國科會專題研究成果報告:
緊緻黎曼曲面之非交換餘調研究

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Chapter 1

報告内容

1.1 前言

The goal of this project is to study and calculate non-abelian cohomology of compact Riemann surfaces. For any manifold X, we may consider the set of all equivalent representations of the fundamental group of X as the first non-abelian cohomology [10]. That is, we may define

$$\mathrm{H}^1_{\mathrm{G}}(\mathrm{X}) = \mathrm{Hom}(\pi_1(\mathrm{X}), \mathrm{G})//,$$

where G is a linear group.

A major reason to study non-abelian cohomology is to see some geometry properties which are not visible by classical cohomology. We found there are some interesting monodromy phenomena in non-abelian cases which we will elaborate in Section 1.5.

In general, it is difficult to compute non-abelian cohomology. We do calculate some interesting special cases, but cannot apply the same technique to all other cases. However, we will propose a promising method to calculate the non-abelian cohomology and will continue the study in the following years.

1.2 研究目的

Our main motivation is to study degenerations of manifolds.

Definition 1. *We say that f is a degeneration of a Kähler manifold if f is a proper map from a Kähler manifold X onto the unit disk Δ such that f is of maximum rank for all $s \in \Delta$ except at the point $s = 0$. Let $\Delta^* = \Delta \setminus \{0\}$. We call $X_t = f^{-1}(X_t)$ a* smooth fiber or generic fiber *when $t \in \Delta^*$ and $X_0 = f^{-1}(0)$ the* singular or degenerated fiber.

We assume the singularity in X_0 is of normal crossing. If it is not the case, we can do some blow-ups and base changes to make X_0 normal crossing. An important tool to study degenerations is the Clemens-Schmid exact sequence, which give a link between singular and nonsingular fibers. The main part of the Clemens-Schmid exact sequence is introduced by the Picard-Lefschetz transformation. An analog of classic Picard-Lefschetz in non-ablian sense is a map

$$T\colon \mathsf{H}^1_G(X_t) \to \mathsf{H}^1_G(X_t),$$

which is introduced by

$$\tilde{T}\colon \pi_1(X_t) \to \pi_1(X_t).$$

We are interested in calculating $\mathsf{H}^1_G(X_t)$, especially classes in $\mathsf{H}^1_G(X_t)$ which are fixed under the Picard-Lefschetz transformation. Those elements have significant geometric meanings. For instance, they play an important role in the Clemens-Schmid exact sequence.

1.3 文獻探討

1.3.1 The Clemens-Schmid Exact Sequence

In this section, we quickly review the (abelian) Clemens-Schmid exact sequence in classical Hodge theory.

Through this report, we assume that $f : X \to \Delta$ is a degeneration of compact Riemann surfaces of genus g. Denote X_t as a generic fiber and X_0 a singular fiber. Fix a smooth fiber X_t, the generator of the fundamental group of the punched unit disk rises to a map:

$$T : \mathrm{H}^1(\mathrm{X_t}) \to \mathrm{H}^1(\mathrm{X_t}).$$

The map is called the Picard-Lefschetz transformation.

Now, we want links between X_t and the singular fiber X_0. There are two maps in Clemens-Schmid exact sequence which provide the links. The first one is

$$\nu : \mathrm{H}^1(\mathrm{X_0}) \to \mathrm{H}^1(\mathrm{X_t}).$$

It is induced by a map $c : X_t \to X_0$ which we shall explain now. A degeneration f is a strong deformation so there is a map from $X \to X_0$, composing with the inclusion $X_t \to X$, one get the map c:

$$c : X_t \hookrightarrow X \to X_0.$$

The map c is called the *Clemens map*, see [3], [6], or [8] for detail construction of this map. The Clemens map c induces the map ν.

The second link from the smooth fiber to the singular fiber is from the Poincaré duality:

$$\psi : \mathrm{H}^1(\mathrm{X_t}) \to \mathrm{H}_1(\mathrm{X_0}).$$

Define $N = I - T$, where I is the identity map on $H^1(X_t)$. Under our assumption that each smooth fiber is a Riemann surface, the Clemens-Schmid exact sequence is

$$1 \to \mathrm{H}^1(\mathrm{X_0}) \xrightarrow{\nu} \mathrm{H}^1(\mathrm{X_t}) \xrightarrow{\mathrm{N}} \mathrm{H}^1(\mathrm{X_t}) \xrightarrow{\psi} \mathrm{H}_1(\mathrm{X_0}) \to 1. \tag{1.1}$$

Clemens proved the sequence 1.1 is exact. Moreover, all maps in Clemens-Schmid exact sequence are actually morphisms of mixed Hodge structures. Therefore, one can put a mixed Hodge structure on the singular fiber if there is a mixed Hodge structure on the smooth one. We refer [6, Section 5.4] for explicit examples.

1.3.2 The Non-abelian Hodge Theory

To construct a non-abelian Clemens-Schmid exact sequence, we construct an exact sequence of "non-abelian cohomologies" instead of classical cohomologies. Simpson suggests that one can regard the moduli space of the fundamental group as the first nonabelian cohomology [10][11]. That is to say we can consider the first non-abelian cohomology to be the set of all equivalent classes of representations of fundamental groups $\pi_1(X_t)$ into some non-abelian "coefficient group" G. To have more geometric meaining, we will take G to be some linear algebriac group over C. For instance, we can take $G = \mathrm{GL}(2, \mathbb{C})$. The equivalent classes are defined by conjugation. For a space Y, we define the first non-abelian cohomology as

$$\mathsf{H}^1_G(Y) = \mathrm{Hom}(\pi_1(X_t), G)//.$$

When we take G as $\mathbb{C}$, it is exactly the standard cohomology if we consider the group operation on $\mathbb{C}$ is the addition operator.

Simpson [9] suggest a way to . A *Higgs bundle* is defined as the following.

Definition 2 (Higgs Bundles)**.** *A* Higgs bundle *over X is a holomorphic vector bundle E together with a holomorphic map*

$$\theta : E \to E \otimes \Omega^1_X.$$

We write a Higgs bundle as a pair (E, θ).

Then, we can consider the moduli space of Higgs bundles, denoted as $\mathcal{M}_{\mathrm{Higgs}}(Y, G)$. Simpson proves that there is a one-to-one correspondence between irreducible representations of $\pi_1(Y)$ and stable Higgs bundles with vanishing Chern classes [9]:

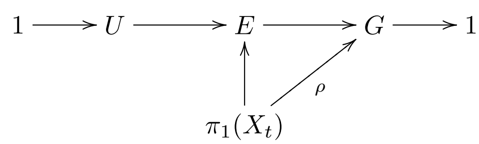
$$\left\{\begin{matrix}\text{irreducible representations} \\ \text{of } \pi_1(Y)\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\text{stable Higgs bundle with} \\ \text{vanishing Chern classes}\end{matrix}\right\}.$$

There is a natural action of $\mathbb{C}^*$ on the set of Higgs bundles. For any $t \in \mathbb{C}^*$, the action will send a Higgs bundle (E, θ) to the Higgs bundle $(E, t\theta)$. An equivalent statement of having a Hodge decomposition on some space is having a a $\mathbb{C}^*$ action on it. Therefore, one can consider the $\mathbb{C}^*$ action on Higgs bundles as a non-abelian pure Hodge structure. Because the one-to-one correspondence between the moduli space of Higgs bundle $\mathcal{M}_{\mathrm{Higgs}}(Y, G)$ and the moduli space of representations of the fundamental group $\mathcal{M}_{\mathrm{Rep}}(Y, G)$, one can define a pure Hodge structure on the moduli space we consider, $\mathcal{M}_{\mathrm{Rep}}(Y, G)$.

Simpson actually provides an "extended version" of correspondence . He shows a correspondence between semistable Higgs bundles with vanishing Chern class and some subset of representations. The extended correspondence allows us to put a $\mathbb{C}^*$ action (hence a Hodge structure) on the nilpotent completion of the fundamental groups. One significant part of this result is that this Hodge structure coincides with Hain's construction [4].

Higgs bundles provides a natural Hodge structure as we have seen. Howerver, it does not make sense of defining Higgs bundles on a singular manifold. Therefore, Higgs bundle cannot provide mixed Hodge structures, at least not in a direct way. We will turn the hope to Hain's construction of mixed Hodge structures on fundamental groups. Given any representation ρ from $\pi_1(X_t)$ to G with Zariski dense image, Hain puts a mixed Hodge structure on the "relative Malcev completion" [5] of $\pi_1(X_t)$ with repect to ρ.

To get the relative Malcev completion, first we consider all possible extension E of our coefficient group G and a unipotent U, and this E should fit the following diagram.



Then, we take the inverse limit to get a proalgebraic group $\mathcal{G}$. This proalgebraic group is the relative Malcev completion of $\pi_1(X_t)$ with repect to the representation ρ.

In theory, relative Malcev completion give a hope to find a non-abelian Clemens-Schmid exact sequence. However, it is hard to calculate the relative Malcev completion in general. In the following subsection, we will propose another direction to calculate non-abelian cohomology.

1.3.3 Tropical Geometry Methods

Let $\mathbb{K}$ be a field with non-Archmidean valuation ν. We can define a norm on $\mathbb{K}$ as

$$\|x\| = e^{-\nu(x)},$$

for all x in $\mathbb{K}$. Define a Log function on $\mathbb{K}^n$ as following

$$\begin{aligned}\text{Log}(\mathbf{x}) &= (\log(\|x_1\|), \log(\|x_2\|), \ldots, \log(\|x_n\|)) \\ &= (-\nu(x_1), -\nu(x_2), \ldots, -\nu(x_n)).\end{aligned}$$

For any algebraic variety V, define the *amoeba* of V as the image of Log of V. There are some interesting structures (tropical structures) on the image side as we will describe now.

Definition 3 (tropical semiring)**.** *A tropical semiring is* $(\mathbb{R}, \oplus, \odot)$*. For any* $x, y \in \mathbb{R}$*, define*

- $x \oplus y := \max(x, y)$

- $x \odot y := x + y$

It is easy to see, the tropical zero is $-\infty$ and the tropical one is 0. Thus, we usually add $-\infty$ to the tropical semiring, and write $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. A tropical polynomial is just a polynomial with tropical addition and tropical multiplication. The following example revels the motivation of the definition of tropical hyperplanes.

Example 1. *Consider "tropicalized" polynomial $f(x) = x^2 + 3x + 4$. By definition, we have*

$$\begin{aligned} f(x) &= x^2 \oplus 3 \odot x \oplus 4 \\ &= \max\{2x, 3 + x, 4\}. \end{aligned}$$

Figure 1.1 shows the "graph" of f. The graph of this tropical polynomial is

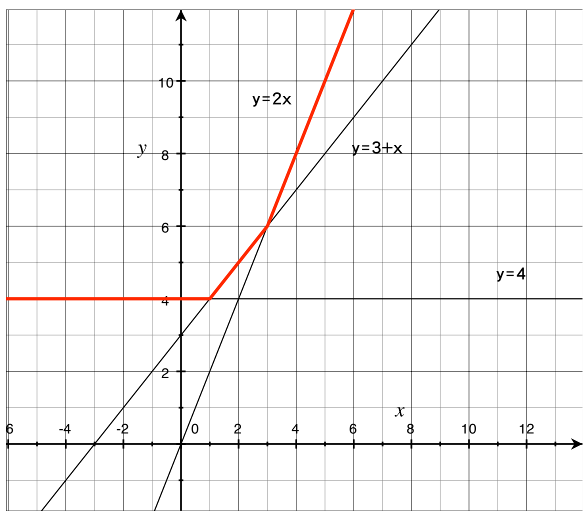


Figure 1.1: graph of $f(x) = x^2 + 3x + 4$

piecewise-linear and it fails to be linear at $x = 1$ and $x = 3$. Note that we can also "factor" this polynomial as following:

$$\begin{aligned} f(x) &= x^2 \oplus 3 \odot x \oplus 4 \\ &= (x \oplus 1) \odot (x \oplus 3). \end{aligned}$$

Therefore, $x = 1$ and $x = 3$ looks like the "roots" of $f(x)$.

The example motivates the following definition of tropical hypersurfaces.

Definition 4 (Tropical Hypersurface)**.** *Let $f(\mathbf{x}) = \sum^{\oplus} \alpha_{\mathbf{i}} \odot \mathbf{x}^{\mathbf{i}}$ be a tropical polynomial. Evaluate this polynomial is to find the maximum of the linear forms $\alpha_{\mathbf{i}} + \langle \mathbf{x}, \mathbf{i} \rangle$. A point in the tropical hypersurface H_f is the maximum of the linear forms achieve at least twice and it is exactly where the graph fails to be linear.*

An advantage of tropical geometry is that many problems can be break down to a combinatoric ones. Therefore, one may use combinatoric to solve classical algebraic geometry or complex geometry problems.

1.4 研究方法

The fundamental group of a compact Riemann surface X is well-known and so we have a chance to calculate $\mathrm{H}^1_\mathrm{G}(\mathrm{X})$. Suppose X is a compact Riemann surface of genus g. The fundamental group $\pi_1(X)$ is generated by $2g$ elements, say $\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_1, \beta_2, \ldots, \beta_g$, satisfying one relation:

$$\prod_{i=1}^{g}[\alpha_i, \beta_i] = 1.$$

We will start with a relative simple example. Let $f\colon X \to \Delta$ be a degeneration with generic fiber X_t being a compact Riemann surface of genius 2. Suppose the fundamental group $\pi_1(X_t)$ is generalized by $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfying

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} = 1.$$

The Picard-Lefschetz transformation $\tilde{T}$ on the level of fundamental groups of X_t induce a map

$$T\colon \mathsf{H}^1_G(X_t) \to \mathsf{H}^1_G(X_t).$$

We are mainly interested in those elements that are fixed under the Picard-Lefschetz transformation. In classical cases, by the local invariant theorem or the Clemens-Schmid exact sequence, the cohomology elements that are fixed under the Picard-Lefschetz transformation can be extended to an element in the global cohomology, namely $\mathsf{H}^1_G(X)$. One interesting phenomenon is that there are some elements which are fixed under the Picard-Lefschetz transformation that cannot extend to global.

We develop a method to find some elements in $\mathsf{H}^1_G(X_t)$ that is fixed by T but cannot extend to global. Observe that if $\gamma \in \pi_1(X_t)$ is the vanishing cycle, the non-abelian cohomology $[\rho] \in \mathsf{H}^1_G(X_t)$ that is fixed by T should send γ to the identity, otherwise it cannot be extended to global.

Write

$$A_1 = \rho(\alpha_1) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix},$$

$$A_2 = \rho(\alpha_2) = \begin{bmatrix} a_5 & a_6 \\ a_7 & a_8 \end{bmatrix},$$

$$B_1 = \rho(\beta_1) = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \text{ and}$$

$$B_2 = \rho(\beta_2) = \begin{bmatrix} b_5 & b_6 \\ b_7 & b_8 \end{bmatrix},$$

and the matrix representing the vanishing cycle by Γ, so $\Gamma = A_1^{-1}B_1^{-1}A_1B_1$. The matrices A_1, A_2, B_1, B_2 satisfy the following relations.

1. $\det A_i = \det B_i = 1$, for $i = 1, 2$.
2. One relation $A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_1^{-2} B_1^{-2} = I$.
3. It is fixed under the Picard-Lefschetz transformation. From our construction this gives us

$$\begin{aligned} A_1 &= H^{-1} A_1 \Gamma^{-1} H, \\ B_1 &= H^{-1} B_1 \Gamma^{-1} H, \\ A_2 &= H^{-1} A_2 \Gamma^{-1} H, \\ B_2 &= H^{-1} B_2 \Gamma^{-1} H, \end{aligned}$$

where H is a change of base matrix, and we write

$$H = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}.$$

Each relation above give us several polynomial equations in

$$\mathbb{C}[a_1, \ldots, a_8, b_1, \ldots, b_8, h_1, \ldots, h_4],$$

and we get a polynomial system. We can use the Gröbner bases to find solutions to the polynomial system.

1.5 結果與討論

By the method described in the previous section, we found a cohomology class $[\rho]$ in $\mathsf{H}^1_G(X_t)$ that is fixed under the Picard-Lefschetz transformation but cannot be extended to global. Suppose $\alpha_1, \alpha_2, \beta_1, \beta_2$ are generators of $\pi_1(X_t)$, and $A_1 = \rho(\alpha_1), A_2 = \rho(\alpha_2), B_1 = \rho(\beta_1), B_2 = \rho(\beta_2)$, we can write these elements explicitly as followings.

$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -4 & \sqrt{31} \\ 0 & -\frac{1}{4} \end{pmatrix}, B_1 = \begin{pmatrix} 4 & -\sqrt{31} \\ 0 & \frac{1}{4} \end{pmatrix}, B_2 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, and $H = \begin{pmatrix} \sqrt{2 - \frac{1}{16}} & -1 - \frac{4}{\sqrt{31}} \\ 0 & \frac{15}{16} \end{pmatrix}$.

The representation matrix of the vanishing cycle will not be identity. Suppose Γ is the image matrix of the vanishing cycle. We have

$$\Gamma = A_1^{-1} B_1^{-1} A_1 B_1 = \begin{pmatrix} 1 & -\frac{15}{16} \\ 0 & 1 \end{pmatrix}.$$

We can easily found examples in higher genius cases, as illustrated in [7]. Note that the examples are different from previous results [7, 12]. However, the image of ρ is still not Zariski dense. Therefore, there is a hope that if we take only those classes $[\rho]$ in $\mathsf{H}^1_G(()X_t)$, one might be able to use Hain's relative Malcev completion to explicitly find a correct analog of the Clemens-Schmid exact sequence. Unfortunately, the relative Malcev completion is hard to compute in general, so we might need to use some other approaches.

We found a promising new field which is called tropical geometry. The technique in tropical geometry allows us to convert algebraic geometry problems into combinatoric ones. In order to apply tropical geometry to the non-abelian Hodge theroy, we need two important backgrounds. The first is that we will have to be able to "slice" a huge manifold X into hyperplane sections X_t. Therefore, we have to use intersection theory in terms of tropical geometry. Lars Allermann and Johannes Rau [2] give the first step in this direction. Moreover, we will need to "tropicalization" $\mathsf{H}^1_G(X_t)$, and a new paper by Alessandrini [1] will help us in this direction. Applying these tropical geometry techniques to the non-abelian Hodge theory will be our main research topics in the following years.

Chapter 2

成果自評

Let $f\colon X \to \Delta$ be a degeneration with generic fiber X_t being a compact Riemann surface of genus n. We explicitly construct some chomology that are fixed under the Picard-Lefschetz transformation but cannot extend to global. The examples are different from our previous results [7, 12]. The new examples give a hope that with some proper restriction, there will a correct analog to the Clemens-Schmid exact sequence in non-abelian cases. We will submit our new results as long as some of our previous works soon.

To do deeper analysis of the non-abelian cohomolgy, we study Simpson's and Hain's results, but find them are difficult to calculate. Therefore, we propose another direction: using the theory of tropical geometry. There are some very important tools available to us, namely the tropicalization of group representations [1] and tropical intersection theory [2]. We will survey on these theories and try to apply them to the non-abelian cohomology.

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