行政院國家科學委員會專題研究計畫 成果報告

熱帶幾何相交理論與熱帶循環之研究
研究成果報告(精簡版)

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國科會專題研究成果報告:
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計畫編號: NSC97-2115-M-004-003
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December 27, 2009

Chapter 1

報告內容

1.1 Introduction

The project survey recent results on tropical geometry especially those related to intersection theory. When we start the project, there is no tropical intersection in general, but in the past year, some exciting results toward this direction have been carried out. We study these results and present the ideas behind these somehow technical results. To elaborate the concepts, we choose to present here in a bit more intuitively. However, the references for rigors statements and proofs are provided.

1.2 Basics of Tropical Geometry

Tropical geometry is the geometry with underline "field" (actually, semifield or semiring) be the tropical semiring.

Definition 1 (tropical semiring)**.** A tropical semiring is $(\mathbb{T}, \oplus, \odot)$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. For any $x, y \in \mathbb{T}$, define

- $x \oplus y := \max\{x, y\}$
- $x \odot y := x + y$

One can then of course define tropical polynomials: they are just classical polynomials with tropical operations. More precisely, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, we denote $x_1^{\odot i_1} \odot x_2^{\odot i_2} \odot \cdots \odot x_n^{\odot i_n}$ by $\mathbf{x}^{\odot \mathbf{i}}$. A tropical polynomial is of the form

$$f(\mathbf{x}) = \sum_{\mathbf{i} \in I}^{\oplus} \alpha_{\mathbf{i}} \odot \mathbf{x}^{\odot \mathbf{i}},$$

where I is a finite subset of $(\mathbb{N} \cup \{0\})^n$.

We can then define the tropical hypersurface defined by the "roots" of a tropical polynomial.

Definition 2 (Tropical Hypersurface)**.** Let $f(\mathbf{x})$ be a tropical polynomial. Evaluate this polynomial is to find the maximum of the linear forms $\alpha_{\mathbf{i}} + \langle \mathbf{x}, \mathbf{i} \rangle$. A point in the tropical hypersurface H_f is the maximum of the linear forms achieve at least twice and it is exactly where the graph fails to be linear.

1.3 Tropical Curves

We work with some examples to see what really happen in tropical world. First, let us see an example of tropical plane curve, which is defined by a two variables tropical polynomial.

Example 1. Let $f(x, y) = x \oplus y \oplus 1 = \max\{x, y, 1\}$ be a tropical polynomial which defines a tropical line. To get the graph of this tropical line, we draw the following rays:

- $x = y \geq 1$,
- $x = 1 \geq y$, and
- $y = 1 \geq x$.

The result picture is as in Figure 1.1.

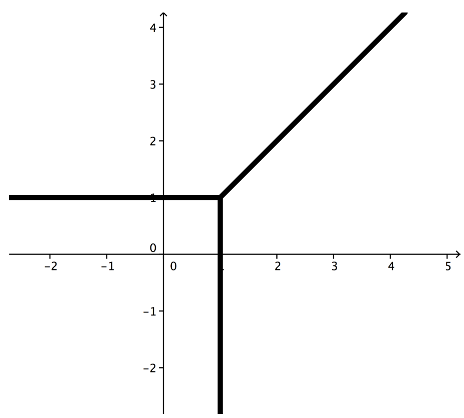

Figure 1.1: The tropical line defined by $f(x, y) = x \oplus y \oplus 1$.

Note that a tropical curve satisfies so called *balancing condition*. In Example 1, the direction of three rays are given by the vector $v_1 = (1, 1)$, $v_2 = (-1, 0)$, $v_3 = (0, -1)$, respectively. We have

$$v_1 + v_2 + v_3 = (0, 0).$$

Of course, the direction given by $(1, 1)$ can be directed by $(2, 2)$. Here we want to take *primitive vector*, namely the first integer-coordinate vector in that direction. Moreover, sometimes adding the primitive vectors up dose not give us the zero vector, we might need to consider "weighted" rays. Hence for any vertex in a tropical curve, and $v_1, v_2, \ldots, v_k$ are primitive vectors, the balancing condition means

$$w_1 v_1 + w_2 v_2 + \cdots + w_k v_k = (0, 0),$$

where $w_1, w_2, \ldots, w_k$ are corresponding weights. For introduction to tropical geometry with more details, please see [3, 4, 6, 7]

One can easily find out all tropical lines look exactly like the one in Figure 1.1, except the vertex might move. We also have two tropical lines in general positions intersect at one point as in figure 1.3, just like classical cases. However, it is easy to see that but sometimes two tropical line intersect at infinitely many points (a ray). The unexpected behavior indicates the tropical intersection theory need to be more careful to make it really work.

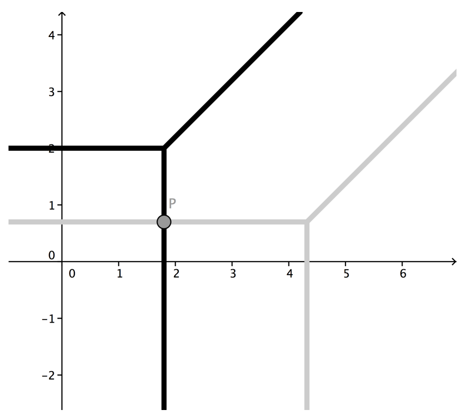

Figure 1.2: two tropical lines in general positions

1.4 Stable Intersection

Two tropical lines might intersect "incorrectly," which means they intersect in infinitely many points, as L_1, L_2 in Figure 1.3.

However, we can move L_1 a bit, call the new line L_1^ε, and move L_2 a bit, call the new line L_2^ε such that $L_1^\varepsilon \cap L_2^\varepsilon$ correctly. That is, they intersect at exactly one point.

It is easy to see, we have

$$P = \lim_{\varepsilon \to 0} L_1^\varepsilon \cap L_2^\varepsilon.$$

We will call P the *stable intersection* of L_1 and L_2, as defined in [6]. One can actually prove tropical Bézout's Theorem in the sense of stable intersection.

In mathematics, what we usually want to do is the following. We consider a class of curves $[C_1]$ and a class of curves $[C_2]$, which we want the correct

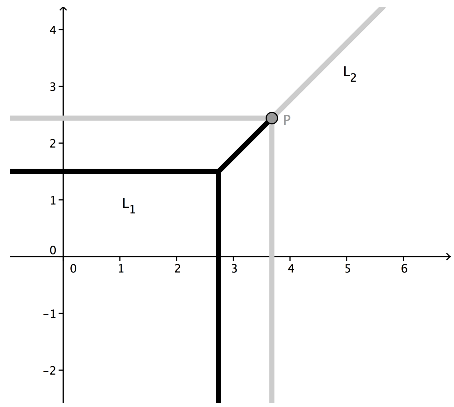

Figure 1.3: two tropical lines intersect in infinitely many points

intersection number. What we want is the stable intersection of C_1 and C_2. On the other hand, we would like to have "good" representatives of $[C_1]$ and $[C_2]$ such that C_1 and C_2 intersect correctly.

Stable intersections seem a correct way to do tropical intersection, but it is a bit ambiguous and not always easy to work with.

1.5 Tropical Cycles

Allermann and Rau [1] tried to develop general tropical intersection theory. The idea is to mimic classical algebraic geometry, defining intersections on divisors. That is, to develop the intersection theory in the sense of Fulton [2].

First, we can generalize the definition of tropical hypersurfaces a bit to define what a *tropical fan* is.

Definition 3. A tropical fan (X, ω_x) of dimension k is a weighted fan satisfying the balancing condition for all $\tau \in X^{(k-1)}$.

Of course, different tropical fans might basically "the same," which means both can have the same *refinement*. Please see [1] for detailed definitions. Thus we can consider a class of tropical fans. As usual, $[(X, \omega_x)]$ denotes the equivalent class of (X, ω_x). We call $[(X, \omega_x)]$ an *affine cycle*.

Definition 4. The element of

$$Z_k^{\text{aff}}(V) = \{[(X, \omega_x)] \mid (X, \omega_x) \text{ a tropical fan of dimension } k \text{ in } V\ \}$$

is called an (tropical) affine k-cycle in V. One can put a structure of abelian group on $Z_k^{\text{aff}}(V)$.

We can then define *affine tropical varieties* be the tropical cycles with non-negative weights.

Definition 5. Let C be a tropical cycle. A Weil divisor on C is an integer linear combination of $Z^{\text{aff}}_{\dim C-1}(C)$. An affine Cartier divisor on C is a piecewise integer affine linear function modulo globally affine linear function.

One can then define intersection product of two tropical affine Cartier divisors, as in [1].

Allermann and Rau allow us to deal with tropical intersection in more general settings. However, it is not clear if it coincide with the stable intersection. Recently, Katz [5] proved that two definitions are indeed equivalent.

Chapter 2

成果自評

In this project, we learned how do we deal with tropical intersections by using an analogue theory to classical intersection theory in algebraic geometry. During the project, there was a graduate student summer school in MSRI, Berkeley, USA. We send Ph.D. student Cheng-Wei Chen there, and he brought many notes also new ideas back and help us to set more clear direction in the future.

In the mean time, we found out that it is necessary to have more solid understanding on basic stuff of tropical geometry in order to convert classical theory into tropical ones. For instance, we have to be able to work on tropical polynomials, rational functions, and meromorphic functions. We survey the related work and prove some classical properties still hold in tropical settings. All results will be on our forthcoming paper [8].

We are in a position to actually carry out some calculation of intersections of tropical curves, which we will continue to work on the next project.

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Visiting Report
08.24.2009 08.28.2009
Introductory Workshop: Tropical Geometry

Department of Mathematical Science, Chen-Wei Chen

December 7, 2009

1 Information of the Seminar

I joined the seminars held by MSRI (Mathematical Sciences Research Institute) this summer and I learnt a lot from them. MSRI held four seminars this summer, respectively: Connections for Women: Tropical Geometry (08, 22, 2009 - 08, 23, 2009), Introductory Workshop: Tropical Geometry, (08, 24, 2009 - 08, 28, 2009),Tropical Geometry in Combinatorics and Algebra, (10, 12, 2009 - 10, 16, 2009) , and Tropical Structures in Geometry and Physics (11 30, 2009 - 12, 4, 2009).
This workshop was to lay the foundations for the upcoming program. Mini-courses comprising lectures and exercise/discussion sessions covered the foundational aspects of tropical geometry as well as its connections with adjacent areas: symplectic geometry, several complex variables, algebraic geometry (in particular enumerative and computational aspects) and geometric combinatorics. The mini-courses were augmented by research talks on current tropical developments to open the scene and set up new goals in the beginning semester. I attended the Introductory Workshop: Tropical Geometry, since it was in the summer vacation, and there are many scholars and graduate students all over the world to join this seminar. Hence I could have more opportunity to share views with them. For the reason that MSRI is an important institute of UC Berkeley in researching Tropical Geometry, many famous professors in this field came here. For instance, Grigory Mikhalkin, Andreas Gathmann, and Bernd Sturmfels attended this seminar. Especially, Grigory Mikhalkin is the most well-known scholar in Tropical Geometry, because he has proven that tropical geometry can be used to compute the numbers of plane curves of given genus g and degree d

through $3d + g + 1$ general points, cf. [5]. This seminar covers the foundational aspects of tropical geometry as well as its connections with adjacent areas: symplectic geometry, several complex variables, algebraic geometry and geometric combinatorics.

2 Thoughts of the seminar

It was a five-day seminar, containing five courses. We had three courses and one speech each day. What I was most interested in was the course named Complex amoebas and (co)amoebas, it was given by professor Mikael Passare. The course focused on the theory of complex (co)amoebas, especially on the method to combine the Tropical Geometry and amoeba spines. This theory contains many methods in analysis. When it came to Ronkin function, I realized that Tropical Geometry could be widely used in many areas. I thought it must be interesting if we apply it to the Nevanlinna Theory. However, there are already some people using this idea few years ago. So when I came back to Taiwan, I talked to professor Yen-Lung Tsai about information and my ideas of this conference. Then he appreciated what I had learnt and gave me more papers to read. They are all about Tropical Nevanlinna theory, which are Tropical Nevanlinna theory and ultra-discrete equations [1], Tropical versions of Clunie and Mohonko lemmas [3], Tropical Nevanlinna theory and second main theorem [2]. Hence now I have gained some ideas in Tropical Nevanlinna Theory. Moreover, this seminar had broadened my horizon. I realized that there are still many areas that we can apply Tropical Geometry to analyze things. Take symplectic geometry for example, many researchers take effort in the connection of Tropical Geometry to this field.
The second course that I was interested in was Introduction to tropical algebraic geometry given by Diane Maclagan. In this course, the tropical arithmetic operations are $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, which is different from the normal mathematic operations. Under the new operation, I learned many new things and had a new perspective on tropical variety. Tropical variety is similar to conventional variety in some parts, however, they are different in essence. For example, the intersection of a variety under conventional concept is still a variety. Nevertheless, in Tropical Geometry, this property only holds in tropical basis, and other arbitrary intersection is not a tropical variety. It also mentioned some relationship between valuation, normal toric variety and initial form. Besides that, I was fascinated in the connection of subdivision and tropical curve. Finally, the speaker refered to the tropical moduli space, which I had learnt a lot from it. In this space, let $M_{o,n}$ be the set of all tropical curves of genus zero with n distinct marked points, then its tropical variety,denoted by $trop(M_{0,n})$, is a

$(n-3)-dim$ fan in $R^{\binom{n}{2}-n}$ with a coarsest fan structure , where is the space of phylogenetic trees. For example, in $M_{0,5}$, the global picture of adjacency and rays is the well-known Petersen graph , cf. [6] and [7].

After I was back to Taiwan, MSRI sent me a book written by Diane Maclagan and Bernd Sturmfels, named Introduction to Tropical Geometry ([4]). I found many things I learned from the seminar in this book, and fortunately, I can have many references and opportunity to research in this fascinating field.

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