行政院國家科學委員會專題研究計畫 成果報告

半純函數之值分佈與其相關問題
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 97-2115-M-004-002-
執 行 期 間 ： 97 年 08 月 01 日至 98 年 07 月 31 日
執 行 單 位 ： 國立政治大學應用數學學系

計 畫 主 持 人 ： 陳天進

計畫參與人員： 碩士班研究生-兼任助理人員：郭玲伶

處 理 方 式 ： 本計畫可公開查詢

中 華 民 國 98 年 10 月 16 日

VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS AND ITS RELATED PROBLEMS

TEN-GING CHEN

Abstract. Let f and g be two distinct non-constant meromorphic functions sharing four distinct values a_1, a_2, a_3, a_4 IM. Mues showed that if f and g share a_1 CM and $\tau(a_2) > 2/3$, then f and g share all the four values CM. Wang showed that if f and g share a_1 CM and $\tau(a_i) > \frac{1}{2}$ for $i = 2, 3, 4$, then the above conclusion also holds. In this paper, we extend Wang's result to the case of small functions and give somewhat generalizations of meromorphic functions sharing small functions.

1. Introduction

Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ and a be a value in $\mathbb{C}_\infty$. We say that f and g share a CM (IM) if $f - a$ and $g - a$ have the same zeros counting (ignoring) multiplicities when a is finite and we say that f and g share ∞ CM (IM) if $1/f$ and $1/g$ share 0 CM (IM). The unicity of meromorphic functions sharing values has been studied for a long time. In 1929, Nevanlinna [8] proved the following.

Theorem 1.1. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct values. If f and g share the four values CM, then f is a Möbius transformation of g.*

In 1979 and 1983, Gundersen [1, 2] proved that if f and g share either three values CM and the other one IM, or two values CM and the other two IM, then the conclusion of Theorem 1.1 also holds. In fact, Gundersen proved the following.

Theorem 1.2. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct values. If f and g share a_1, a_2, a_3 CM and share a_4 IM, then f and g share all the four values CM.*

Theorem 1.3. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct values. If f and g share a_1, a_2 CM and share a_3, a_4 IM, then f and g share all the four values CM.*

The remaining case that f and g share one value CM and the other three values IM is still open. However, if we add some conditions on the other three values, then the result also holds. In order to study the remaining case, we recall some important properties for two meromorphic functions sharing four values IM which was proved by Mues [7].

Theorem 1.4. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct values. If f and g share the four values IM, then*

(i) $T(r, f) = T(r, g) + S(r, f)$, $T(r, g) = T(r, g) + S(r, g)$.

(ii) $\sum_{i=1}^{4} \overline{N}(r, \frac{1}{f-a_i}) = 2T(r, f) + S(r, f)$.

(iii) $N_0(r, \frac{1}{f'}) = S(r, f)$, $N_0(r, \frac{1}{g'}) = S(r, g)$, *where* $N_0(r, \frac{1}{f'})$ *counts the zeros of* f' *but not the zeros of* $f - a_i$.

(iv) $\sum_{i=1}^{4} N^*(r, a_i) = S(r, f)$, *where* $N^*(r, a_i)$ *counts the multiple common zeros of* $f - a_i$ *and* $g - a_i$.

Let f and g be two non-constant meromorphic functions that share a value a IM. Mues introduced the quantity $\tau(a)$ to measure the degree of f and g share a CM. In fact,

$$\tau(a) = \begin{cases} \liminf\limits_{r\to\infty} \dfrac{\overline{N}_E(r, a)}{\overline{N}(r, \frac{1}{f-a})} & \text{if } \overline{N}(r, \frac{1}{f-a}) \neq 0, \\ 1 & \text{if } \overline{N}(r, \frac{1}{f-a}) = 0, \end{cases}$$

where $\overline{N}_E(r, a)$ is the reduced counting function which counts the common zero of $f - a$ and $g - a$ with the same multiplicities. Note that $0 \leq \tau(a) \leq 1$, and if f and g share a CM, then $\tau(a) = 1$. In some sense, if $\tau(a)$ is much more close to 1, then so is the proportion $\overline{N}_E(r, a)/\overline{N}(r, \frac{1}{f-a})$. In other words, the degree of f and g sharing a CM is much more. Mues [7] and Wang [9] proved the following results.

Theorem 1.5. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct values. If f and g share a_1 CM and share a_2, a_3, a_4 IM with $\tau(a_2) > 2/3$, then f and g share all the four values CM.*

Theorem 1.6. *Let f and g be two non-constant meromorphic functions sharing four distinct values a_1, a_2, a_3, a_4 IM. If f and g share a_1 CM and $\tau(a_i) > \frac{1}{2}$ for $i = 2, 3, 4$, then f and g share all the four values CM.*

In this report, we will show that Theorem 1.6 still holds if we replace sharing values by sharing small functions. Throughout the report, we will use the standard notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [3].

2. Meromorphic functions sharing small functions

Let f, g and a be meromorphic functions. If $T(r,a) = o(T(r,f))$ as $r \to \infty$ possible outside a set of finite linear measure, then we say that a is a small function of f. We say that f and g share a IM* if

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) - \overline{N}(r,a) = S(r,f) \text{ and } \overline{N}\Big(r, \frac{1}{g-a}\Big) - \overline{N}(r,a) = S(r,g),$$

where $\overline{N}(r,a)$ is the reduced counting function which counts the common zeros of $f-a$ and $g-a$. We say that f and g share a CM* if

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) - \overline{N}_E(r,a) = S(r,f) \text{ and } \overline{N}\Big(r, \frac{1}{g-a}\Big) - \overline{N}_E(r,a) = S(r,g),$$

where $\overline{N}_E(r,a)$ is the reduced counting function mentioned in section 1. Most results in section 1 remain valid for small functions. For example, Ishizaki [4] proved the analogues for Theorem 1.1 and Theorem 1.2.

Theorem 2.1. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct small functions. If f and g share the four small functions CM*, then f is a quasi-Möbius transformation of g.*

Theorem 2.2. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct small functions. If f and g share a_1, a_2, a_3 CM* and share a_4 IM*, then f and g share all the four small functions CM*.*

Li [6] proved the analogue for Theorem 1.3.

Theorem 2.3. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct small functions. If f and g share a_1, a_2 CM* and share a_3, a_4 IM*, then f and g share all the four small functions CM*.*

Besides, Yao [10] proved that Theorem 1.4 (iv) still holds for small functions.

Theorem 2.4. *Let f and g be two distinct non-constant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct small functions. If f and g share the four small functions IM*, then*

$$\sum_{i=1}^{4} N^*(r, a_i) = S(r,f).$$

However, we do not have Theorem 1.4 (iii) for small functions. In fact, the proof of Theorems 1.4 is base on the Nevanlinna's second fundamental theorem which is not available for small functions. In general, the error term $m\big(r, \frac{f'}{f-a}\big)$ is not $S(r,f)$ when a is a small function. For example, let $f(z) = e^z + 2z$, $g(z) = e^{-z} + 2z$ and

$a_1(z) = 2z$, $a_2(z) = 1 + 2z$, $a_3(z) = -1 + 2z$, $a_4 = \infty$. Clearly, f and g share a_1, a_2, a_3, a_4 CM and a_1, a_2, a_3, a_4 are small function of f and g. But

$$m\Big(r, \frac{f'}{f - a_1}\Big) = m\Big(r, \frac{e^z + 2}{e^z}\Big) = T(r, f) + S(r, f)$$

and

$$N_0\Big(r, \frac{1}{f'}\Big) = N\Big(r, \frac{1}{e^z + 2}\Big) = T(r, f) + S(r, f).$$

Hence the sharing small function problems become difficult and different auxiliary functions must be introduced. In the following, we will use some techniques in the paper of Ishizaki [5].

3. Lemmas

To generalize Theorem 1.6, we need extend the definition of $\tau(a)$ for small function.

Definition 3.1. *Let f and g be two non-constant meromorphic functions that share a small function a IM*. We define*

$$\tau(a) = \begin{cases} \liminf\limits_{r\to\infty} \dfrac{\overline{N}_E(r,a)}{\overline{N}(r,a)} & \text{if } \overline{N}(r,a) \neq 0, \\ 1 & \text{if } \overline{N}(r,a) = 0. \end{cases}$$

Note that if a is a value, then $\tau(a)$ coincides with Mues' original definition.

Let f and g be two non-constant meromorphic functions sharing four distinct small functions a_1, a_2, a_3, a_4 IM*. Consider the functions

$$F(z) = \frac{f(z) - a_2(z)}{f(z) - a_1(z)} \frac{a_3(z) - a_1(z)}{a_3(z) - a_2(z)}, \quad G(z) = \frac{g(z) - a_2(z)}{g(z) - a_1(z)} \frac{a_3(z) - a_1(z)}{a_3(z) - a_2(z)},$$
$$a(z) = \frac{a_4(z) - a_2(z)}{a_4(z) - a_1(z)} \frac{a_3(z) - a_1(z)}{a_3(z) - a_2(z)}.$$

Then F and G share ∞, 0, 1, a IM*. Let $b_1 = \infty$, $b_2 = 0$, $b_3 = 1$ and $b_4 = a$. Note that f and g share a_i CM* if and only if F and G share b_i CM*, and $\tau_{f,g}(a_i) > \alpha$ if and only if $\tau_{F,G}(b_i) > \alpha$. Therefore, without loss of generality, we may assume that f and g share ∞, 0, 1, a IM*, where a is a non-constant small function of f and g. In order to prove our main results, we need some lemmas.

Lemma 3.2. *Let f and g be two non-constant meromorphic functions sharing $\infty, 0, 1$ and a IM*, where a is a non-constant small function. Let*

$$\psi = \frac{\Delta(f - g)}{\Pi},$$

where

$$\Delta = \begin{vmatrix} f^2 & g^2 & a^2 & 1 \\ ff' & gg' & aa' & 0 \\ f & g & a & 1 \\ f' & g' & a' & 0 \end{vmatrix} \quad \text{and } \Pi = f(f-1)(f-a)g(g-1)(g-a),$$

then ψ *is a small function of* f *and* g.

Lemma 3.3. *Let* f *and* g *be two non-constant meromorphic functions sharing* $\infty, 0, 1$ *and a small function* a *IM* and let*

$$\beta = \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}, \qquad \beta_1 = \frac{f_1'(f_1 - 1/a)}{f_1(f_1-1)} - \frac{g_1'(g_1 - 1/a)}{g_1(g_1-1)},$$

$$\beta_2 = \frac{f_2'\big(f_2 + 1/(a-1)\big)}{f_2(f_2-1)} - \frac{g_2'\big(g_2 + 1/(a-1)\big)}{g_2(g_2-1)},$$

where

$$f_1 = \frac{f}{a}, \qquad g_1 = \frac{g}{a}, \qquad f_2 = \frac{f-1}{a-1}, \qquad g_2 = \frac{g-1}{a-1}.$$

If any one of β, β_1, *and* β_2 *is identically zero, then* f *and* g *share* $\infty, 0, 1$ *and* a *CM*.*

With some analysis, we can prove the following.

Lemma 3.4. *Let* f *and* g *be two non-constant meromorphic functions sharing* $\infty, 0, 1$ *and a small function* a *IM* and let* β, β_1, β_2 *be the functions defined in Lemma 3.3. If* $\beta \cdot \beta_1 \cdot \beta_2 \not\equiv 0$, *then*

$$\overline{N}(r,a) \le \overline{N}(r,0) - \overline{N}_E(r,0) + \overline{N}(r,1) - \overline{N}_E(r,1) + \overline{N}(r,\infty) - \overline{N}_E(r,\infty) + S(r,f),$$

$$\overline{N}(r,1) \le \overline{N}(r,0) - \overline{N}_E(r,0) + \overline{N}(r,a) - \overline{N}_E(r,a) + \overline{N}(r,\infty) - \overline{N}_E(r,\infty) + S(r,f),$$

$$\overline{N}(r,0) \le \overline{N}(r,1) - \overline{N}_E(r,1) + \overline{N}(r,a) - \overline{N}_E(r,a) + \overline{N}(r,\infty) - \overline{N}_E(r,\infty) + S(r,f).$$

4. Main Theorem

Now, we can prove the analogue of Theorem 1.6 for small function.

Theorem 4.1. *Let* f *and* g *be two non-constant meromorphic functions sharing four distinct small functions* a_1, a_2, a_3, a_4 *IM*. If* f *and* g *share* a_1 *CM* and* $\tau(a_i) > \frac{1}{2}$ *for* $i = 2, 3, 4$, *then* f *and* g *share all the four small functions CM*.*

Proof. Without loss of generality, we assume that $a_1 = \infty$, $a_2 = 0$, $a_3 = 1$, $a_4 = a$. If any one of β, β_1, β_2, defined in Lemma 3.4, is identically zero, then, by

Lemma 3.3, f and g share $\infty, 0, 1$ and a CM*. Now, we assume that $\beta \cdot \beta_1 \cdot \beta_2 \not\equiv 0$. Since f and g share ∞ CM*, by Lemma 3.4, we get

$$\overline{N}(r,a) \le \overline{N}(r,0) - \overline{N}_E(r,0) + \overline{N}(r,1) - \overline{N}_E(r,1) + S(r,f),$$
$$\overline{N}(r,1) \le \overline{N}(r,0) - \overline{N}_E(r,0) + \overline{N}(r,a) - \overline{N}_E(r,a) + S(r,f),$$
$$\overline{N}(r,0) \le \overline{N}(r,1) - \overline{N}_E(r,1) + \overline{N}(r,a) - \overline{N}_E(r,a) + S(r,f).$$

Hence,

$$2\overline{N}_E(r,0) - \overline{N}(r,0) + 2\overline{N}_E(r,1) - \overline{N}(r,1) + 2\overline{N}_E(r,a) - \overline{N}(r,a) \le S(r,f).$$

By the hypothesis, $\min\{\tau(0), \tau(1), \tau(a)\} > \frac{1}{2}$, there exists $c > 0$ and $R > 0$ such that, for all $r \ge R$,

$$\begin{aligned} c\big[\,\overline{N}(r,0) + \overline{N}(r,1) + \overline{N}(r,a)\,\big] &\le 2\overline{N}_E(r,0) - \overline{N}(r,0) + 2\overline{N}_E(r,1) - \overline{N}(r,1) \\ &\quad + 2\overline{N}_E(r,a) - \overline{N}(r,a) \\ &\le S(r,f), \end{aligned}$$

which implies that f and g share 0, 1, a CM*. Therefore, f and g share ∞,0, 1, a CM*. ❑

References

[1] G. G. Gundersen, *Meromorphic functions that share three or four values*, J. London Math. Soc., **20**(1979), 457-466.

[2] G. G. Gundersen, *Meromorphic functions that share four values*, Trans. Amer. Math. Soc., **277**(1983), 545-567.

[3] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[4] K. Ishizaki, *Uniqueness problems on meromorphic functions that share four small functions*, Proceedings of Second ISAAC Congress, Kluwer 2000, 467-472.

[5] K. Ishizaki, *Meromorphic functions sharing small functions*, Arch. Math. 77(2001), 273-277.

[6] P. Li, *Meromorphic functions that share four small functions*, J. Math. Anal. Appl., **263**(2001), 316-326.

[7] E. Mues, *Meromorphic functions sharing four values*, Complex Variables, **12**(1989), 169-179.

[8] R. Nevanlinna, *Le thérème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthiers-Villars, Paris, 1929.

[9] J. P. Wang, *Meromorphic functions sharing four values*, Indian J. Pure Appl. Math., **32**(2001), 37-46.

[10] W. H. Yao, *Meromorphic functions sharing four small functions IM**, Indian J. pure appl. Math., **34**(2003), 1025-1033.