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碩 士 學 位 論 文

Utility Indifference Pricing in Incomplete Markets

效用無差異價格於不完全市場下之應用

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Abstract

In incomplete markets, prices of a contingent claim can be obtained between the upper and lower hedging prices. In this thesis, we will use utility indifference pricing to find an initial payment for which the maximal expected utility of trading the claim is indifferent to the maximal expected utility of no trading. From the central duality result, we show that the gap between the seller's and the buyer's utility indifference prices is always smaller than the gap between the upper and lower hedging prices under the exponential utility function.

中文摘要

在不完全市場下，衍生性金融商品可利用上套利和下套利價格來訂出價格區間。我們運用效用無差異定價於此篇論文中，此定價方式爲尋找一個初始交易價，會使在起始時交易商品和無交易商品於商品到期日之最大期望效用相等。利用主要的對偶結果，我們證明在指數效用函數下，效用無差異定價區間會比上套利和下套利定價區間小。

Contents

1 Introduction

Pricing contingent claims, also called derivative securities or options, are very important topics in finance. Every contingent claim can admit a unique arbitrage-free price in complete markets. On the other hand, most contingent claims only acknowledge intervals of arbitrage-free prices in incomplete markets. The main result of the thesis is to price smaller intervals of arbitrage-free prices for bounded European contingent claims in incomplete markets.

In the fundamental financial market, we describe risky assets by process

$$dS(t) = S(t)[\mu(t)dt + \sigma(t) \cdot dB(t)],$$

where $B(t)$ is a Brownian motion. If there exists an equivalent local martingale measure with respect to the normalized market, then the market has no arbitrage. We say that the contingent T-claim G is attainable if there exists an admissible portfolio $\pi(t)$ and initial capital $x \in \mathbb{R}$ such that

$$G(\omega) = x + \int_0^T \pi(t) \cdot dS(t) \quad \textbf{a.s.}.$$

In complete markets, every contingent T-claim can be attainable by some admissible portfolios at maturity time T. Moreover, the contingent claim price at initial time is taking the expectation of a risky free asset discount of it's maturity value with respect to the (local) martingale measure. We can accept the unique (local) martingale measure by the Girsanov's theorem.

In incomplete markets, there are infinite equivalent local martingale measures, so that it isn't definite to take expectation. There have been some studies of contingent claim prices in an incomplete market. We compare superreplication, subreplication and utility indifference pricing in this thesis. Superreplication is defined as a minimal price of the initial capital such that the maturity capital is more than the contingent claim for an investment strategy. The price is also called the seller's price or the upper hedging price of the contingent claim. Similarly, subreplication is to consider a maximal payment for this security such that the buyer's maturity capital adding the contingent claim for an investment strategy is always to make money. The payment is also called the buyer's price or the lower hedging price of the contingent claim. From the seller's and buyer's hedging prices, we can determine the price interval of the contingent claim.

In general, the gap between the upper and lower hedging prices is too broad. Other famous studies are about the utility function. We study the idea of utility indifference pricing. The pricing formula was introduced by Hodges and Neuberger (1989). It is based on an initial payment for which the maximal expected utility of trading the claim is indifferent to no trading. As utility indifference pricing, Øksendal and Sulem (2009) introduced risk indifference pricing. It follows from a convex risk measure

$$\rho : \mathbb{F} \to \mathbb{R},$$

where $\mathbb{F}$ is the set of some random variables; see Föllmer and Schied (2002). Therefore, we can get an initial payment for which the risk of trading the claim is indifferent to no trading. Moreover, Øksendal and Sulem (2009) proofed the price interval of risk indifference pricing belongs to the price interval of the upper and lower hedging prices in jump diffusion markets. The purpose of this thesis is to prove that the price interval of utility indifference pricing also falls between the upper and lower hedging prices for every bounded T-claim in the fundamental market. To verify the main result, we use the relative entropy which measures the difference between two probability measures in probability theory and information theory. Under an exponential utility function, the relative entropy can lead to a special duality, called the central duality result. The central duality result can help us to solve the utility indifference price.

The framework of the thesis is organized as follows: in section 2, we consider the fundamental finical market model and describe definitions of superreplication, subreplication and utility indifference pricing. In section 3, we give some principles of local martingale and prove that the upper and lower hedging prices can be represented by the maximal and minimal local martingales of bounded claims. In section 4, we introduce the relative entropy and verify the price interval of utility indifference pricing falls between the price interval of the upper and lower hedging prices, based on the central duality result. In section 5, we prove that the central duality result of the maximal expect utility by applying the relative entropy. Finally, we give the conclusions and further researches in section 6.

2 The Fundamental Financial Market Model

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$, where we define $\mathcal{F}_t = \mathcal{F}_t^{(m)}$ to be the σ-algebra generated by the m-dimensional Brownian motion $B_t(\omega)$. Let a financial market $S(t)$ with $n+1$ assets. The price processes of assets are $\mathcal{F}_t$-adapted and have the forms:

(1) a risk free asset with unit price, $S_0(t) = 1$ for every time $0 \le t \le T$.

(2) risky asset prices $S_i(t)$ for $i = 1, ..., n$ where we take a risk free asset as the unit of account are given by

$$\begin{aligned} dS_i(t) &= S_i(t)[\,\mu_i(t)dt + \sigma_i(t)\cdot dB(t)] \quad \text{for } t \in [0,T],\\ S_i(0) &= s_i. \end{aligned}$$

We consider $\mu_i(t)$ and $\sigma_i(t)$ are $\mathcal{F}_t$-adapted processes for $i = 1, ..., n$ with

$$\int_0^T \left\{ \sum_{i=1}^n |\mu_i(s)| + \sum_{j=1}^m \sum_{i=1}^n |\sigma_{ij}(s)|^2 \right\} ds < \infty \;\; \textbf{a.s.}.$$

A portfolio in the market $\{S(t)\}_{t\in[0,T]}$ is an $(n+1)$-dimensional (t,ω)-measurable and $\mathcal{F}_t$-adapted stochastic process. The value process with initial value x at time t of a portfolio $\pi(t) = (\pi_0(t), \pi_1(t), ..., \pi_n(t))$ is defined by

$$X_x^{(\pi)}(t,\omega) = \pi(t)\cdot S(t) = \sum_{i=0}^n \pi_i(t) S_i(t)$$

where $X_x^{(\pi)}(0) = x$.

Definition 2.1 *(1) The portfolio $\pi(t)$ is called self-financing if*

$$\int_0^T \left\{ |\sum_{i=1}^n \pi_i(s)\,\mu_i(s)| + \sum_{j=1}^m \left[\sum_{i=1}^n \pi_i(s)\,\sigma_{ij}(s) \right]^2 \right\} ds < \infty \;\; \boldsymbol{a.s.} \tag{2.1}$$

and

$$X_x^{(\pi)}(t) = x + \int_0^t \pi(s)\cdot dS(s) \quad \text{for } t \in [0,T].$$

(2) A portfolio $\pi(t)$ which is self-financing is called admissible if the value process $X_x^{(\pi)}(t)$ is (t,ω) a.s. lower bounded.

Note that we say a portfolio with (2.1) is S-integrable. Moreover, if the initial value x and $\pi_1(t), ..., \pi_n(t)$ are given such that (2.1) holds, then there always exists a $\pi_0(t)$ such that $\pi(t)$ is a self-financing portfolio. A bounded European contingent T-claim G is a bounded $\mathcal{F}_T$-measurable random variable and taken a risk free asset as the unit of account. It represents the net payoff of a derivative security at maturity T. In order to study incomplete market, we introduce pricing formula below.

Superreplication and subreplication. Let $\mathcal{P}$ be the set of all admissible portfolios. Superreplication of a T-claim G is to consider a minimal price as the initial capital x such that the maturity capital is always more than G for an investment strategy. The minimal price which the seller is willing to sell is

$$P_{up}(G) = \inf\{x|\text{ there exits } \pi \in \mathcal{P} \text{ such that } X_x^{(\pi)}(T) \geq G \textbf{ a.s. }\}$$

and also called the seller's price or the upper hedging price of G. Similarly, subreplication is to consider a maximal payment for this security G such that the buyer's maturity capital adding the contingent claim for an investment strategy is always to make a profit. The maximal price which the buyer is willing to pay is

$$P_{low}(G) = \sup\{x|\text{ there exits } \pi \in \mathcal{P} \text{ such that } X_{-x}^{(\pi)}(T) + G \geq 0 \textbf{ a.s. }\}$$

and also called the buyer's price or the lower hedging price of G. In the complete market, it follows from the Girsanov's theorem there exists a martingale measure Q such that

$$P_{low}(G) = E_Q[G] = P_{up}(G) :$$

see [10, Theorem 12.3.2].

Utility indifference pricing. Given a utility function $U : \mathbb{R} \to \mathbb{R}$. Let x be the initial capital before the claim G is being traded.

(1) If we give the price p_1 for the guarantee to the seller, then the seller can use $x+p_1$ as the initial value to invest and needs to pay the guarantee G at maturity time. The maximal expected utility for the seller is

$$V_s(G, x+p_1) = \sup_{\pi \in \mathcal{P}} E[U(X^{(\pi)}_{x+p_1}(T) - G)].$$

(2) There is no claim G to trade. The maximal utility expected utility at maturity time T is

$$V(G, x) = \sup_{\pi \in \mathcal{P}} E[U(X^{(\pi)}_{x}(T))].$$

(3) If we give the payment p_2 for the guarantee to the buyer, then the buyer use an initial fortune $x - p_2$ to invest and can gain the guarantee G at maturity time. The maximal expected utility for the buyer is

$$V_b(G, x-p_2) = \sup_{\pi \in \mathcal{P}} E[U(X^{(\pi)}_{x-p_2}(T) + G)].$$

The seller's and buyer's utility indifference prices are to find initial prices $p_1 = p^u_s$ and $p_2 = p^u_b$ with $V_s(G, x+p^u_s) = V(G, x)$ and $V_b(G, x-p^u_b) = V(G, x)$. Denote $P_{seller}(G) := p^u_s$ and $P_{buyer}(G) := p^u_b$ for the T-claim G. In this study, we given an exponential utility

$$U(x) = -e^{-\gamma x},$$

where $\gamma > 0$ is the risk aversion parameter.

3 Superreplication and Subreplication

Itô integrals $\int_0^t \pi(s) \cdot dB(s)$ exist and are martingales under the condition

$$E\left[\int_0^T |\pi(t)|^2 dB(t)\right] < \infty.$$

In fact, Itô integrals $\int_0^t \pi(s) \cdot dB(s)$ exist under the weak condition

$$\int_0^T |\pi(t)|^2 \, dt < \infty \quad \textbf{a.s.}.$$

However, we can't pledge $\int_0^t \pi(s) \cdot dB(s)$ are martingales under the weak condition. In general, we discuss local martingale rather than martingale in stochastic differential equations.

Definition 3.1 *An $\mathcal{F}_t$-adapted stochastic process $M(t) \in \mathbb{R}^n$ is called a local martingale with respect to $\{\mathcal{F}_t\}$ if there exists an increasing sequence of $\mathcal{F}_t$-stopping times τ_k such that*

$$\tau_k \to \infty \text{ a.s.} \quad as \quad k \to \infty$$

and $M(t \wedge \tau_k)$ is an $\mathcal{F}_t$-martingale for all k.

We can get following properties:

Property 3.2 *Let a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, P)$ and $M(t)$ be a local martingale with respect to $\{\mathcal{F}_t\}$.*

(1) If $M(t)$ is lower bounded, then $M(t)$ is a supermartingale.

(2) Let $\phi(t, \omega)$ be a $\mathcal{F}_t$-adapted process such that $\int_0^t \phi(s) \cdot dM(s)$ exist. Then,

$$Z(t) := \int_0^t \phi(s) \cdot dM(s)$$

is a local martingale for $0 \le t \le T$.

Proof. (1). Given $t \geq s$. Suppose τ_k is an increasing sequence of $\mathcal{F}_t$-stopping times such that $M(t \wedge \tau_k)$ is an $\mathcal{F}_t$-martingale for all k. From Fatou's lemma,

$$E[\liminf_{k\to\infty} M(t \wedge \tau_k)] \leq \liminf_{k\to\infty} E[M(t \wedge \tau_k)]. \tag{3.1}$$

A process $M(t \wedge \tau_k)$ is an $\mathcal{F}_t$-martingale and

$$E[M(t \wedge \tau_k)] = E[E[M(t \wedge \tau_k)|\mathcal{F}_0\,]\,] = E[M(0 \wedge \tau_k)] = E[M(0)].$$

Since $M(t \wedge \tau_k)$ exists as $k \to \infty$ and (3.1), we get

$$E[M(t)] = E[\lim_{k\to\infty} M(t \wedge \tau_k)] \leq E[M(0)]$$

and $M(t)$ is integrable. It follows from Fatou's lemma for conditional expectations,

$$E[\liminf_{k\to\infty} M(t \wedge \tau_k)|\mathcal{F}_s] \leq \liminf_{k\to\infty} E[M(t \wedge \tau_k)|\mathcal{F}_s] = \liminf_{k\to\infty} M(s \wedge \tau_k)$$

and hence $E[M(t)|\mathcal{F}_s] \leq M(s)$ almost surely.

(2). Since $M(t)$ is a local martingale with respect to $\{\mathcal{F}_t\}$, there exists an increasing sequence of stopping times τ'_k such that definition 3.1 holds. Suppose $\{U_k\}$ is an increasing sequence of bounded open sets in $\mathbb{R}^n$. Let

$$\tau_k = \inf\{t \geq 0 |\, \phi(t) \notin U_k\} \wedge \tau'_k \quad \text{for} \quad k \in \mathbb{N}.$$

Since $\phi(t \wedge \tau_k)$ is bounded and $M(t \wedge \tau_k)$ is a martingale for $t \in [0, T]$, we obtain

$$Z(t \wedge \tau_k) := \int_0^{t\wedge\tau_k} \phi(s) \cdot dM(s)$$

is a martingale for $t \in [0, T]$. The proof is complete. □

A probability measure Q is called a local martingale measure with respect to P, if it is equivalent to P and the process $\hat{S} = (S_1(t), ..., S_n(t))$ is a Q-local martingale. By $\mathcal{M}$ we denote the set of all equivalent local martingale measures for process $\hat{S}$. In this section, we show the upper hedging price and lower hedging price can be represented by

$$P_{up}(G) = \sup_{Q\in\mathcal{M}} E_Q[G],$$

$$P_{low}(G) = \inf_{Q\in\mathcal{M}} E_Q[G].$$

The following theorem 3.3 and property 3.4 are adapted from the paper of Kramkrov (1996).

Theorem 3.3 *(*Optional Decomposition*, [8, Theorem 2.1]) Let $F(t) \in \mathbb{R}$ be a positive $\mathcal{F}_t$-adapted process. Then F is a super-martingale for each measure $Q \in \mathcal{M}$ if and only if there exists an S-integrable predictable process $H(t) \in \mathbb{R}^n$ and an $\mathcal{F}_t$-adapted increasing positive process C such that*

$$F(t) = F(0) + \int_0^t H(s) \cdot d\hat{S}(s) - C(t),\ t \in [0, T].$$

Proposition 3.4 *([8, Proposition 4.2]) Let f be a positive variable on $(\Omega, \mathcal{F}, P)$ with $\sup_{Q \in \mathcal{M}} E_Q f < \infty$. There is a right continuous and left limit exists process*

$$F(t) = ess \sup_{Q \in \mathcal{M}} E_Q[f|\mathcal{F}_t], \quad t \geq 0.$$

The process F is a Q super-martingale whatever $Q \in \mathcal{M}$.

We consult theorem 3.5 of Kunita (2004) and proof the main result in this section as follows:

Theorem 3.5 *Let G be a bounded European contingent T-claim. We get*

$$P_{up}(G) = \sup_{Q \in \mathcal{M}} E_Q[G],$$

$$P_{low}(G) = \inf_{Q \in \mathcal{M}} E_Q[G],$$

where $\mathcal{M}$ denotes the set of equivalent local martingale measures Q.

Proof. Without loss of generality, we assume $G \geq 0$. Set $p = \sup_{Q \in \mathcal{M}} E_Q[G]$ and

$$\mathcal{A}_{up} = \{x | \text{ there exits } \pi \in \mathcal{P} \text{ such that } X_x^{(\pi)}(T) \geq G \ \textbf{a.s.}\ \}.$$

We first prove $p \leq P_{up}(G)$. Let $x' \in \mathcal{A}_{up}$, then there exists a $\pi' \in \mathcal{P}$ such that

$$X_{x'}^{(\pi')}(T) = x' + \int_0^T \pi'(t) \cdot dS(t) \geq G \quad \textbf{a.s.}.$$

Given a $Q \in \mathcal{M}$. Since G is lower bounded and $Q \ll P$, we get

$$X_{x'}^{(\pi')}(T) = x' + \int_0^T \pi'(t) \cdot dS(t) \geq G \quad \textbf{a.s.} \quad \text{w.r.t.} \quad Q \tag{3.2}$$

and it is a lower bounded local Q-martingale. By property 3.2, $X_{x'}^{(\pi')}(t)$ is a super-martingale with respect to Q and

$$E_Q[\int_0^T \pi'(s) \cdot dS(s)] = E_Q[E_Q[\int_0^T \pi'(s) \cdot dS(s) | \mathcal{F}_t]] \leq 0.$$

Taking the expectation of (3.2) with respect to Q we obtain

$$x' \geq E_Q[G].$$

Hence

$$p \leq P_{up}(G). \tag{3.3}$$

Next, we verify the reverse inequality. From property 3.4., $ess \sup_{Q \in \mathcal{M}} E_Q[G|\mathcal{F}_t]$ is a super-martingale for $Q \in \mathcal{M}$ and lower bounded. It follows from theorem 3.3 that

$$ess \sup_{Q \in \mathcal{M}} E_Q[G|\mathcal{F}_t] = E_Q[G] + \int_0^t H(s) \cdot d\hat{S}(s) - C(t), \ \ t \in [0, T],$$

where H is an S-integrable process and C is a positive increasing process. There exists a $\pi \in \mathcal{P}$ such that $\pi_i = H_i$ for $i = 1, ..., n$. We get

$$E_Q[G] + \int_0^T \pi(t) \cdot dS(t) \geq ess \sup_{Q \in \mathcal{M}} E_Q[G|\mathcal{F}_T] = G \quad \textbf{a.s.},$$

and $E_Q[G] \in \mathcal{A}_{up}$. Therefore,

$$P_{up}(G) \leq E_Q[G] \leq p.$$

From (3.3), we obtain $P_{up}(G) = p$. It is similar to verify $P_{low}(G) = \inf_{Q \in \mathcal{M}} E_Q[G]$.
$\square$

4 Utility Indifference Pricing

In this section, we introduce the relative entropy and the central duality theorem to show the main result. Denote $\mathbb{P}_a$ the set of local martingale measures for $\hat{S}$ absolutely continuous to P. The relative entropy $H(Q|P)$ is defined by

$$H(Q|P) = \begin{cases} E\left[\dfrac{dQ}{dP}\ln\dfrac{dQ}{dP}\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

In probability theory and information theory, the quantity $H(Q|P)$ measures the difference between two probability measures Q and P. Define $\mathbb{P}_f = \mathbb{P}_f(P)$ to be the set of $Q \in \mathbb{P}_a$ with finite relative entropy, $H(Q|P) < \infty$. We consider the assumption

$$\mathbb{P}_f \cap \mathcal{M} \neq \emptyset. \tag{4.1}$$

There is a unique measure $Q^0 \in \mathbb{P}_f \cap \mathcal{M}$ minimizing $H(Q|P)$ over all $Q \in \mathbb{P}_f$ and call Q^0 the minimal P-entropy martingale measure: see [7, Property 3.1 and Property 3.2].

The main result of this study is obtained from the following theorem called the central duality result. Let G be a bounded T-claim.

Theorem 4.1 *Given $\lambda \in \mathbb{R}$ and consider the function*

$$u(x;\lambda) := -\frac{1}{\gamma}\inf_{\pi\in\mathcal{P}} \ln E[\exp\left(-\gamma(X_x^{(\pi)}(T) + \lambda G)\right)]. \tag{4.2}$$

Then, we can get

$$u(x;\lambda) = x + \inf_{Q\in\mathbb{P}_f}\{\lambda E_Q[G] + \frac{1}{\gamma}H(Q|P)\}. \tag{4.3}$$

Moreover, the infimum in (4.3) is attained for a unique $Q^\lambda \in \mathbb{P}_f \cap \mathcal{M}$ whose Radon-Nikodym derivative is given by

$$\frac{dQ^\lambda}{dP} = \exp\left(c_\lambda - \gamma\left(\int_0^T \pi^\lambda \cdot dS + \lambda G\right)\right),$$

where π^λ is a self-financing portfolio and c_λ is a constant. In particular, if $\int_0^t \pi^\lambda \cdot dS$ is (t,ω) a.s. lower bounded, then $\pi^\lambda \in \mathcal{P}$ attains the infimum in (4.2).

Because $\ln x/\gamma$ is a continuous function on $(0,\infty)$, we can formulate the central duality result to utility indifference pricing. The proof of theorem 4.1 is obtained in section 5. Moreover, we improve the following result of İlhan, Jonsson and Sircar (2005) by theorem 4.1.

Denote Θ to be the set of $\mathcal{F}_t$-adapted, S-integrable $\mathbb{R}^{n+1}$-valued processes θ such that θ is self-financing and $X_0^{(\theta)}(T)$ are Q-martingales for all $Q \in \mathbb{P}_f$.

Theorem 4.2 *([6, Theorem 4.1], [1, Theorem 2.2] and [7, Theorem 2.1]) Given $\lambda \in \mathbb{R}$ and consider the function*

$$u'(x;\lambda) := -\frac{1}{\gamma}\inf_{\theta\in\Theta} \ln E[\exp\left(-\gamma(X_x^{(\pi)}(T) + \lambda G)\right)]. \tag{4.4}$$

Then, we can get

$$u'(x;\lambda) = x + \inf_{Q\in\mathbb{P}_f}\{\lambda E_Q[G] + \frac{1}{\gamma}H(Q|P)\}. \tag{4.5}$$

Moreover, the infimum in (4.5) is attained for a unique $Q^\lambda \in \mathbb{P}_f \cap \mathcal{M}$ whose Radon-Nikodym derivative is given by

$$\frac{dQ^\lambda}{dP} = \exp\left(c_\lambda - \gamma\left(\int_0^T \theta^\lambda \cdot dS + \lambda G\right)\right),$$

where $\theta^\lambda \in \Theta$ attains the infimum in (4.4), and c_λ is a constant.

The seller's and buyer's utility indifference price of the bounded claim G is the solution p_s^u and p_b^u of the stochastic differential equations

$$V_s(G, x + p_s^u) = V(G,x) = V_b(G, x - p_b^u). \tag{4.6}$$

Put an exponential utility $U(x) = -e^{-\gamma x}$ into (4.6), where $\gamma > 0$ is the risk aversion parameter. We get equations

$$\begin{aligned} -\inf_{\pi\in\mathcal{P}} E[\exp(-\gamma(X_{x+p_s^u}^{(\pi)}(T) - G))] &= -\inf_{\pi\in\mathcal{P}} E[\exp(-\gamma X_x^{(\pi)}(T))], \\ -\inf_{\pi\in\mathcal{P}} E[\exp(-\gamma(X_{x-p_b^u}^{(\pi)}(T) + G))] &= -\inf_{\pi\in\mathcal{P}} E[\exp(-\gamma X_x^{(\pi)}(T))]. \end{aligned} \tag{4.7}$$

Since $\ln x$ is a continuous function on $(0, \infty)$, it follows from (4.2) and (4.7) that we rewrite (4.6) by

$$u(x + p_s^u; -1) = u(x; 0) = u(x - p_b^u; 1). \tag{4.8}$$

Hence, utility indifference pricing for bounded T-claim G are solutions p_s^u and p_b^u of (4.8) and we consider $P_{seller}(G) = p_s^u$ and $P_{buyer}(G) = p_b^u$.

Lemma 4.3 *Let G be a bounded T-claim. Then*

$$P_{buyer}(G) \leq P_{seller}(G).$$

Proof. From the Theorem 4.1 and (4.8), we get

$$P_{seller}(G) = \sup_{Q \in \mathbb{P}_f} \{E_Q[G] - \frac{1}{\gamma}(H(Q|P) - H(Q^0|P))\}$$
$$P_{buyer}(G) = \inf_{Q \in \mathbb{P}_f} \{E_Q[G] + \frac{1}{\gamma}(H(Q|P) - H(Q^0|P))\}$$

where Q^0 is a unique measure in $\mathbb{P}_f \cap \mathcal{M}$ and minimize $H(Q|P)$ over all $Q \in \mathbb{P}_f$. Let a function $\zeta : \mathbb{P}_f \to \mathbb{R}$ with

$$\zeta(Q) = \frac{1}{\gamma}(H(Q|P) - H(Q^0|P)) \geq 0.$$

We obtain

$$\begin{aligned} P_{seller}(G) - P_{buyer}(G) &= \sup_{Q \in \mathbb{P}_f} \{E_Q[G] - \zeta(Q)\} - \inf_{Q \in \mathbb{P}_f} \{E_Q[G] + \zeta(Q)\} \\ &= \sup_{Q \in \mathbb{P}_f} \{E_Q[G] - \zeta(Q)\} + \sup_{Q \in \mathbb{P}_f} \{-E_Q[G] - \zeta(Q)\} \\ &\geq \sup_{Q \in \mathbb{P}_f} \{-2\zeta(Q)\} = 0. \end{aligned}$$

□

Using lemma 4.3 we can get the following inequality.

Theorem 4.4 *Let G be a bounded T-claim. Then*

$$P_{low}(G) \leq P_{buyer}(G) \leq P_{seller}(G) \leq P_{up}(G).$$

Proof. It suffices to verify $P_{seller}(G) \leq P_{up}(G)$. From theorem 4.1 and lemma 4.3, we get

$$\begin{aligned} P_{seller}(G) &= \sup_{Q \in \mathbb{P}_f \cap \mathcal{M}} \{E_Q[G] - \zeta(Q)\} \\ &\leq \sup_{Q \in \mathcal{M}} E_Q[G] = P_{up}(G). \end{aligned}$$

It is similar to show $P_{low}(G) \leq P_{buyer}(G)$.

□

5 Proof of the Central Duality Result

We consult the idea of Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002) to the setting of theorem 4.1. Consider a bounded contingent T-claim G and define a probability measure P_G equivalent to P by

$$\frac{dP_G}{dP} := C_G \, e^{\gamma G}, \tag{5.1}$$

where $C_G^{-1} := E[e^{\gamma G}] \in (0, \infty)$. Since G is bounded, we can get for $Q \ll P$

$$H(Q|P) = E_Q\left[\ln \frac{dQ}{dP_G} + \ln C_G + \gamma G\right] = H(Q|P_G) + \ln C_G + E_Q[\gamma G] \tag{5.2}$$

and $H(Q|P) < \infty$ implies $H(Q|P_G) < \infty$. In particular, denote the set

$$\begin{aligned}\mathbb{P}_f(P_G) &:= \{Q \in \mathbb{P}_a | H(Q|P_G) < \infty\} \\ &= \{Q \in \mathbb{P}_a | H(Q|P) < \infty\} \\ &= \mathbb{P}_f(P)\end{aligned}$$

and write $\mathbb{P}_f$ simply. Recall the assumption (4.1), there is a unique measure $Q_G^0 \in \mathbb{P}_f \cap \mathcal{M}$ minimizing $H(Q|P_G)$ over all $Q \in \mathbb{P}_f$ and call Q_G^0 the minimal P_G-entropy martingale measure.

To verify the central duality result, we need one additional arrangement. Under assumption (4.1), the density function of the minimal entropy Q^0 with respect to P has the form

$$\frac{dQ^0}{dP} = K \, \exp\left(X_0^{(\theta)}(T)\right) = K \, \exp \int_0^T \theta \cdot dS \tag{5.3}$$

for some constant $K > 0$ and some self-financing θ such that

$$X_0^{(\theta)}(T) = \int_0^T \theta \cdot dS$$

is a Q^0-martingale: see [3, Corollary 2.1] and [4, Proposition 3.2 and the proof of Theorem 4.13]. Taking the natural logarithm and the expectation under Q^0, we could rewrite (5.3) by

$$\ln K = E_{Q^0}[\ln \frac{dQ^0}{dP}] = H(Q^0|P). \tag{5.4}$$

In the same way for Q_G^0, we get the density function

$$\frac{dQ_G^0}{dP_G} = K_G \exp\left(X_0^{(\theta_G)}(T)\right) = K_G \exp\left(\int_0^T \theta_G \cdot dS\right) \tag{5.5}$$

for some constant $K_G > 0$ and some self-financing θ_G such that

$$X_0^{(\theta_G)}(T) = \int_0^T \theta_G \cdot dS$$

is a Q_G^0-martingale. Let

$$\theta^* = -\frac{1}{\gamma}\theta_G \tag{5.6}$$

be a self-financing portfolio such that

$$\exp\left(-\gamma X_0^{(\theta^*)}(T)\right) = \exp\left(X_0^{(\theta_G)}(T)\right) = \frac{1}{K_G}\frac{dQ_G^0}{dP_G} \tag{5.7}$$

is exactly in $L^1(P_G)$.

Proof of theorem 4.1. Since we give $\lambda \in \mathbb{R}$, λG is always bounded. Without loss of generality, we suppose $\lambda = -1$ and consider an initial value $x \in \mathbb{R}$. Writing $u_1(x)$ and $u_2(x)$ for equations (4.2) and (4.3) imply

$$\begin{aligned}
u_1(x) &= -\frac{1}{\gamma}\inf_{\pi\in\mathcal{P}} \ln E[\exp\left(-\gamma(x + \int_0^T \pi \cdot dS - G)\right)] \\
&= x - \frac{1}{\gamma}\inf_{\pi\in\mathcal{P}} \ln E_{P_G}[\frac{1}{C_G}\exp\left(-\int_0^T \gamma\pi \cdot dS\right)] \\
&= x + \frac{\ln C_G}{\gamma} - \frac{1}{\gamma}\inf_{\pi\in\mathcal{P}} \ln E_{P_G}[\exp\left(-\int_0^T \pi \cdot dS\right)]
\end{aligned}$$

and

$$\begin{aligned}
u_2(x) &= x + \inf_{Q\in\mathbb{P}_f}\{-E_Q[G] + \frac{1}{\gamma}H(Q|P)\} \\
&= x + \inf_{Q\in\mathbb{P}_f}\{-E_Q[G] + \frac{1}{\gamma}(H(Q|P_G) + \ln C_G + E[\gamma G])\} \\
&= x + \frac{\ln C_G}{\gamma} + \frac{1}{\gamma}\inf_{Q\in\mathbb{P}_f} H(Q|P_G)
\end{aligned}$$

by (5.1) and (5.2).

As argued above and further including the definition of Q_G^0, we can verify that

$$-\inf_{\pi\in\mathcal{P}} \ln E_{P_G}[\exp\left(-\int_0^T \pi \cdot ds\right)] = \inf_{Q\in\mathbb{P}_f} H(Q|P_G) = H(Q_G^0|P_G).$$

From (5.5), we can get

$$H(Q_G^0|P_G) = E_{Q_G^0}[\ln \frac{dQ_G^0}{dP_G}] = \ln K_G.$$

For any $\pi \in \mathcal{P}$, it follows from $Q_G^0 \in \mathbb{P}_f \cap \mathcal{M}$ and property 3.2 that $\int_0^T \pi \cdot dS$ is a Q_G^0-supermartingale. Since $\int_0^T \theta_G \cdot dS$ is a Q_G^0-martingale, (5.5) and Jensen's inequality, we obtain

$$\begin{aligned}
\ln E_{P_G}[\exp(-\int_0^T \pi \cdot dS)] &= \ln E_{Q_G^0}[\frac{1}{K_G}\exp(-\int_0^T \pi \cdot dS - \int_0^T \theta_G \cdot dS)] \\
&\geq -\ln K_G + E_{Q_G^0}[-\int_0^T \pi \cdot dS - \int_0^T \theta_G \cdot dS] \\
&\geq -\ln K_G.
\end{aligned}$$

Therefore,

$$-\inf_{\pi \in \mathcal{P}} \ln E_{P_G}[\exp(-\int_0^T \pi \cdot dS)] \leq \ln K_G.$$

Next, we verify the reverse inequality. From (5.5), there exists a self-financing portfolio θ_G such that

$$E_{P_G}[\exp(\int_0^T \theta_G \cdot dS)] = \frac{1}{K_G}.$$

Choose a self-financing portfolio π_G such that for $t \in [0, T]$, $\omega \in \mathcal{F}_t$ consider

$$\pi_G(t, \omega) = \begin{cases} \theta_G(t, \omega) & \text{if } \int_0^t \theta_G(s, \omega) \cdot dS(s) > 0, \\ 0 & \text{if } \int_0^t \theta_G(s, \omega) \cdot dS(s) \leq 0 \end{cases}$$

and hence

$$\ln E_{P_G}[\exp(-\int_0^T \pi_G \cdot dS)] \leq \ln E_{P_G}[\exp(-\int_0^T \theta_G \cdot dS)] = -\ln K_G.$$

Since π_G is lower bounded, we can get $\pi_G \in \mathcal{P}$ and

$$-\inf_{\pi \in \mathcal{P}} \ln E_{P_G}[\exp(-\int_0^T \pi_G \cdot dS)] \geq \ln K_G.$$

Choose $\pi^\lambda = \theta^* = -\theta_G/\gamma$, $Q^\lambda = Q_G^0$ and $c_\lambda = \ln(K_G C_G)$. Thus the proof is complete. □

6 Conclusion

Summarizing the thesis, it follows from theorem 3.5 and theorem 4.1 that we transform superreplication, subreplication and the maximal expect utility into representations of equivalent local martingale measures with respect to given risky asset prices. Therefore, we can compare pricing formulas and determine the main result.

Note that this study took a step in the exponential utility function and bounded claims in the fundamental financial market. If the contingent claim is unbounded, the proof of theorem 4.1 is not complete. But the central dual result of the maximal expect exponential utility still holds under some assumptions; see Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002). On the other hand, it is possible of course to consider other utility functions or risky asset processes.

In additional, risk indifference pricing is established by convex risk measure

$$\rho : \mathbb{F} \to \mathbb{R},$$

satisfying some axioms, where $\mathbb{F}$ is the set of $\mathcal{F}_T$ measurable random variables. Based on theorem 2.2 of Øksendal and Sulem (2009), the convex risk measure has the representation about a family $\mathcal{L}$ of probability measures $Q \ll P$ on $\mathcal{F}_T$, for which $\mathcal{L}$ is given. Therefore, further research is required to compare risk indifference pricing with utility indifference pricing for some suitable $\mathcal{L}$.

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