function of X_n has the simple form

$$g(t; X_n, b_{n,+}) = \prod_{j=1}^{n} (1 - itd_{nj})^{-b_{nj}}.$$

One can easily use Theorem 2.5 of Jiang, Dickey, and Kuo (2004) to obtain any moment of X_n. For example,

$$E(X_n) = \frac{\sum_{j=1}^{n} b_{nj} d_{nj}}{b_{n,+}} \text{ and } E(X_n^2) = \frac{(\sum_{j=1}^{n} b_{nj} d_{nj})^2 + \sum_{j=1}^{n} b_{nj} d_{nj}^2}{b_{n,+}(b_{n,+}+1)}.$$

In addition,

$$g(t; \xi_\mu(h), c) = \exp\left[-\int_\Omega \ln[1 - ith(y)]\, d\mu(y)\right].$$

Definition 4. *Let $v = (a_2 - a_1)u + a_1$, a linear transformation of u from $[0,1]$ to $[a_1, a_2]$ where u has a beta distribution with parameters b_1 and b_2, then v is said to have a generalized beta distribution on $[a_1, a_2]$, denoted by Gbeta$(b_1, b_2; a_1, a_2)$, with parameters b_1 and b_2. The PDF of v has the form*

$$\frac{1}{B(b_1, b_2)} \frac{(v - a_1)^{b_1 - 1}(a_2 - v)^{b_2 - 1}}{(a_2 - a_1)^{b_1 + b_2 - 1}}, \qquad \text{for } a_1 \le v \le a_2.$$

Theorem 5. *Let L be any integer greater than 1, and $X_L = \int_{-1}^{1} y\, dU_L(y)$ where $U_L \sim D(\mu_L)$ on $(-1, 1)$ and μ_L is a probability measure corresponding to Gbeta$((L-1)/2, (L-1)/2; -1, 1)$. Then, for $-1 < x < 1$,*

(i) $f_{X_2}(x) = \dfrac{2\sqrt{1-x^2}}{\pi}$;

(ii) $f_{X_3}(x) = \dfrac{e}{\pi}(1+x)^{-(1+x)/2}(1-x)^{-(1-x)/2} \cos \dfrac{\pi x}{2}$;

(iii) $f_{X_4}(x) = \dfrac{2}{\pi} e^{1/2 - x^2} \cos(x\sqrt{1-x^2} + \arcsin x)$.

In general, for $-1 < x < 1$,

(iv) *when L is an odd integer and $L \ge 4$,*

$$f_{X_L}(x) = \frac{\sin\left(\int_{-1}^{x} \pi\, d\mu_L(y)\right)}{\pi} \exp\left\{ \frac{-1}{B(1/2, (L-1)/2)} \sum_{k=0}^{(L-3)/2} \binom{\frac{L-3}{2}}{k} \frac{(-1)^k}{(2k+1)} \times \left[(1 + x^{2k+1}) \ln(1+x) + (1 - x^{2k+1}) \ln(1-x) - 2 \sum_{m=0}^{k} \frac{x^{2k-2m}}{2m+1} \right] \right\};$$

(v) *when L is an even integer and $L \ge 5$,*

$$f_{X_L}(x) = \frac{2\sin\left(\int_{-1}^{x} \pi\, d\mu_L(y)\right)}{\pi} \exp\left[\frac{2^{3-L}\pi}{B(1/2, (L-1)/2)} \sum_{k=0}^{(L-4)/2} \binom{L-2}{k} \frac{\cos[(L-2-2k)\arcsin x]}{L-2-2k} \right].$$

4 Conclusions

The study of the random moments of the Ferguson-Dirichlet process has drawn the attention of statisticians for decades. The use of c-characteristic function together with its inversion formulas provide a useful method to study the random moments of the Ferguson-Dirichlet process. Here, we provide the PDF, which are new in the literature, of the random moments of a Ferguson-Dirichlet process with some interesting parameter measures.

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行政院國家科學委員會補助國內專家學者出席國際學術會議報告

2009 年 9 月 8 日

附件三

報告人姓名	姜志銘	服務機構及職稱	國立政治大學應用數學系教授
會議時間地點	2009 年 8 月 1-6 日	本會核定補助文號	NSC-97-2118-M-004-003
會議名稱	(中文)2009 聯合統計季會議 (英文)2009Joint Statistical Meetings		
發表論文題目	(中文) (英文)Some new application results of the c-characteristic functions		

報告內容應包括下列各項：

一、參加會議經過

由美國統計學會及其他多個國際知名的統計學會，如 IMS、ENAR、WNAR 及 SSC 聯合主辦的 2009 年聯合統計會議，為全球最大型的統計會議，今年在美國首府(DC)舉辦，吸引來自全球各地約六千名的統計專家學者們，共同為追求及交換新的統計理論與方法而來參加，本人亦抱持著這種態度，希望藉由這個機會與其他的統計學者專家作學術上的交流。

二、與會心得

我的演講題目為「Some new application results of the c-characteristic functions」。 c-特徵函數可以解決某些不容易利用傳統特徵函數來解決的機率或統計問題。在這次演講中，除了介紹 c-特徵函數一些特別性質外，並也提供了一些它在應用上的有趣新結果，演講中也引起與會者的興趣與發問。跟往年一樣，2009 年聯合統計會議涵蓋的領域非常廣泛，從理論到計算，從貝式統計到生物統計，甚至目前大家所關心的區域性甚至國際性的議題，如溫度變化等等，都可以看到與會者所展現的最新研究成果。透過參與這些會議，以及會議期間與所遇到的部份學者、專家，互相交換最近一些研究領域方向的看法，使得參加這次會議得益不少。最後，謝謝國科會給予這次機會參加這個有意義的會議。

三、考察參觀活動(無是項活動者省略)

四、建議

無

五、攜回資料名稱及內容

JSM 2009 Abstracts(CD)-研討會摘要

六、其他

表 Y04