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博士論文

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人壽保險人之資產負債管理：有效存續期間/有效凸性之分析與模擬最佳化

Asset and Liability Management for Life Insurers: Effective Duration and Effective Convexity Analysis and Simulation Optimization

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中文摘要

本研究的第一部份是利用有效存續期間與有效凸性來衡量人壽保險人的利率風險。我們發現 Tsai (2009)指出的壽險保單準備金之有效存續期間結構並非一般化的結果。當長期利率水準高於保單預定利率及保單解約率敏感於利差時，準備金之有效存續期間會呈現與 Tsai (2009)相反的結構。我們進一步發現準備金之有效凸性會亦有可能呈現負值，且不易依照保單到期期限歸納出一般化的結構。負值的有效凸性起因於準備金並非利率的單調函數，且準備金與利率的函數關係隨保單到期期限而不同。我們的研究結果可以幫助人壽保險人執行更為精確的資產負債管理。

本研究的第二部分是利用模擬最佳化的方法，幫助銷售傳統壽險保單的保險人求解出適切的業務槓桿與資產配置策略。我們假設保險人在考量破產機率與報酬率的波動之下，將資本與淨保費收入投資於資本市場中，以追求較高的業主權益報酬率。以業務槓桿與資產配置相互影響為前提，我們求解出適切的業務槓桿與多期資產配置策略，並分析在不同的業務槓桿之下，保險人多期資產配置的差異。

關鍵字：有效存續期間、有效凸性、保單準備金、資產配置、業務槓桿、模擬最佳化、人壽保險人

ABSTRACT

In the first part of this doctoral dissertation, we focus on a proper measurement on interest rate risk of life insurer's liabilities, policy reserves, by incorporating the general effective duration and effective convexity measures. Tsai (2009) identified a term structure of the effective durations of life insurance reserves. We find that his results are not general. When the long-run mean of interest rates is higher than the policy crediting rate and the surrender rate is sensitive to the spread, the term structure would exhibit an opposite pattern to the one in Tsai (2009). We further find that the effective convexities might be negative and the term structure of the effective convexities exhibits no general pattern. The irregularities originate from negative effective convexities result from the relationship between mean reserves and initial short rate for different years to maturity. Our results can help life insurers to implement more accurate asset-liability management.

In the second part, we analyze asset allocation and leverage strategies for a life insurer selling traditional insurance products by using a simulation optimization method. We assume that an insurer invests equity capital (from its shareholders) and premiums it receives from policyholders by choosing a portfolio intended to maximize the annual return of equity minus the penalty of insolvencies and risks. We regard the leverage as an internal factor in asset allocation. Based on these assumptions, we get a promising multiple-periods asset allocation and leverage, besides analyzing how leverage affects asset allocation strategies.

Keywords: effective duration, effective convexity, policy reserve, asset allocation, leverage, simulation optimization, life insurer

CONTENT

Part One: Characteristics of the Effective Durations and Effective Convexities of Life Insurance Reserves

INTRODUCTION

Life insurance reserves are significantly exposed to interest rate risk because the policies usually last for long time and have minimal crediting rates. The high leverage ratios of life insurers aggravate the impact of interest rate variations on solvency. Life insurers therefore should manage the interest rate risk associated with policy reserves in a prudent way, and this starts with measuring the risk correctly.

One way to evaluate the interest rate risk of a life insurer's policy reserves is to calculate the effective duration. Unlike Macaulay duration and the modified duration,[1] the effective duration considers the interest sensitivity of cash flows as well as the term structure of interest rates. The dynamics of interest rates being complex stochastic process are well documented in the literatures (e.g., Chan et al., 1992; Chen, 1996; Dahlquist, 1996; Norman, 1997; Ahlgrim, D'Arcy and Gorvett, 1999). Tsai, Kuo, and Chen (2002) and Kuo, Tsai, and Chen (2003) further showed that interest rates are a significant determining factor of surrender rates and the cash flows of life insurance policies are sensitive to interest rates as a result. Using Macaulay duration or the modified duration rather than the effective duration would thus result in erroneous measurement on the interest rate risk of policy reserves (Li and Panjer, 1994; Babble, 1995; Santomero and Babbel, 1997; Briys and Varenne, 1997; 2001).

Tsai (2009) recently identified a term structure of the effective durations of policy reserves. Using a cointegrated vector auto-regression (VAR) model for the relation between the surrender rate and the interest rate, he calculated the effective durations of reserves for policies with different maturities. The calculations brought out some negative and/or extreme values. He then plotted the duration values against the policy maturities and identified a term structure consisting of a pair of curves separated by the so-called zero-reserve line. One curve is in the positive domain and the duration increases with the maturity to infinity; the other is in the negative domain and the duration increases from negative infinity with the maturity. The rationale behind such a term structure is that policy reserves are an increasing function of policy year but start from negative values for non-single-premium policies that have positive net present values (NPVs) to life insurers. The effective durations will be negative when policy reserves are negative and be huge when policy reserves are close to zero.

The findings of Tsai (2009), albeit insightful and reasonable, may not be general. The VAR model specifies the one-year interest rate as an AR(2) (auto-regression of order two) process with little mean reversion. It also specifies a particular interest sensitivity of surrender rates. The difference between the policy

[1] Macaulay duration and the modified duration assume that the yield curve is flat, the curve moves in a parallel fashion, and the cash flows of assets or liabilities are independent of interest rates (discount factors). Unless specified, the durations in this paper are effective durations.

crediting rate and the long-run mean of interest rates assumed in the interest rate model is, apparently, a determining factor to policy reserves and thus the effective durations. So is the interest sensitivity of surrender rates. Tsai (2009) however was not able to conduct the sensitivity analysis on the parameters of the interest rate model and robustness tests across alternative interest rate – surrender rate relations due to the use of an empirical VAR model.

In this paper we choose common and flexible interest rate and surrender rate models to scrutinize the generality of the findings in Tsai (2009). We employ the Cox, Ingersoll, and Ross (1985; CIR) model as the interest rate model and follow Babbel et al. (2002) and Kim (2005) in using arctangent functions to model how the surrender rate reacts to the spreads between market interest rates and the policy crediting rate.[2] The use of the CIR model enables us to investigate how the long-run mean, volatility, and mean reversion of interest rates may affect the term structure of reserve durations.[3] Employing the arctangent function grants us the flexibility in assigning the sensitivity of surrender rates to the spreads and allows us to investigate the impact of thw sensitivity on the reserve duration.

Another contribution of this paper to the literature is analyzing the effective convexities of policy reserves. The importance of convexity in managing the interest rate risk is well known in the finance literature (see Choudhry (2005) and the references therein). In the insurance literature, Babbel and Stricker (1987) were the first to point out how the mismatch of asset convexity and liability convexity could adversely affect a life insurer's surplus. Santomero and Babbel (1997) reported the effective convexities of the reserves for some products. They however disclosed only the final results without the policy specification, interest rate model, surrender behavior, or any other assumptions. The only paper documenting the calculation of the effective convexities for insurance products was Ahlgrim, D'Arcy, and Gorvett (2004), but that was about the property-casualty insurance. The characteristics of the effective convexities of life insurance reserves, despite of their importance in risk management, remain obscure in the present literature.

We find that the results of Tsai (2009) about reserve durations are not general. His results are valid only for the cases in which the long-run mean of interest rates is lower than or equal to the policy crediting rate. When the long-run mean is higher than the policy crediting rate, the term structure of the effective durations may exhibit a different pattern to the one in Tsai (2009). A reverse pattern would even emerge when surrenders are sensitive to the spread and the long-run mean is higher than the policy crediting rate. Contrary to Tsai (2009), we find that the effective duration can be positive even for the policies with negative reserves (i.e., positive NPVs) and can be negative for positive reserves. The rationale behind our findings is that the interest-sensitive surrender behaviors coupled with persisting positive spreads will make policy reserves become increasing functions of interest rate shocks.

Tsai's results are therefore more suitable under the expectation of no

[2] Kim (2005) stated that insurers often fitted surrender rates with an arctangent function of the spreads. Doll et al. (1998) also argued for the arctangent function to depict the interest sensitivities of surrender behaviors.

[3] We also tried other popular interest rate models such as Vasicek and Hull-White models. Our findings are robust across these models.

significant and persisting interest rate rises, and our findings are more applicable when insurers expect significant long-run rises of interest rates. Our findings are particularly relevant to the conventional savings-oriented products including annuities and endowment that have low policy crediting rates and interest-sensitive surrender rates. The results of this paper have significant implications to the asset-liability management of life insurers, especially to the insurers that issued policies with low policy crediting rates during the past low-interest-rate era.

With regard to convexity, we find that the effective convexities of life insurance reserves may be negative. The effective convexities and effective durations may have opposite signs, and the effective convexities are more volatile than the effective durations. Furthermore, we cannot identify any general pattern for the term structure of effective convexities. The effective convexities exhibit irregular patterns because the relation between the initial short rate and reserves changes with policy maturity considerably. The changes in the relation originate from the interest sensitivity of surrender rates. The relation is also affected by long-run mean level of interest rates. Life insurers should pay attention to the irregularities of effective convexities for accurate asset-liability management and the management should be conducted dynamically as a result.

The remainder of this article is organized as follows. Section 2 describes the specifications of the analyzed policies. It also explains how we measure the interest risk of policy reserves by calculating the effective durations and effective convexities. Section 3 describes the term structure model of interest rates and the arctangent function used to model the relation between the surrender rate and the spread between the market interest rate and policy crediting rate. It also specifies the model parameters. Section 4 and Section 5 present the results about the effective durations and effective convexities, respectively. Section 6 summaries our findings and concludes the paper.

POLICY SPECIFICATIONS AND MEASURES OF THE INTEREST RATE RISK

Cash Flows of a Twenty-Year Endowment Policy

We analyze the same twenty-year endowment policies as in Tsai (2009) to facilitate comparisons between his and our results. The policies were issued to 30-year-old males in different years. Death benefits and surrender values are assumed to be paid at the end of the year while premiums and expenses are received and paid at the beginning of the year. The expected net cash flow at time t ($t \in N$) for the policy that is at the beginning of policy year k (i.e., sold k-1 years ago; $1 \le k < 20$ and $1 \le t < 20 - k + 1$) but after the k-th net premium being collected can then be represented as:[4]

[4] Note that the insured is at age 30+k-1 when the policy is at the beginning of policy year k.

$$
\begin{aligned}
&E(NCF_t \mid k) = \\
&[({}_{t-1}p^{(\tau)}_{30+k-1} \times q^{(d)}_{30+k-1+t-1} \times B^d) + ({}_{t-1}p^{(\tau)}_{30+k-1} \times q^{(s)}_t \times B^s_{k-1+t})\ ] \\
&- {}_t p^{(\tau)}_{30+k-1} \times [\pi \times (1 - L^{cm}_{k-1+t} - L^{vcost}) - \lambda_{k-1+t}],
\end{aligned} \tag{1}
$$

where ${}_t p^{(\tau)}_{30+k-1}$ is the probability that the policy for an insured with the age of $30+k-1$ remains valid for t years,[5] $q^{(d)}_{30+k-1+t-1}$ is the probability of the insured with the age of $30+k-1+t-1$ dying within one year, B^d denotes the death benefit paid at the end of the year in which the insured dies, $q^{(s)}_t$ is the probability that the policy is surrendered in year t,[6] B^s_{k-1+t} denotes the cash surrender value paid at the end of policy year $k-1+t$,[7] π denotes the level premium received at the beginning of each surviving year, L^{cm}_{k-1+t} represents the commission rate for the commissions paid at the beginning of policy year $k-1+t$, L^{vcost} stands for the variable cost rate, and λ_{k-1+t} represents the fixed cost paid at the beginning of policy year $k-1+t$.[8]

At the policy due date (i.e., $t = 20-k+1$ and $1 \le k \le 20$), no premiums are paid by the insured. We further assume that neither commissions nor variable and fixed costs are incurred at maturity. Thus, the second term of Equation (1) vanishes and is replaced by the term denoting the expected surviving benefit:

$$
\begin{aligned}
E(NCF_{20-k+1} \mid k) = &({}_{20-k}p^{(\tau)}_{30+k-1} \times q^{(d)}_{49} \times B^d) \\
&+ ({}_{20-k}p^{(\tau)}_{30+k-1} \times q^{(s)}_{20-k+1} \times B^s_{20}) + {}_{20-k}p^{(\tau)}_{30+k-1} \times B^{suvr}.
\end{aligned} \tag{2}
$$

The actuarial assumptions about some of the above variables are shown in Table A1.

Policy Reserves

The present value of the expected net cash flows associated with the policy right after the *k*-th net premium is received, R_k, can then be expressed as:

$$
R_k = \sum_{t=1}^{20-k+1} [E(NCF_t \mid k) \times v_t], \tag{3}
$$

where v_t denotes the discount factor for the expected net cash flow at time t. R_k represents the present value of the expected liability associated with the policy that just collected the *k*-th net premiums, given an interest rate path and the corresponding surrender rate path.[9] Since interest rates are random and cause surrender rates to be random as well, we follow the framework of Wilmott (1998) to simulate the random

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[5] Note that ${}_0 p^{(\tau)}_{30+k-1}$ =1. The upper script (τ) indicate a function referring to all causes or total force of decrement. Two causes of decrement, death and surrender, are considered in this paper and are denoted by the upper scripts (*d*) and (*s*) respectively.

[6] Note that 1 - $q^{(d)}_{30+k-1+t-1}$ - $q^{(s)}_t$ = ${}_1 p^{(\tau)}_{30+k-1+t-1}$. A policy not terminated in a year by death or surrender means that the policy remains valid for a year. Furthermore, ${}_{t-1}p^{(\tau)}_{30+k-1} \times {}_1 p^{(\tau)}_{30+k-1+t-1} = {}_t p^{(\tau)}_{30+k-1}$, i.e., the probability of a policy with an insured age 30+*k*-1 being valid for *t* years equals the probability of the policy being valid for *t*-1 years times the probability of the policy with the insured age 30+*k*-1+*t*-1 remaining valid for one more year.

[7] This is equivalent to saying that the cash surrender value is paid at the end of year *t*.

[8] The time line regarding the above cash flows is plotted in Figure A1 for further clarification.

[9] The expectation is taken over the probabilities of decrement.

variable R_k.

Measures of Interest Rate Sensitivity

We follow Fabozzi (1998) and Hayre and Chang (1997) in calculating the effective duration (*ED*) and effective convexity (*EC*) of a financial product. The *ED* and *EC* of the policy reserve are thus defined as follows: [10]

$$ED = \frac{E(R_k\,|r_0 - \Delta r_0) - E(R_k\,|r_0 + \Delta r_0)}{2\Delta r_0 \cdot E(R_k\,|r_0)}, \text{ and} \tag{4}$$

$$EC = \frac{E(R_k\,|r_0 - \Delta r_0) + E(R_k\,|r_0 + \Delta r_0) - 2 \times E(R_k\,|r_0)}{(2\Delta r_0)^2 \times E(R_k\,|r_0)\cdot}, \tag{5}$$

where r_0 denotes the initial short rate of interest and Δr_0 denotes the change of the initial short rate which is specified as 25 basis points in later calculations.

SURRENDER RATE AND INTEREST RATE MODELS

Interest Rate Model

We choose the famous CIR model to simulate interest rates. The CIR model is a mean-reverting process in which the volatility of the short rate is proportional to the square root of the short rate. The discrete-time version of the model is:

$$r_{s+\Delta s} - r_s = \kappa \cdot [\mu - r_s]\Delta s + \sigma_s \sqrt{r_s}\, Z_s \sqrt{\Delta s}\ , \tag{6}$$

where r_s is the short rate at time s ($s \geq 0$), κ reflects the speed of the mean reversion, μ represents the long-term mean to which r_s tends to revert to over time, Δs denotes the time interval, σ_s indicates the volatility of r_s, and Z_s denotes a random number generated from the standard normal distribution.

Although the continuous-time version of the CIR model guarantees positive short rate, the discrete-time version may generate some negative values due to discretization errors. We resort to Euler discretization (Glasserman, 2003) for the exact transition of the short rate so that our simulated $r_{s+\Delta s}$ are all positive. More specifically, if $r_{s+\Delta s}$ follows the CIR process, then $r_{s+\Delta s}$ is distributed as $\sigma_s{}^2\{1-\exp(-\kappa\Delta s)\}/4\kappa$ times a non-central chi-square random variable with degrees of freedom $d = 4\mu\kappa/\sigma_s{}^2$ and a non-centrality parameter $\lambda = [4\kappa\exp(-\kappa\Delta s)$ $/\sigma_s{}^2\{1-\exp(-\kappa\Delta s)\}]r_s$. This transition allows us to simulate the values of r_s directly from their exact distribution and maintain one the of major appealing feature of the CIR model.

We adopt the parameter values and simulation set up used in Ahlgrim et al. (2004) to simulate paths of annual interest rates and discount factors. They set

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[10] The expectation here is taken over random interest rates and surrender rates. $E(R_k\,|\,r_0)$ can be regarded as the policy reserve marked to the market using the current term structure.

$\kappa = 0.25$, $\sigma_s = 0.08$, and the time interval as a quarter.[11] To see how the long-term mean of interest rates may affect policy reserves and the effective durations, we experiment with different values of μ within the range of 2% to 9%.[12] The initial short rate is assumed to be the same as the policy crediting rate, 4%.[13] Since cash flows are incurred annually, we follow Ahlgrim et al. (2004) to compound the quarterly rates into annual rates of interest, denoted as r_t^a. The discount factor is then calculated as: $v_t = [(1+r_1^a)(1+r_2^a)\cdots(1+r_t^a)]^{-1}$.

Surrender Rate Model

For a policy with saving property, Doll et al. (1995), Babbel et al. (2002) and Kim (2005) proposed to model the surrender rate as an arctangent function of the spread between a market interest rate and the policy crediting rate.[14] They argued that two characteristics of surrender behaviors should be captured by the model.[15] Firstly, policyholders have higher incentives to surrender their policies when the spread gets larger. Secondly, the surrender rate should have a lower bound and an upper bound as implied by the historical data. For instance, Tsai, Kuo, and Chen (2002) and Kuo, Tsai, and Chen (2003) observed the possible existence of a "natural" surrender rate similar to the existence of the natural unemployment rate. They also observed that the surrender rate in the US never exceed 21.1%. We therefore model the surrender rate as a monotonically increasing function of the spread with a lower bound as the following arctangent function:[16]

$$q_t^{(s)} = \max\{lb, p_1 + p_2 \times \tan^{-1}(p_3 \cdot (r_t^a - r_p) - p_4)\}, \quad (7)$$

where $q_t^{(s)}$ denotes surrender rate at time t, p_1, p_2, p_3, and p_4 are model parameters, and r_p is the policy crediting rate.[17]

As a benchmark, the parameters (p_1, p_2, p_3, p_4) are set as (0.07, 0.05, 50, 1) with the lower bound of 3%. The arctangent function specified by these parameters has the saddle point at $r_t - r_p = 2\%$, $q_t^{(s)} = 7\%$, and the resulted probability of receiving surviving benefits is about 17% when μ is set at 6%. The parameters are chosen so that the probability of receiving surviving benefits is close to that implied by the historical surrender rates from 1969 to 1988 (American Council of Life

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[11] Their values are indeed taken from Chen et al. (1992).

[12] We also experimented with different values of κ and σ_s but found that they do not affect the findings about the characteristics of the term structures of *ED* and *EC*.

[13] We also tried other values for the initial short rate. The impacts of such changes are immaterial in almost all cases and leave the term structures of *ED* and *EC* intact.

[14] Be compared with the econometric model, complementary log-log model, in Kim (2005), surrender rates are also fit in with an arctangent model soundly.

[15] Doll et al. also proposed that surrender should be tempered by a surrender charge. We regard the surrender charge as nil to simplify the arctangent function. Also, the results of effective duration and effective convexities are indifferent from the presence of a surrender charge.

[16] Although we do not have an explicit upper bound in the formula, the surrender rate is capped as Figure 3 will show later.

[17] By the first-order and second-order differentiation of $q_t^{(s)}$ at $r_t^a - r_p$ on the differentiable interval, we get the saddle point of the arctangent surrender function which is at $r_t - r_p = p_4 / p_3$, $q_t^{(s)} = p_1$. At the saddle point, the marginal surrender rate is $p_2 p_3$.

Insurers, 1999) and the 1980 CSO male mortality table.[18]

TERM STRUCTURE OF EFFECTIVE DURATION

The effective durations of mean reserves calculated under different levels of μ are reported in Table 1.[19] We see that some *EDs* are negative, and this is consistent with Tsai (2009). He argued that *EDs* are negative because the corresponding mean reserves are negative. Policies with negative mean reserves are indeed assets to the insurance company, and designating these "assets" as liabilities on the balance sheet results in these "negative liabilities" having negative durations. The property that mean reserves decrease with a rise/shock in the interest rate holds whether these policies with negative mean reserves are treated as assets or liabilities.

[Insert Table 1 Here]

We however found counter examples to the above Tsai's argument. The effective durations can be positive while the corresponding mean reserves are negative, and positive mean reserves may have negative *EDs*. The mean reserves corresponding to the *EDs* in Table 1 are shown in Table 2. We can see that the mean reserves maturing twenty years later are -\$39,225, -\$48,419, and -\$52,458 when μ = 5%, 6%, and 7% respectively. This sold policy is an asset to the insurer because the long-run mean of the short rate is higher than the policy crediting rate. Its mean reserves however have positive *EDs* of 3.21, 4.65, and 5.30. On the other hand, the mean reserves maturing eighteen years later are \$6,246 and \$229 when μ = 6% and 7%, but their *EDs* are -10.55 and -420.71. Similar cases can be found for the reserves maturing seventeen years later when μ = 7% and 8%.

[Insert Table 2 Here]

When we plot the results of Table 1 as in Figure 1, we spot the patterns opposite to that of Tsai (2009). In the cases of μ = 6%, 7%, and 8%, the *EDs* increase from zero first but then decrease until becoming negative as the policy's maturity increases from one year to eighteen or nineteen years. The *EDs* jump to the positive domain for longer maturities and then start decreasing. We speculate the general pattern of the term structure of the *EDs* for these cases are as the red curves in Figure 2.

[Insert Figure 1 and Figure 2 Here]

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[18] This twenty-year period is chosen to be within the sampling period used in estimating the parameters of the interest rate model.

[19] The values of many *EDs* in Table 1 are rather small mainly due to the mean-reverting property of the CIR model. A shock to the initial interest rate fades away as (simulation) time goes by and leaves mid-run and long-run interest rate levels almost intact. Mean reserves hence do not change much and have small *EDs*. Experimenting with alternative mean-reverting speeds, we confirmed that the values of most *ED*s decreased with the speeds. The interest sensitivity of surrenders also contributes to the small values of the *EDs*. When we remove the mean-reverting property of the interest rate model as well as the interest sensitivity of surrenders and calculate the modified durations, the values become close the years to maturity. For instance, the modified durations of the policy reserves maturing 1 year and 5 years later are 0.96 and 4.86 respectively when the interest rate and surrender rate are set at 4%.

The three key conditions resulting in the above pattern are: the long-run mean of interest rates being higher than the policy crediting rate, the surrender rate being sensitive to the spread between the market interest rate and policy crediting rate, and the policy being issued few years ago with small mean reserves. When the first two conditions emerge, a positive interest rate shock will induce more policyholders to surrender their policies. Furthermore, the pre-determined cash values paid to these policyholders are larger than the "fair" mean reserves since the cash values are determined under the assumption that μ = 4% while the mean reserves are marked-to-market by a higher μ. These surrenders therefore will increase the reserves. Since these policies have small mean reserves (the third condition), the impact of these surrenders may outweigh the effect of the present value decreasing with the increased initial short rate and thus cause mean reserves to increase.

We illustrate the above reasoning using some examples. At μ = 6%, the mean reserves maturating twenty, nineteen, and eighteen years later are -\$48,419, -\$22,703, and \$6,246 respectively as shown in Table 2. The corresponding surrender values are \$0, \$8,160.77, and \$39,789.39 from Table A1. A 0.25% interest rate shock will increase the surrender probability and causes the mean reserves to increase by \$641, \$508, and \$393 respectively if we hold r_t^a unchanged. On the other hand, a 0.25% interest rate shock will cause the present values of the mean reserves to decline by \$70, \$174, and \$228 when we assume that the surrender probability does not increase. The net changes are \$571, \$334, and \$165 and thus results in the effective durations of 4.72, 5.88, and -10.55 in Table 1, respectively.[20]

Our findings and the above reasoning demonstrate the importance of the long-run mean of interest rates, the interest sensitivity of the surrender rate, and the policy's time to maturity in determining the effective durations of mean reserves.[21] Comparing the *EDs* across the columns in Table 1 and/or examining the graphs in Figure 1, we see clearly the importance of μ in determining the *EDs*. This implies that life insurers should pay special attention to their estimates on the long-run interest rate level when implementing asset-liability management strategies. The importance of the long-run mean is obscure in Tsai (2009).[22]

The importance of the interest-sensitive surrender rate in determining the effective durations of mean reserves was not explored in Tsai (2009) either. We illustrate the importance by setting alternative parameter sets of the arctangent function and examining the resulting term structures of the *EDs*. The parameters (p_1, p_2, p_3, p_4) are set as (0.1, 0.05, 50, 0.5) to indicate the more sensitive behavior and are chosen to be (0.05, 0.05, 50, 2) to represent the less sensitive surrender behavior.[23]

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[20] The duration figures are slightly different from those in Table 2 because we use only the positive interest rate shock to calculate *EDs* here but Table 2 employs Equation (4) that takes into account of both positive and negative shocks.

[21] The importance of time to maturity in determining Macaulay and modified durations is well known and self-evident. Tsai (2009) documented its importance in determining the effective durations of mean reserves. We confirm this importance again in this paper but decide not to elaborate it further for the sake of paper length.

[22] An interesting characteristic of the VAR model in Tsai (2009) is that changes in the initial interest rate cause similar changes in the mean of the simulated interest rates. Their effects on the *EDs* therefore mingle together and are difficult to distinguish from each other.

[23] The lower bound of the surrender rate is kept as 3% for these two types of surrender behaviors.

The arctangent functions associated with these two parameter sets along with the benchmark set are plotted in Figure 3, and the resultant term structures of the *EDs* are shown in Figure 4 and Figure 5.[24]

[Insert Figures 3, 4, and 5 Here]

The patterns of the term structures in Figure 4 are consistent with Figure 1 and conform to the general pattern depicted by the red curves in Figure 2. Figure 5 contains patterns similar to those in Tsai (2009) with a distinction: some recently issued policies have negative mean reserves but positive effective durations. This phenomenon emerges when the long-run mean μ is significantly larger than the policy crediting rate (e.g., μ = 8% or 9%). The large spread induce the surrenders that have cash values much larger than the "fair" policy values and cause mean reserves to increase, even when the surrender rate is sensitive to the spread with a minor degree only. We plot this modified pattern of Tsai (2009) as the blue curves in Figure 2.

The above findings and reasoning are robust across values of κ and σ_r. We experimented with κ = 0.1, κ = 0.18, and σ_s = 0.03. All stories remain intact. We thus can conclude that the pattern identified in Tsai (2009) represent the cases in which the long-run mean of interest rates is not above the policy crediting rate to a certain extent. If the long-run mean is significantly higher than the policy crediting rate, his pattern has to be modified (as the blue curves in Figure 2) even when the surrender rate exhibits low sensitivity to the spread. Higher sensitivities of surrender rates will result in a pattern opposite to the pattern of Tsai (2009) as the red curves in Figure 2 when the long-run means of interest rates are higher than the policy crediting rate. The newly found pattern reflected by some graphs of Figure 1 and Figure 5 results from high sensitivity of surrender rates (and the long-run means of interest rates being higher than the policy crediting rate). In short, the interest sensitivity of the surrender rate and the long-run interest rate level are critical in determining the term structure of the effective durations.

TERM STRUCTURE OF EFFECTIVE CONVEXITY

In addition to identifying new term structure patterns of reserve durations, we further calculating the effective convexities of mean reserves for policies maturing in different years. The results are reported in Table 3 and plotted in Figure 6. They are new to the literature.

[Insert Table 3 and Figure 6 Here]

In Table 3 and Figure 6, we find three features of effective convexities. Firstly, many *ECs* are negative that were not seen in the insurance literatures. Secondly, the sign of *ECs* may not be the same as that of *EDs*. It may not be the same as that of mean reserves either. Thirdly, the term structure of *ECs* does not exhibit a general pattern and thus does not have the same pattern as that of *EDs*.

The negative *ECs* emerge when mean reserves are concave functions of initial short rate. For different years to maturity, the mean reserves function of initial short

[24] The corresponding values of these *EDs* are displayed in Table A2 and Table A3.

rate might be concave upward/downward or even U-shaped. For example, at μ = 2%, the mean reserves function maturing twenty years is concave downward in the positive range. Then we have a negative *EC* (-2.50) accompanied with a positive *ED* (8.48). At μ = 7%, the mean reserves function maturing eighteen years is concave downward from negative to positive range. At this case, we have a negative *EC* (-613.58) as well as a negative *ED* (-420.71). At μ = 4%, the mean reserves function maturing twenty years is U-shaped with negative range. At this case, we derive a negative *EC* (-41.27) as well as a negative *ED* (-1.17). The mean reserves function in the above three cases are shown in Figure 7.

[Insert Figures 7 Here]

Comparing Table 1 and 3, we find the sign of *ED*s and *EC*s are not consistent. For example, at μ = 5%, we have negative *ECs* (-33.66 and -22.84) accompanied with positive *EDs* (4.12 and 3.21) when policy maturing nineteen and twenty years. At μ = 6%, we have positive *ECs* (4.75) accompanied with negative *EDs* (-10.55) when policy maturing eighteen years. At μ = 7%, 8% and 9%, we have more examples that negative *ECs* are accompanied with positive *EDs* when policy maturing from seven and fifteen years. The inconsistency of the sign of *ECs* and *EDs* results from interest-sensitive surrenders and long-run mean of interest rates. Both factors lead the mean reserves function being increasing/decreasing or even not monotonic with initial short rate. Then sign of *ED*s and of *EC*s are not consistent.

The negative *ECs* are new to insurance literatures but not to finance literatures. Corporate bonds with callable option are accompanied with negative *ECs*. Douglas (1990) founded that the callable bonds have both positive and negative *ECs*. The call features, expected trend in interest rates, and market yield volatility would determine whether the *EC* is positive or negative. Douglas' conclusions can be evidences for our explanations of negative *ECs* of mean reserves of endowment policies. Meanwhile, the surrender feature of endowment and the long-run mean of interest rates determine the positive or negative *ECs* of mean reserves of policy.

We are not able to spot a general pattern of *ECs* across different long-run mean of interest rates and years to maturity. In the cases of μ = 2% to 5%, the *ECs* increase from zero for early maturities and increase from negative range when mean reserves turn to be negative. In the cases of μ = 6%, the *ECs* increase from zero for early maturities but jump to negative range. Meanwhile, when mean reserves turn to be negative, the *ECs* become decreasing. In the cases of μ = 7% to 9%, the *ECs* decrease from zero to negative range for early maturities. When mean reserves turn to be negative, the *ECs* of mean reserves then decrease from positive range. Under these cases, the term structure of ECs exhibits an irregularity. Thus, due to the negative *ECs* and the inconsistency of the sign of *ECs* and *EDs*, we are unable to generalize the term structure of *ECs* across different long-run mean of interest rates and years to maturity.

The irregularities of the term structures of *ECs* originate from the interest sensitivity of surrender rates as well as the long-run mean of interest rates and the former dominates the irregularities. In the case of more-sensitive surrenders, the irregularity of term structures of *ECs* become more severe. The term structures of

ECs for more-sensitive surrenders are shown in Figure 8. Higher sensitivities of surrender rates will result in an irregular pattern. In the case of less-sensitive surrenders, the term structures of *ECs* turn to be regular and are similar to the term structures of convexities of mean reserves. The term structures of *ECs* for less-sensitive surrenders are shown in Figure 9 and the term structures of convexities of mean reserves are shown in Figure 10.[25] Under fixed interest rates and surrender rates, the term structures of convexities are similar to the term structures of modified duration addressed in Tsai (2009). This is because fixed interest rates and no interest-sensitive surrenders lead the mean reserves function to be monotonic and simplified.

CONCLUSIONS

The policy reserves of life insurance are exposed to significant interest rate risk due to the long-run protection nature of life insurance. The insurance literatures pinpointed the significance of interest-sensitive cash flows in determining the interest rate risk of policy reserves and argued strongly for the usage of effective duration and effective convexity rather than the simpler Macaulay and modified measures. Recently, Tsai (2009) identified a term structure of the effective duration of policy reserves using a specific VAR model of interest rates and surrender rates.

We extend the literature by two ways. Firstly, we re-examine the term structure pattern identified by Tsai (2009) through utilizing more general and flexible models. The use of the popular CIR model enables us to examine how the characteristics of interest rates such as the long-run mean, mean-reverting speed, and volatility may affect the pattern. Using the arctangent function to model how the surrender rate reacts to the spread between the policy crediting rate and market interest rates renders us the flexibility in specifying the interest sensitivity of the surrender rate. Secondly, we illustrate the term structure of the effective convexity of policy reserves. The importance of convexity in managing fixed-income security investments is well known, and our illustration is new to the insurance literature.

We found that the term structure pattern of reserve durations identified by Tsai (2009) is not universal. His pattern is valid when the long-run mean of short rates is not above the policy crediting rate and/or the surrender rate is not sensitive to the interest spread. The term structure pattern changes radically, as demonstrated in Figure 2, when the long-run mean is higher than the policy crediting rate and the surrender rate exhibits certain degree of sensitivity to the interest spread. Tsai's result needs to be revised even when the surrender rate is not sensitive to the spread if the long-run mean is significantly higher than the policy crediting rate.

The reason why Tsai (2009) did not detect our newly identified patterns is because his VAR model consists of an AR(2) process of the one-year interest rate with little mean reverting and a surrender rate process featuring only moderate interest sensitivity. Such an interest rate model obscures the distinction between a short-term interest rate shock and the change in the long-run mean. Being stuck with the surrender rate model specified by the vector-autoregression structure, Tsai (2009) was

[25] The *ECs* under more-sensitive surrenders and less-sensitive surrenders are listed in Appendix Table 4 and 5.

not able to explore alternative sensitivities of surrender rates to interest rates. His findings, therefore, do not represent universal cases.

Our findings signify the critical roles played by the long-run mean of interest rates and the interest sensitivity of surrender rates in determining the term structure pattern of reserve durations. They have material implications to the life insurers that sell savings-oriented products with fixed crediting rates that are popular in annuity markets and in Asia life insurance markets during low interest rate eras. These life insurers will suffer severely from the disintermediation happened in high interest periods, if they do not have correct estimates on the effective durations of their products and implement appropriate asset-liability management. The damages may be brought by the awaiting recoveries from the recent economic downturns that accompany interest rate rises.

Our findings about the effective convexities of policy reserves have practical implications as well, in addition to the contribution to the literature. We find that the effective convexities of mean reserves might be negative for some years to maturity and term structure of effective convexities can not be generalized. Life insurers hence should pay attention to effective convexities when implementing asset-liability management. The irregular term structure patterns of the effective convexities further call for a dynamic management.

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TABLES AND FIGURES

Table 1: Effective Durations of Mean Reserves

	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
4	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
5	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02
6	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.03
7	0.06	0.06	0.06	0.05	0.04	0.04	0.04	0.04
8	0.09	0.08	0.08	0.07	0.06	0.05	0.05	0.05
9	0.12	0.12	0.10	0.09	0.08	0.07	0.07	0.07
10	0.17	0.16	0.14	0.12	0.10	0.08	0.08	0.08
11	0.23	0.22	0.19	0.15	0.12	0.10	0.10	0.10
12	0.32	0.31	0.26	0.20	0.15	0.12	0.12	0.13
13	0.44	0.43	0.36	0.26	0.18	0.14	0.13	0.15
14	0.62	0.60	0.50	0.34	0.21	0.14	0.13	0.16
15	0.87	0.85	0.70	0.44	0.21	0.07	0.06	0.11
16	1.25	1.23	1.00	0.55	0.09	-0.19	-0.24	-0.14
17	1.82	1.85	1.50	0.60	-0.54	-1.42	-1.70	-1.53
18	2.78	3.00	2.57	0.00	-10.55	-420.71	37.19	20.43
19	4.51	5.82	10.39	4.12	5.85	6.29	6.17	5.71
20	8.48	31.77	-1.17	3.21	4.65	5.30	5.50	5.43

Table 2: Mean Reserves under Different Long-Run Interest Rates

	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	980,343	970,743	961,246	951,851	942,588	933,427	924,376	915,434
2	920,948	903,010	885,571	868,700	852,516	836,866	821,689	806,901
3	862,527	837,393	813,363	790,619	769,391	749,338	730,263	711,931
4	805,170	773,883	744,460	717,213	692,471	669,653	648,369	628,193
5	748,792	712,297	678,529	647,940	620,930	596,628	574,409	553,623
6	693,508	652,697	615,530	582,598	554,330	529,535	507,325	486,823
7	639,209	594,893	555,164	520,741	492,042	467,528	446,024	426,430
8	585,861	538,795	497,267	462,087	433,614	409,963	389,667	371,412
9	533,521	484,410	441,773	406,468	378,754	356,402	337,666	321,036
10	482,159	431,662	388,520	353,627	327,107	306,389	289,453	274,617
11	431,721	380,447	337,364	303,359	278,377	259,520	244,521	231,562
12	383,366	331,859	289,288	256,479	233,191	216,246	203,173	192,053
13	335,779	284,552	242,902	211,573	190,115	175,115	163,944	154,614
14	289,012	238,555	198,207	168,599	149,072	136,028	126,720	119,126
15	243,090	193,867	155,169	127,489	109,966	98,861	91,360	85,432
16	198,995	151,432	114,661	89,020	73,478	64,223	58,419	54,059
17	156,260	110,735	76,125	52,599	38,985	31,454	27,208	24,285
18	115,004	71,836	39,532	18,091	6,246	229	-2,673	-4,340
19	77,691	37,101	7,129	-12,369	-22,703	-27,519	-29,394	-30,100
20	44,189	6,317	-21,411	-39,225	-48,419	-52,458	-53,748	-53,938

Table 3: Effective Convexities of Mean Reserves

	Long-Run Means of the Short Rate μ							
Year to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.00
8	0.01	0.00	0.01	0.00	0.00	0.00	-0.00	-0.00
9	0.01	0.01	0.01	0.01	0.00	-0.00	-0.00	-0.00
10	0.01	0.02	0.02	0.01	0.00	-0.00	-0.00	-0.01
11	0.02	0.03	0.04	0.02	0.01	-0.01	-0.01	-0.01
12	0.04	0.05	0.06	0.04	0.01	-0.02	-0.03	-0.03
13	0.05	0.09	0.12	0.07	0.01	-0.05	-0.08	-0.08
14	0.08	0.17	0.23	0.14	0.00	-0.12	-0.18	-0.19
15	0.12	0.31	0.46	0.31	-0.01	-0.31	-0.48	-0.50
16	0.21	0.62	0.98	0.75	-0.01	-0.88	-1.38	-1.50
17	0.36	1.30	2.39	2.18	0.01	-2.98	-5.40	-6.36
18	0.58	3.18	7.57	11.61	4.75	-613.58	91.31	65.40
19	0.62	9.46	70.08	-33.66	-6.83	4.69	11.47	15.46
20	-2.50	82.09	-41.27	-22.84	-12.37	-4.31	1.93	6.74

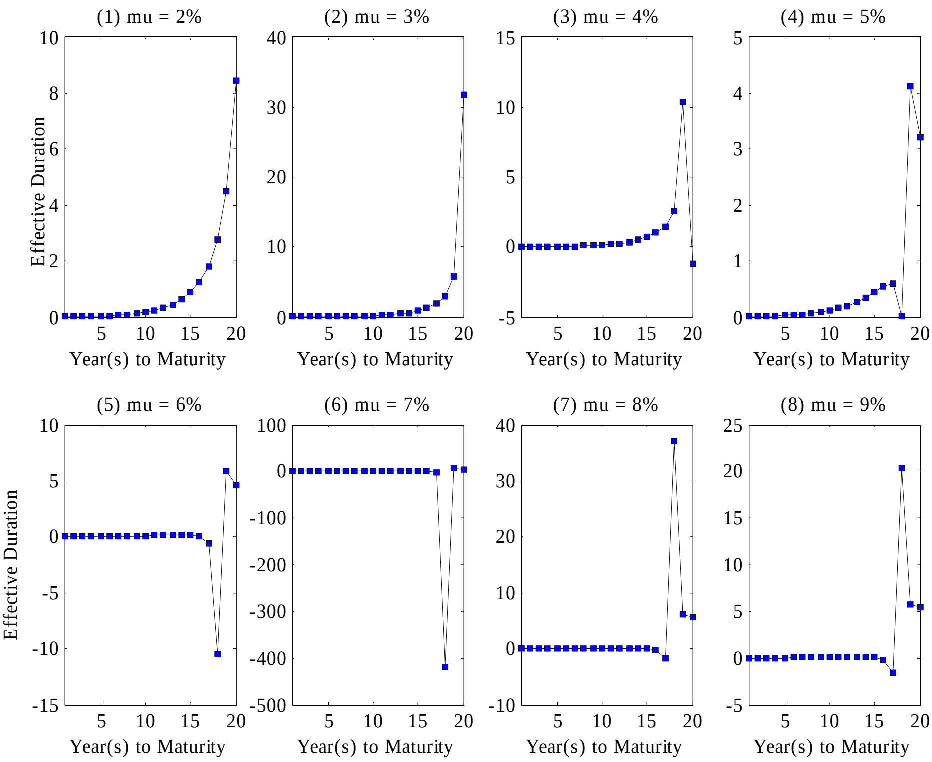


Figure 1: Effective Durations of Mean Reserves

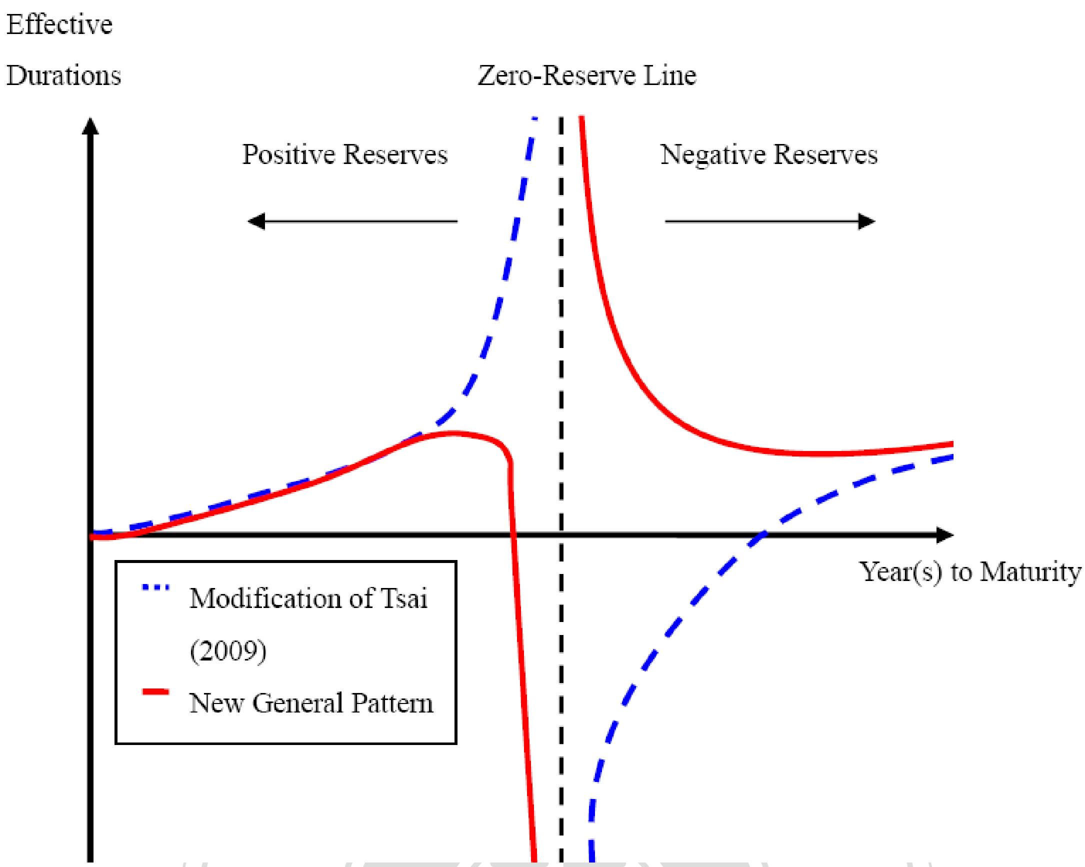


Figure 2: The General Pattern(s) of the Term Structure of Effective Durations

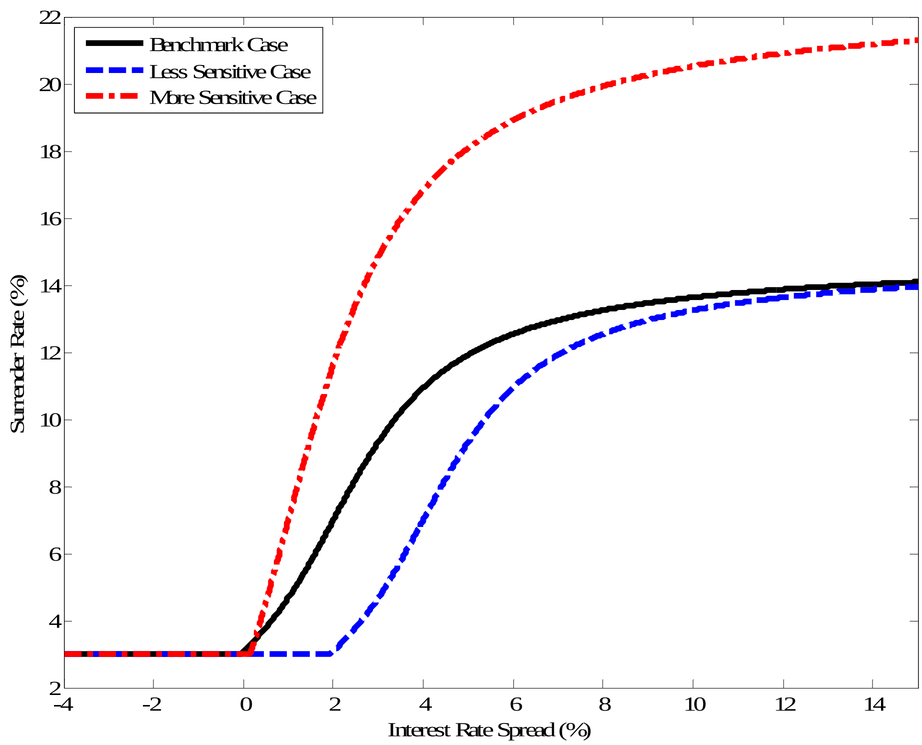


Figure 3: Arctangent Functions of Surrender Rate to Interest Rate Spread

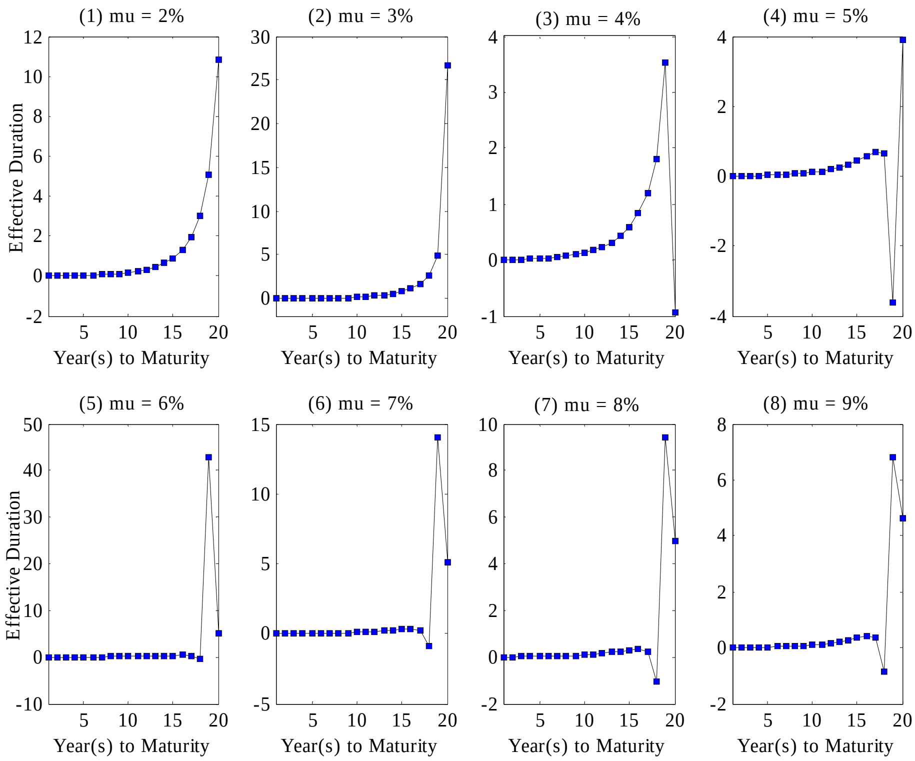


Figure 4: Effective Durations of Mean Reserves for More-Sensitive Surrenders

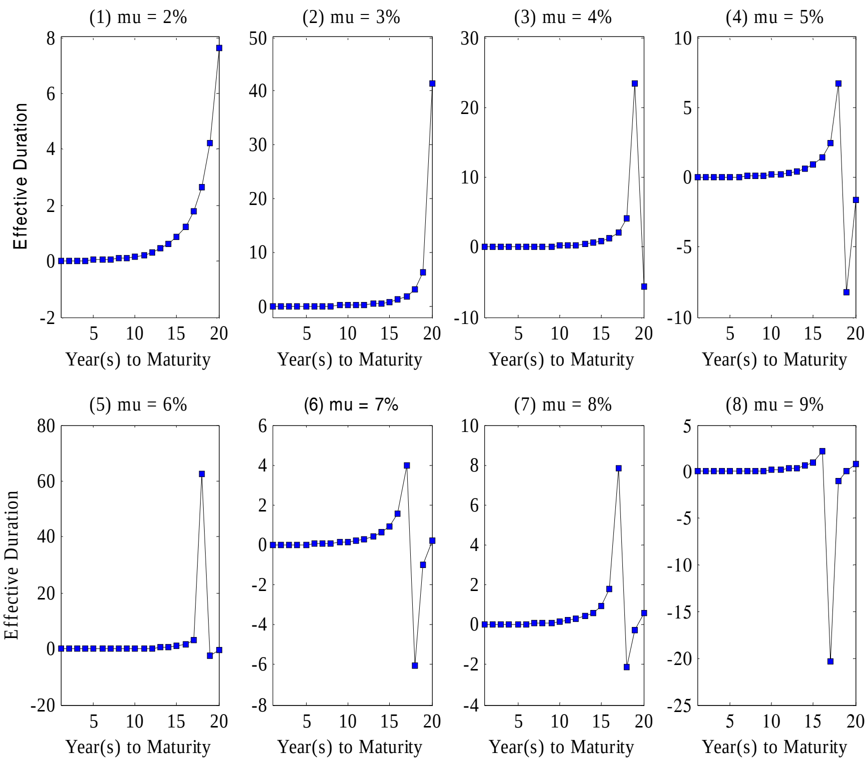


Figure 5: Effective Durations of Mean Reserves for Less-Sensitive Surrenders

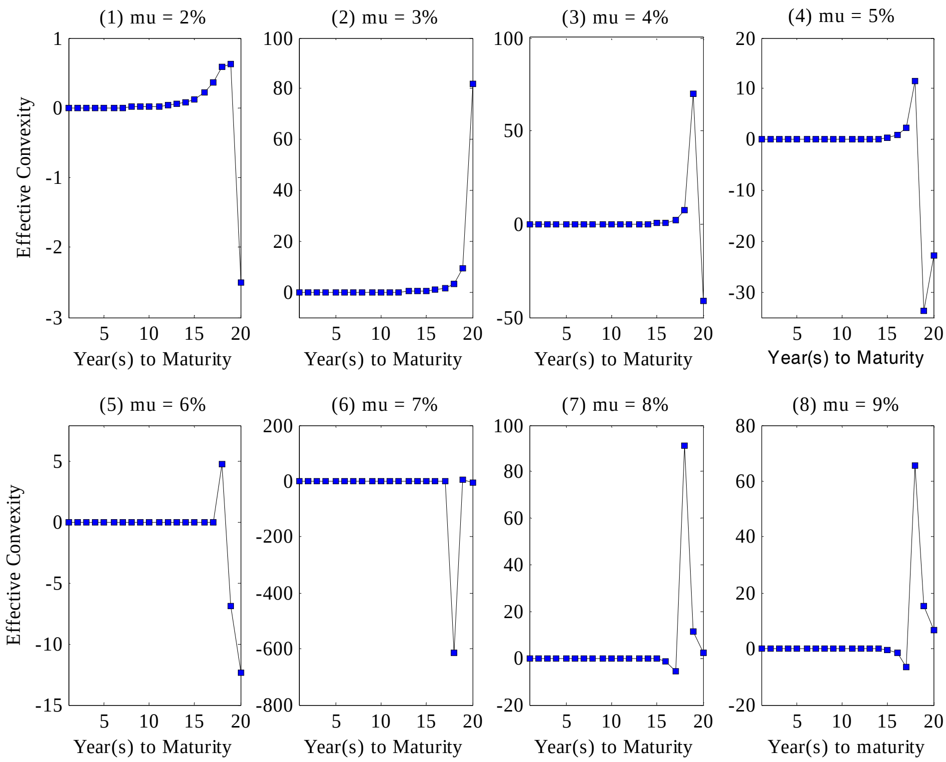
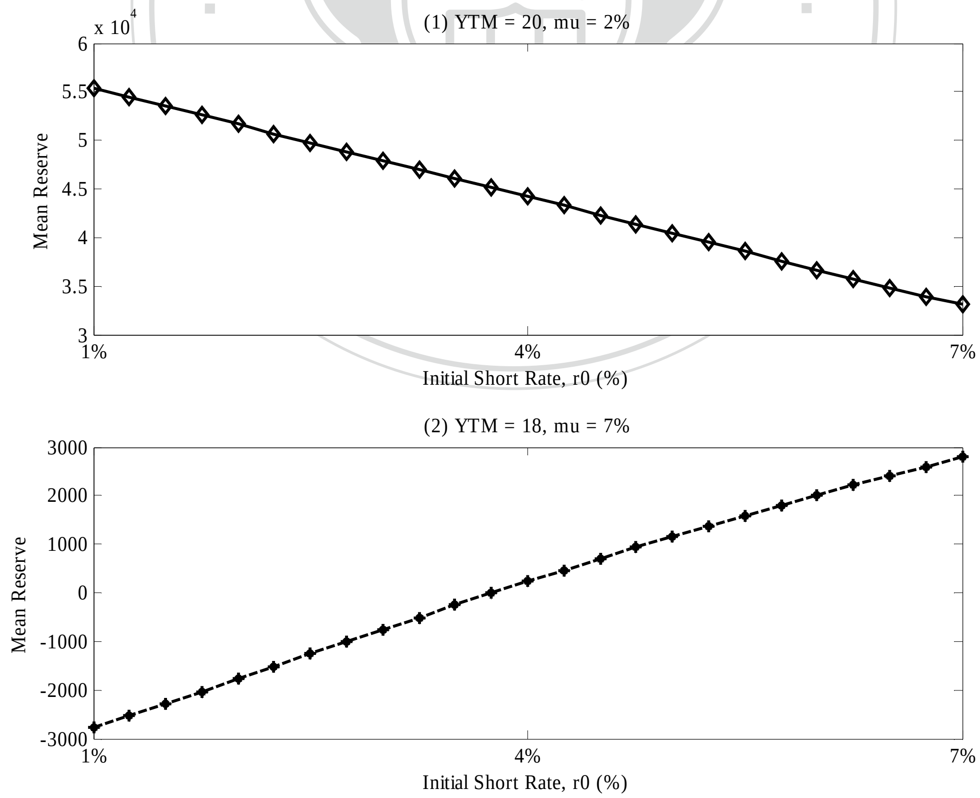


Figure 6: Effective Convexities of Mean Reserves



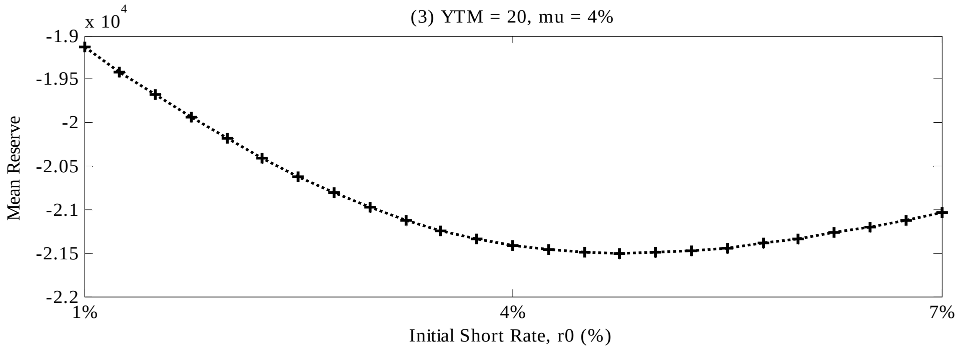


Figure 7: Mean Reserve Curve

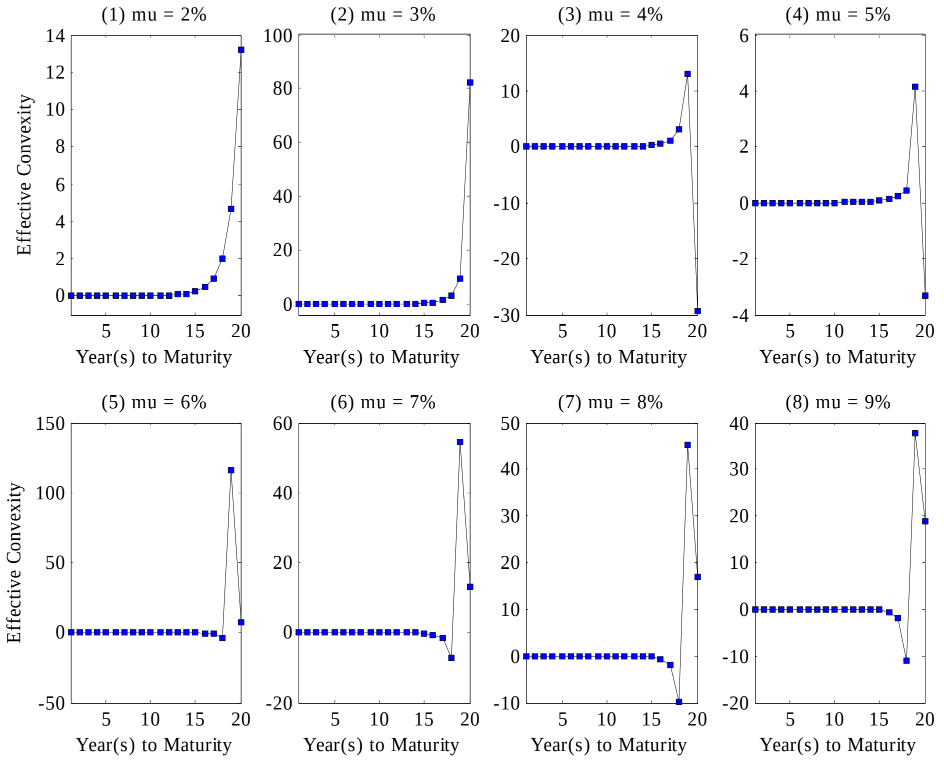


Figure 8: Effective Convexities of Mean Reserves for More-Sensitive Surrenders

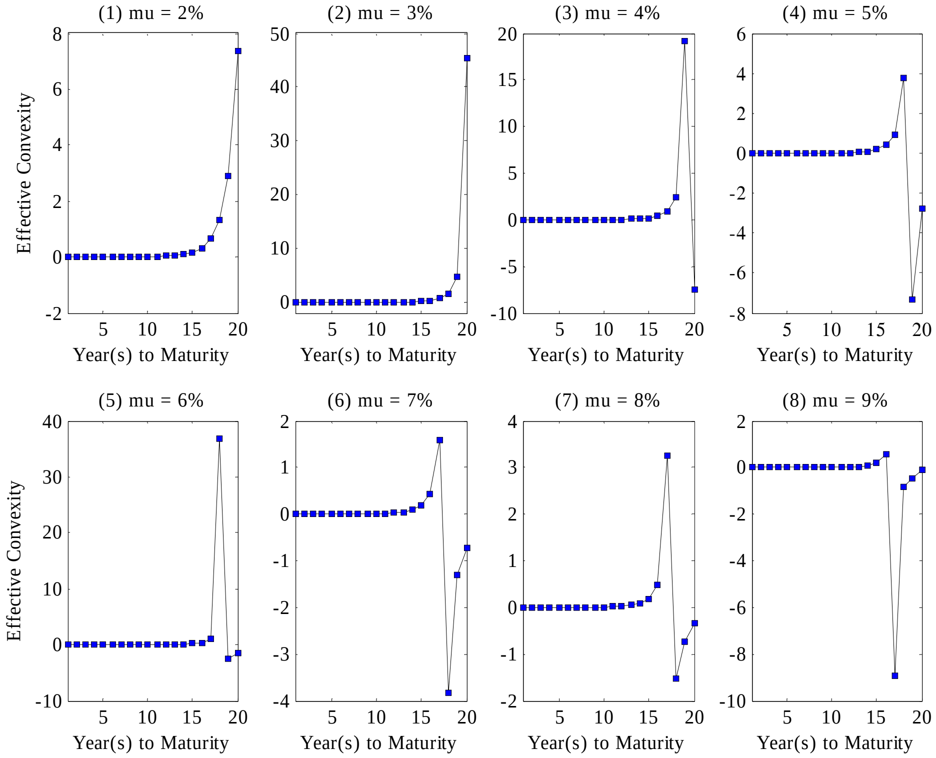


Figure 9: Effective Convexities of Mean Reserves for Less-Sensitive Surrenders

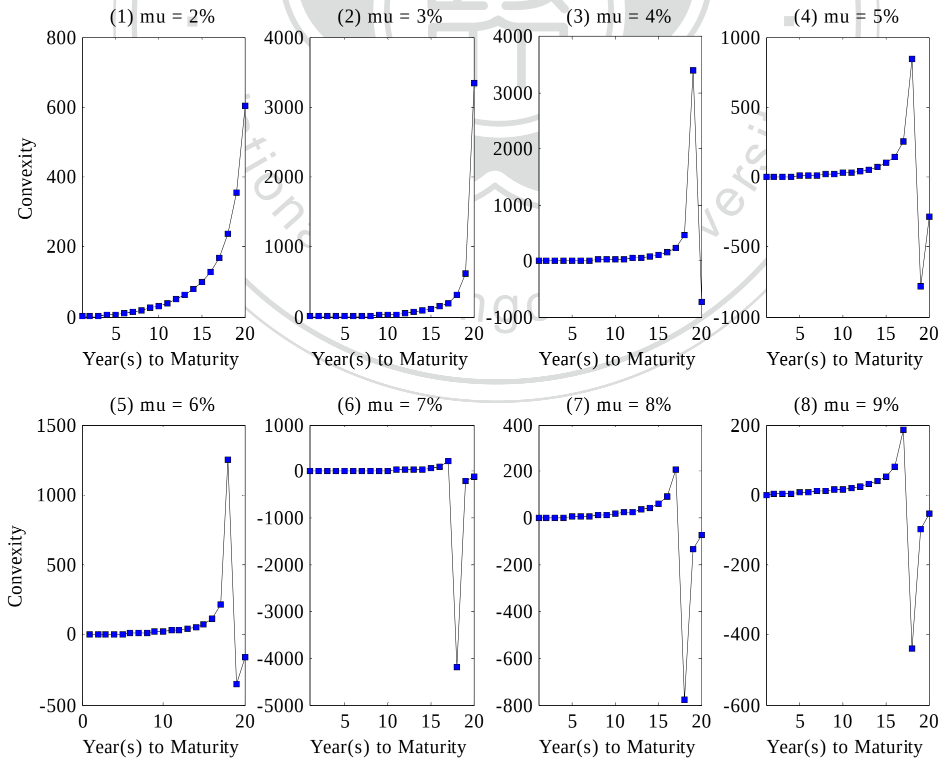


Figure 10: Convexities of Mean Reserves

APPENDICES

Table A1: Actuarial Assumptions of the Twenty-Year Endowment Policy

Insured's Age	Mortality Rate of age *a*	At the Beginning of Policy Year	Surrender Value	Commission Rate	Fixed Expense	Variable Cost Rate
a	$q_a^{(d)}$	k	B_k^s	L_k^{cm}	λ_k	$L^{v\cos t}$
30	0.0009790	1	N/A	62.40%	4,530	0.001
31	0.0010055	2	8,161	27.00%	1,359	0.001
32	0.0010481	3	39789	20.60%	1,359	0.001
33	0.0011075	4	73,767	14.00%	1,359	0.001
34	0.0011826	5	110,192	13.00%	1,359	0.001
35	0.0012712	6	149,173	12.00%	1,359	0.001
36	0.0013711	7	190,831	10.00%	1,359	0.001
37	0.0014807	8	235,294	10.00%	1,359	0.001
38	0.0015989	9	282,707	10.00%	1,359	0.001
39	0.0017291	10	333,223	10.00%	1,359	0.001
40	0.0018749	11	387,004	7.00%	1,359	0.001
41	0.0020407	12	437,655	7.00%	1,359	0.001
42	0.0022297	13	490,342	7.00%	1,359	0.001
43	0.0024446	14	545,163	7.00%	1,359	0.001
44	0.0026795	15	602,227	7.00%	1,359	0.001
45	0.0029268	16	661,664	7.00%	1,359	0.001
46	0.0031784	17	723,620	7.00%	1,359	0.001
47	0.0034268	18	788,259	7.00%	1,359	0.001
48	0.0036671	19	855,760	7.00%	1,359	0.001
49	0.0039091	20	926,314	7.00%	1,359	0.001
50	N/A	20*	1,000,000	N/A	N/A	N/A

1. The death benefit and survival benefit is $1,000,000. The policy is issued to a 30 year-old male, and the annual premium expected to pay at the beginning of each surviving year is $45,300 under the policy crediting rate of 4%.

2. The notation 20* is used to denote the end of policy year 20.

3. The policy surrendered at the beginning of the first policy year has no surrender value. Neither mortality nor expenses apply any more when the policy matures. We denote all these values as N/A.

4. The variable cost is assumed to be 0.1%.

Table A2: Effective Durations for Less-Sensitive Surrenders

	Less-Sensitive Surrenders							
	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
4	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
5	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
6	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
7	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
8	0.09	0.09	0.08	0.08	0.08	0.08	0.08	0.08
9	0.12	0.12	0.12	0.11	0.11	0.11	0.11	0.11
10	0.17	0.17	0.16	0.16	0.16	0.15	0.15	0.15
11	0.23	0.23	0.23	0.22	0.22	0.21	0.21	0.21
12	0.32	0.32	0.31	0.31	0.30	0.30	0.30	0.29
13	0.44	0.45	0.44	0.43	0.43	0.42	0.42	0.42
14	0.61	0.63	0.62	0.61	0.61	0.61	0.62	0.63
15	0.86	0.89	0.89	0.90	0.92	0.94	0.97	1.02
16	1.22	1.29	1.34	1.38	1.47	1.59	1.78	2.12
17	1.76	1.96	2.15	2.40	2.94	3.95	7.80	-20.40
18	2.66	3.23	4.21	6.69	62.44	-6.04	-2.14	-0.97
19	4.22	6.43	23.40	-8.27	-2.38	-0.97	-0.30	0.08
20	7.63	41.35	-5.65	-1.62	-0.37	0.21	0.56	0.79

Table A3: Effective Durations for More-Sensitive Surrenders

	More-Sensitive Surrenders							
	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
4	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
5	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02
6	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.03
7	0.06	0.06	0.05	0.05	0.04	0.04	0.04	0.04
8	0.09	0.08	0.07	0.06	0.06	0.05	0.05	0.05
9	0.12	0.11	0.09	0.08	0.07	0.07	0.07	0.07
10	0.17	0.15	0.13	0.11	0.10	0.09	0.09	0.09
11	0.23	0.20	0.17	0.14	0.12	0.12	0.12	0.12
12	0.32	0.28	0.23	0.19	0.16	0.15	0.15	0.16
13	0.45	0.39	0.31	0.25	0.21	0.19	0.20	0.21
14	0.63	0.55	0.43	0.33	0.27	0.25	0.25	0.27
15	0.90	0.77	0.59	0.43	0.33	0.30	0.31	0.34
16	1.30	1.11	0.83	0.55	0.38	0.33	0.35	0.40
17	1.93	1.66	1.18	0.68	0.34	0.21	0.24	0.35
18	3.00	2.64	1.79	0.65	-0.36	-0.92	-1.04	-0.86
19	5.09	4.97	3.52	-3.62	42.71	14.04	9.44	6.82
20	10.85	26.71	-0.95	3.89	4.94	5.12	4.95	4.63

Table A4: Effective Convexities for More-Sensitive Surrenders

	More-Sensitive Surrenders							
	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00
9	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00
10	0.02	0.02	0.02	0.01	0.00	0.00	0.00	0.00
11	0.03	0.03	0.03	0.01	0.00	-0.01	-0.01	0.00
12	0.05	0.05	0.04	0.02	0.00	-0.01	-0.01	-0.01
13	0.08	0.09	0.08	0.03	-0.01	-0.03	-0.03	-0.03
14	0.14	0.17	0.14	0.05	-0.03	-0.08	-0.08	-0.07
15	0.25	0.33	0.27	0.08	-0.10	-0.19	-0.21	-0.19
16	0.47	0.65	0.54	0.14	-0.28	-0.52	-0.59	-0.54
17	0.94	1.37	1.17	0.24	-0.87	-1.62	-1.91	-1.86
18	2.01	3.22	3.01	0.45	-3.53	-7.24	-9.73	-10.97
19	4.66	9.32	12.96	4.17	116.18	54.70	45.13	37.63
20	13.20	82.57	-29.37	-3.29	7.00	13.11	16.84	18.86

Table A5: Effective Convexities for Less-Sensitive Surrenders

	Less-Sensitive Surrenders							
	Long-Run Means of the Short Rate μ							
Year(s) to Maturity	2%	3%	4%	5%	6%	7%	8%	9%
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00
10	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
11	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.01
12	0.04	0.03	0.03	0.03	0.02	0.02	0.02	0.02
13	0.06	0.05	0.06	0.05	0.04	0.04	0.04	0.04
14	0.10	0.10	0.11	0.09	0.09	0.09	0.09	0.09
15	0.18	0.18	0.20	0.18	0.18	0.18	0.19	0.20
16	0.33	0.36	0.39	0.39	0.40	0.43	0.49	0.60
17	0.67	0.74	0.86	0.94	1.14	1.58	3.24	-8.95
18	1.33	1.69	2.36	3.79	36.87	-3.84	-1.53	-0.84
19	2.88	4.78	19.18	-7.31	-2.44	-1.29	-0.75	-0.44
20	7.36	45.32	-7.43	-2.79	-1.38	-0.73	-0.35	-0.09

The time line below describes the relations among policy year k, insured's age 30+k-1, and the evaluation time t and where the net cash flows are.

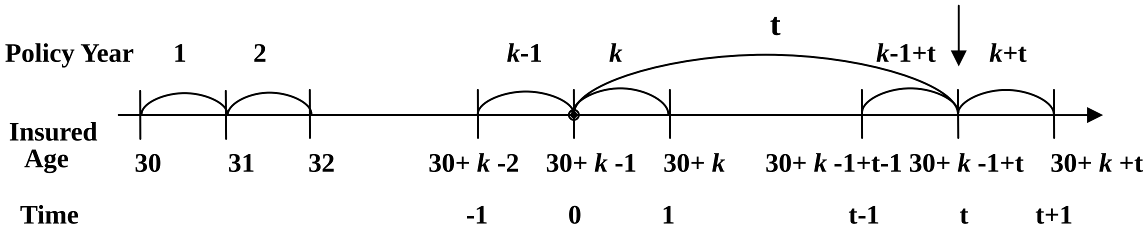


Figure A1: Illustrative Time Line

Part Two: A Promising Asset Allocation and Leverage Strategy for a Life Insurer by Simulation Optimization

INTRODUCTION

Managing investments is important for life insurers to ensure that funds are available to pay claims when they fall due, in the future. However, conflicts of interest between shareholders, regulators and policyholders make investment decisions difficult. Shareholders of insurers urge them to generate higher returns from investments and underwriting but regulators and policyholders ask them to maintain risk at acceptable levels. Life insurers thus have to manage divergent expectations emanating from both assets and liabilities sides. In this study, we re-think the classic asset allocation problem specifically for life insurers. We propose a non-linear simulation model with stochastic variables to derive promising asset allocations and leverage strategies.

Extant literature has covered the asset allocation problem quite extensively. In financial literature, there are two categories of methods to address this problem. One is the mean-variance analysis of Markowitz (1952), which suggests the efficient frontier representing the best portfolios in terms of return-risk tradeoff. The mean-variance analysis, however, is prone to two fundamental flaws: the single-period framework, and the inappropriate utility function assumed for the investor (Brennan et al., 1997). The solution to a static portfolio choice problem can be different from the solution to a multi-period dynamic problem (Campbell, 2000). The other method to construct optimal portfolios originated from Merton (1971; 1990). The literature along this line formulates the asset allocation problem as a stochastic optimal control problem; solutions are characterized by Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDE) but it is difficult to get a closed-form solution from a high-dimensional PDE. Also, numerical solutions of PDE can be obtained only in rare cases. Cox and Huang (1989) made conceptual progress by showing that one can apply the Martingale representation theory to reduce the stochastic dynamic programming problem to a static problem in complete markets. However, few closed-form solutions have been available, except for the simplest cases, and complex hedging terms are difficult to evaluate numerically.

We need a powerful tool to integrate the asset allocation problem and leverage strategies for efficient asset and liability management by life insurers. A company-wide simulation model is one such tool (Browne, Carson and Hoyt, 1999; Browne, Carson and Hoyt, 2001; Kaufmann et al., 2001; and Hardy, 1993, 1996).[1] It is a "systemic approach" to financial modeling which projects financial results under a variety of possible scenarios, showing how outcomes might be affected by changing business, competitive and economic conditions." The system starts with two fundamental equations, as follows:

[1] A company-wide simulation system is often named as "Dynamic Financial Analysis" (DFA) system in the non-life insurance industry. What is called DFA in non-life insurance is also known as "Asset Liability Management" (ALM) in life insurance.

$$\sum_i A_{i,t} - \sum_j L_{j,t} = S_t \text{, and} \quad (1)$$

$$\sum_i \Delta A_{i,t} - \sum_j \Delta L_{j,t} = \Delta S_t \text{,} \quad (2)$$

where $A_{i,t}$ and $L_{j,t}$ represent values of individual asset and liability items, respectively, at time *t*, and $\Delta(.)$ denotes the change of the variable. Equation (1) depicts the fundamental relations among financial variables at any given point of time; Equation (2) captures the dynamic relations among the variables across time. The system can specify models for values of individual asset and liability items at time *t*. These models are supposed to reflect the stochastic nature of financial markets and insurance underwriting.[2]

A company-wide simulation model, though powerful, is merely a descriptive model. It only helps us understand the dynamics of, and complex interactions among, the elements of the system, and this system lacks optimization capability. In other words, a simulation model helps us to know which proposed strategy is better but is unable to determine what the optimal strategy is. It does not have the mechanism/algorithm to search for the optimum. We, therefore, have to make educated guesses on what the optimal strategy could be like, and employ the trial-and-error method to determine the right strategy. Trying all possible strategies to seek the optimum is infeasible due to the large number of decision variables. A simulation model without an optimization mechanism is, therefore, incapable of helping managers maximize shareholder value.

In this study, we apply the techniques of simulation optimization to address the asset allocation problem by simulating the system of a life insurer. Comparatively few researchers have used a company-wide simulation model for optimization of a life insurer's asset allocation. We have found two articles in insurance and financial literature that focus on investment management for servicing participating policies with minimum guarantees. Iwaki and Yumae (2004) analyze trading strategies in a continuous time economy by utilizing the Martingale method. They derive an efficient frontier for the company, as well as trading strategies for efficient portfolios. Consiglio, Saunders, and Zenios (2006) examine asset-liability management associated with single-premium participating policies with minimum guarantees.

In our company-wide simulation system for a life insurance company, we incorporate four types of assets and three types of insurance products. Assets include default-free zero-coupon bonds, stock index, real estate index, and alternative investment characterized by "high-return and high-risk." Insurance products include 20-year non-participating term life insurance, endowment, and pure endowment. We assume that leverage represents the premiums received at the beginning of the first policy year, divided into the initial equity of shareholders. Without loss of generality, for simplicity, we assume no new business comes after the second policy year. The objective function of our simulation and optimization problem is maximization of the

[2] The system's major outcome is the insurer's surplus/equity distribution at some point of time in the future. Managers can employ the simulated surplus distribution to make choices among alternative strategies. Life insurers can use a company-wide simulation system to assess asset allocation strategies by examining the impacts of alternative strategies on the surplus distribution over a target time horizon. The simulation system therefore can help managers make investment and business decisions in a comprehensive and robust way.

expected annual rate of return of equity minus the risk and insolvency penalty. We maximize this function through re-allocating investments in different assets every five periods.

We find the promising asset allocation and leverage strategies by particle swarm optimization (PSO). The PSO is a novel computational method that can solve difficult problems efficiently and reliably (Kendall and Su, 2005). Eberhart and Kennedy (1995) introduced PSO, which is based on the analogy of birds flocking and fish schooling. PSO has been shown to be powerful, easy to implement, rapid to converge, and computationally efficient (Poli, 2008). Dissimilar to evolutionary algorithms, such as GA, PSO considers parameters of crossover probability, mutation probability, and population size, and it is more implementable.

The promising leverage is 20 (i.e. the total premiums in the first year are twenty times the initial equity) and the corresponding asset allocation varies across periods. The objective value is 24.63% with 5 insolvencies in the simulation. We find no consistency in composition of the portfolio in terms of risky assets (stock, real estate and alternative investment) and fix income securities (default-free zero coupon bonds) at the time of different re-allocations. In the first and sixteenth periods, weights of fix income securities are higher than weights of risky assets. But it is the opposite in sixth and eleventh periods. Allocations for the first and the last re-allocation period are relatively conservative, when weights of fix income securities are 70.75% and 67.62%, respectively. At the sixth and the eleventh re-allocation, the insurer needs to hold more risky assts to improve the objective value. The ratio of risky assets to fix income securities can even go up to 2.43. However, weights of fix income securities at each re-allocation period are higher than 29% due to life insurers' investment strategies being subject to penalty for violation of stipulated norms for asset allocation.

Among risky assets, weights of equities and real estate dominate alternative investment. Asset allocations show that equities have a prominent share in the investment portfolio in each period. At each re-allocation, weight of stocks is higher than 13%. Weight of real estate undergoes significant change after the fifth period, in the sixth period (rising from 4.78% to 28.15%), as well as after fifteenth, in the sixteenth period, when it plunges from 36.61% to 3.43%. The rationale is that the price of real estate follows a jump diffusion process with the average number of jumps being 0.1 per year. Alternative investment is not the first choice of investment because of the high risk it entails. Even in the sixth and the eleventh period, to improve the objective value, the insurer prefers stock and real estate rather than alternative investment since the risks of real estate (0.18) and stock (0.25) are smaller than alternative investment (0.5). When determining the composition of risky assets, insurers would concern more about risk than the return of each type of asset.

Another contribution of this study is that we investigate asset allocation strategies under different leverages. Insurance literatures indicate that capital structure (leverage) affects the risk-taking behavior of insurers. Michaelsen and Goshay (1967), Hammond et al. (1976), and Harrington and Nelson (1986) found some degree of support for the hypothesis that insurers with higher portfolio risk operate with lower leverage ratios (measured by the ratio of net premium written to equity, which is similar to its definition in our study). Cummins and Sommer (1996) indicated that property and liability (P&L) insurers prefer to operate at finite levels of leverage (capital to asset ratio) and risk to avoid bankruptcy cost under the cost-based hypothesis, as in Shrieves and Dahl (1992). Baranoff and Sager (2002; 2003; 2004) indicate support for finite risk hypothesis, that is, for life insurers, leverage (total liabilities to total assets) and the proportion of risky assets (stock) in portfolio are negatively interrelated. We assume two different leverages (12 and 16) to compare promising asset allocations under these two scenarios with our promising asset allocation and leverage strategies.

The results show that when leverage increases, insurers need to hold more fix income securities in the first period. In other words, the finite risk hypothesis holds conditionally in a multi-period asset allocation. When leverage is 12, the insurer re-allocates more investment to fix income securities in the first, sixth, and eleventh periods. But in the sixteenth period, the insurer re-allocates to risky assets in full to increase the objective value. When leverage is 16, the insurer holds more fix income securities only in the first and the sixth periods. From the 11^{th} period onwards, the insurer increases weights of risky assets to increase the objective value. Asset allocation at these two points is different from the optimal because leverage strategies are not optimal in the simulation. Thus, the insurer's investment decisions are far away from the promising strategies. Besides, volatility of annual rate of return of equity is decreasing as the leverage increases. Numbers of insolvencies in simulation paths exhibit no correlation with leverage. The objective value is then increasing, as the leverage increases, to reach closer to the promising strategy.

The remainder of this paper is structured as follows. Section 2 presents our simulation model, including the setting of asset and liability sides. Section 3 presents balance sheets of assets and liabilities for each period and formulates asset allocation and leverage strategy as a high-dimensional constrained optimization problem. Section 4 presents the results and exhibits how leverage affects asset allocation strategy. Section 5 presents conclusions.

COMPANY-WIDE SIMULATION MODEL

The Investment Markets

We assume the assets of a life insurer are allocated, in different proportions, to default-free zero-coupon bonds, stocks, real estate and alternative investment characterized by "high-return and high-risk." The time to maturity of default-free zero-coupon bonds ranges from one to fifteen years. The insurer is, therefore, able to invest in 18 securities.

We assume the dynamics of the yearly interest rate r_t follow the CIR model (Cox, Ingersoll, and Ross, 1985). The discrete-time CIR model is:

$$\Delta r_t = \kappa[\mu - r_t]\Delta t + \sigma_r \sqrt{r_t}\varepsilon_r \sqrt{\Delta t}\,, \quad (3)$$

$$\varepsilon_r \sim N(0,1)\,, \quad (4)$$

where Δt equals one year, μ is the long-term average of the short-term rate, κ reflects the speed of mean reversion, σ_r is the volatility parameter of the process, and ε_r is a random number drawn from a standardized normal distribution.[3] Then we get the price dynamics of a default-free zero-coupon bond at time t for delivery of \$1 at different maturity times $T > t$, which was derived in Cox, Ingersoll, and Ross (1985) as follows:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{5}$$

where $B(t,T) = \dfrac{2(e^{(T-t)h}-1)}{2h+(\kappa+h)(e^{(T-t)h}-1)}$, $A(t,T) = [\dfrac{2he^{\frac{(\kappa+h)(T-t)}{2}}}{2h+(\kappa+h)(e^{(T-t)h}-1)}]^{\frac{2\kappa\mu}{\sigma^2}}$, $T > t > 0$, and $h=\sqrt{\kappa^2+2\sigma_r^2}$.

We assume the return of equities evolves according to a discrete-time version of interest-rate-adjusted geometric Brownian motion:

$$\frac{\Delta S_t}{S_t} = (r_t + \pi_S)\Delta t + \sigma_S \varepsilon_S \sqrt{\Delta t}, \tag{6}$$

where ΔS_t denotes the change in the stock index at time t, Δt equals one year, the constant parameter π_S denotes the risk premium on equities investment, σ_S is volatility of the index return, and ε_S has a standard normal distribution.

The return of investment in real estate (index) is specified by the jump diffusion model of Merton (1976). Let μ_{RE} be the expected return from real estate, λ be the average number of jumps during time interval Δt (one year), and β is the average jump size measured as a percentage of the index, and σ_{RE} is volatility of real estate index return. The process for the real estate index is then:

$$\frac{\Delta RE_t}{RE_t} = (\mu_{RE} - \lambda\beta)\Delta t + \sigma_{RE}\varepsilon_{RE}\sqrt{\Delta t} + N_{PSN}(\Delta t), \tag{7}$$

where N_{PSN} is a Poisson process with parameter λ, independent of ε_{RE}. Jump size β is assumed to follow uniform distribution (-0.5, 0.5).

We assume that price of one alternative investment follows geometric Brownian motions. Alternative investment with high return and high risk can be regarded as a hedge fund, and is denoted by χ_t^{hh}. Price dynamics of the alternative investment is then as follows:

$$\frac{\Delta \chi_t^{hh}}{\chi_t^{hh}} = \mu_{\chi^{hh}}\Delta t + \sigma_{\chi^{hh}}\varepsilon_{\chi^{hh}}\sqrt{\Delta t}, \tag{8}$$

where $\mu_{\chi^{hh}}$ and $\sigma_{\chi^{hh}}$ are the expected return and volatility, respectively, $\varepsilon_{\chi^{hh}}$ is drawn from standard normal distribution. To incorporate the correlation between price dynamics of each type of asset, we specify a matrix R to describe the correlation of random terms of each type of asset. Parameter values of the above asset models are shown in Appendix (Table 2).

Cash Flow Specification of Insurance Products

[3] Whenever the simulated r is negative due to the discretization, we substitute zero for the negative r.

We assume that liabilities of a life insurer are met from its aggregate reserves of traditional insurance products, including 20-year term life insurance, 20-year endowment, and 20-year pure endowment.

To focus on leverage and asset allocation problems, we assume all policies are newly issued to 30-year-old males, and no further business will be solicited. Death benefits and surrender values are assumed to be payable at the end of the year, while premiums and expenses are received and paid at the beginning of the year. The expected net cash flow at time $t\ (t \in N \cup \{0\})$, for a policy that is at the end of policy year k (denoted by k', $1 \le k' \le 19$ and $0 \le t < 20 - k'$), but before the $k+1$-th net premium is collected, can then be represented as:[4]

$$E(NCF_t^j \mid k') = \begin{cases} -{}_{t+1}p_{30+k}^{(\tau)} \times [\pi^j \times (1 - L_{cm,k+t+1}^j - L_{vcost}^j) - \lambda_{k+t+1}^j], & \text{if } t = 0 \\ [({}_t p_{30+k}^{(\tau)} \times q_{30+k+t-1}^{(d)} \times B_d^j) + ({}_t p_{30+k}^{(\tau)} \times q_{t+1}^{(s),j} \times B_{s,k+t}^j)\ ] & \\ -{}_t p_{30+k+1}^{(\tau)} \times [\pi^j \times (1 - L_{cm,k+t+1}^j - L_{vcost}^j) - \lambda_{k+t+1}^j] & \text{if } t \ge 1, \end{cases} \tag{9}$$

where $j \in \{tm,\ ed, ped\}$ indicates the type of policy, ${}_t p_{30+k}^{(\tau)}$ is the probability that the policy for an insured male of age $30+k$ remains valid for t years,[5] $q_{30+k+t-1}^{(d)}$ is the probability of the insured of age $30+k+t-1$ dying within one year, B_d^j denotes the death benefit paid at the end of the year in which the insured dies, $q_{t+1}^{(s),j}$ is the probability that the policy is surrendered in year $t+1$,[6] $B_{s,k+t}^j$ denotes the cash surrender value paid at the end of policy year $k+t$,[7] π^j denotes the premium received at the beginning of each surviving year , $L_{cm,k+t+1}^j$ represents the rate of commission paid at the beginning of policy year $k+t+1$, L_{vcost}^j stands for the variable cost rate, and λ_{k+t+1}^j represents the fixed cost incurred at the beginning of policy year $k+t+1$.

On the day a policy comes into force, the reserve will be incurred, after the first premium being collected. The expected net cash flow at time $t\ (t \in N)$, for the policy that is at the beginning of the first policy year ($k=1$ and $1 \le t < 20$), can then be represented as:

$$E(NCF_t^j \mid k=1) = ({}_{t-1}p_{30}^{(\tau)} \times q_{30+t-1}^{(d)} \times B_d^j) + ({}_{t-1}p_{30}^{(\tau)} \times q_t^{(s),j} \times B_{s,t}^j) - {}_t p_{30}^{(\tau)} \times [\pi^j \times (1 - L_{cm,t}^j - L_{vcost}^j)] - \lambda_t^j\,] \tag{10}$$

.

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[4] Note that the insured is at age 30+k-1 when the policy is at the beginning of policy year k.

[5] Note that ${}_0 p_{30+k-1}^{(\tau)}$=1. The upper script (τ) indicate a function referring to all causes or total force of decrement. Two causes of decrement, death and surrender, are considered in this paper and are denoted by the upper scripts (d) and (s) respectively.

[6] Note that $1 - q_{30+k-1+t-1}^{(d)} - q_t^{(s)} = {}_1 p_{30+k-1+t-1}^{(\tau)}$. A policy not terminated in a year by death or surrender means that the policy remains valid for a year. Furthermore, ${}_{t-1}p_{30+k-1}^{(\tau)} \times {}_1 p_{30+k-1+t-1}^{(\tau)} = {}_t p_{30+k-1}^{(\tau)}$, i.e., the probability of a policy with an insured age 30+k-1 being valid for t years equals the probability of the policy being valid for t-1 years times the probability of the policy with the insured age 30+k-1+t-1 remaining valid for one more year.

[7] This is equivalent to saying that the cash surrender value is paid at the end of year t.

Survival benefits for endowment and pure endowment policies are counted for the purpose of expected net cash flow only in policy due for two cases. One is at the beginning of the first policy year, when the planning horizon time t = 20. The other is at the end of the k^{th} policy year (i.e. $1 \le k' \le 19$), when the planning horizon time t = $20-k'$. At the beginning of the first policy, the expected net cash flows for $t=20$, including death, surrender, and survival benefit payments, without counting incoming net premium, can be represented as:

$$E(NCF_{20}^{j} \mid k=1) = ({}_{19}p_{30}^{(\tau)} \times q_{49}^{(d)} \times B_{d}^{j}) + ({}_{19}p_{30}^{(\tau)} \times q_{20}^{(s),j} \times B_{s,20}^{j}) + {}_{19}p_{30}^{(\tau)} \times B_{surv}^{j} \quad . \tag{11}$$

At the end of the k^{th} policy year also, the expected net cash flow for $t=20-k'$, including death, surrender, and survival benefit payments, without counting incoming net premium, can be represented as:

$$E(NCF_{20-k'}^{j} \mid k') = ({}_{20-k}p_{30+k}^{(\tau)} \times q_{49}^{(d)} \times B_{d}^{j}) + ({}_{20-k}p_{30+k}^{(\tau)} \times q_{k+1}^{(s),j} \times B_{s,k+1}^{j}) + {}_{20-k}p_{30+k}^{(\tau)} \times B_{surv}^{j} \quad , \tag{12}$$

where B_{surv}^{j} denotes the survival benefit for policy j.

Actuarial assumptions about some of the above variables are shown in Tables A2 to A4.

Policy Reserves

The present value of the expected net cash flows associated with policy j after the first net premium being received, R_1^j, can then be expressed as:

$$R_1^j = \sum_{t=1}^{20} [E(NCF_t^j \mid k=1) / P(1,t)] \,, \tag{13}$$

and the present value of the expected net cash flows associated with policy j before the k + 1 net premium being received, $R_{k'}^j$, can be expressed as:

$$R_{k'}^j = \sum_{t=0}^{20-k'} [E(NCF_t \mid k') / P(k',t)], \quad \text{if } 1 \le k' \le 19 \,, \tag{14}$$

where $P(k',0)=1$ equals the face value of zero-coupon bonds. In short, we calculate the reserves at the beginning of the first policy year, and at the end of each policy year, from the first to the nineteenth year.

Aggregate Reserves

We now know the reserves for each policy j in the first year, after net premium being collected, and at the end of each policy year, before the next net premium being received. The only thing we need to know, to calculate aggregate reserves, is the number of policies j issued at the beginning of the first policy year. Aggregate reserves for k = 1 and $1 \le k' \le 19$ are then expressed as:

$$AGR = \begin{cases} AGR_1 = \sum_j Q_j \times R_1^j \\ AGR_{k'} = \sum_j Q_j \times R_{k'}^j \end{cases} , \tag{15}$$

where Q_j is the number of policies j issued at the beginning of the first policy year.

ASSET ALLOCATION PROBLEM

The Dynamics of the Insurer's Financial Status

At the beginning, business comes in. The insurer receives premiums from writing twenty-year term life insurance, endowment, and pure endowment, pays the associated underwriting expenses, and then allocates net premiums along with the capital at the beginning of period one (E_1), among four asset classes considered in the previous section. The insurer can set the leverage to capital (L) ratio to determine the number of policies that can be issued (to maintain the leverage),[8] and then calculate the number of policies as $Q_j = (1/3) \times E_1 \times L / \pi^j$. Let s_1, re_1, ai_1, and b_1 denote allocations of funds to equities, real estate, alternative investment, and risk-free zero-coupon bonds,[9] respectively, where $0 \le s_1 \le 1$, $0 \le re_1 \le 1$, $0 \le ai_1 \le 1$, $0 \le b_1 \le 1$ and $s_1 + re_1 + ai_1 + b_1 = 1$. In other words, borrowings and short sales are not allowed.

Investment returns and underwriting outcomes are realized at the end of a period. We assume that losses incurred are paid by selling assets proportionally, at market value. Asset allocation of the insurer will hence remain unaffected by sale of assets. The insurer's positions at the end of year *k* (i.e. k') will then be as follows:
stock positions

$$SP_{k'} = (SP_{(k-1)'} + F_k \times s_k / S_k - CL_{k'} \times SP_k / TA_{k'}^{before}, \tag{16}$$

real estate positions

$$REP_{k'} = (REP_{(k-1)'} + F_k \times re_k / RE_k) - CL_{k'} \times REP_k / TA_{k'}^{before}, \tag{17}$$

alternative investment positions

$$AIP_{k'} = (AIP_{(k-1)'} + F_k \times ai_k / AI_k) - CL_{k'} \times AIP_k / TA_{k'}^{before}, \tag{18}$$

positions in *L*-year ($L \in N$ and $L < 15$) default-free bonds

$$B_{k'}^{L} = B_{(k-1)'}^{L+1} + (1/15) \times F_k \times b_k / P(k,L) - CL_{k'} \times B_k^{L+1} / TA_{k'}^{before}, \tag{19}$$

positions in fifteen-year default-free bonds

$$B_{k'}^{15} = (B_{(k-1)'}^{1} + F_k \times b_k \times (1/15)) / P(k,1)) / P(k',15) - CL_{k'} \times B_k^{1} / TA_{k'}^{before}, \tag{20}$$

where F_k denotes funds available for investment at time *k*, $0 \le s_k \le 1$, $0 \le re_k \le 1$, $0 \le ai_k \le 1$, $0 \le b_k \le 1$, $k = 2, \cdots, 20$ denote weights of each asset class, $TA_{k'}^{before}$ denote total asset value before paying claims at the end of the year, and $CL_{k'}$ denotes claim payments. Taking stock positions as example, positions at the beginning of year *k* (SP_k) are equal to $SP_{(k-1)'} + F_k \times s_k / S_k$, that is, positions after paying claims at the end of year *k-1* plus new positions taken by new investments. Positions in other asset classes are calculated in the same way, at the end of each year, after paying the claims.

As the position in each asset class is known, total asset value of the insurer,

---

[8] Without loss of generality, we assume $L \in N$.

[9] For simplicity, we assume that the insurer invests in the one-year, two-year, ... , and fifteen-year bonds with an equal weight of 1/15. The amount of the matured bonds is further assumed to be re-invested onto the fifteen-year bond to ensure that the longest maturity of the invested bonds remains to be 15 across time.

after paying the claims at the end of each year, is as follows:

$$TA_{k'}^{after} = SP_{k'} \times S_{k'} + REP_{k'} \times RE_{k'} + AIP_{k'} \times AI_{k'} + \sum_{L=1}^{15} B_{k'}^{L} P(k', L). \qquad (21)$$

Whenever $AGR_{k'} > TA_{k'}^{after}$, the insurer is deemed insolvent, and we stop simulating that path.

The number of paths to be simulated is set to be 10,000. For simplicity, without loss of generality, we assume that the insurer makes re-allocation decisions at the beginning of years 1, 6, 11, and 16 only. More specifically, $\vec{\theta}_k = \vec{\theta}_1$ for k = 1 – 5, $\vec{\theta}_k = \vec{\theta}_6$ for k = 6 – 10, $\vec{\theta}_k = \vec{\theta}_{11}$ for k = 11 – 15, $\vec{\theta}_k = \vec{\theta}_{16}$ for k = 16 – 20, where $\vec{\theta}_k^{'} = \left[s_k \quad re_k \quad ai_k \quad b_k \right]$. We further assume that the insurer re-allocates all its assets according to $\vec{\theta}_k$ at the beginning of periods 1, 6, 11, and 16. We are interested in the financial condition of the insurer at the end of year 20.

The Problem

The insurer has to maximize its objective function over the time horizon [0, 20]. The objective function contains three components: expected annual rate of return on surplus (investment assets), volatility of the rate of annual return, and ruin penalty. The insurer prefers a high annual rate of return on its surplus but low volatility and ruin penalty. More specifically, the optimization problem of the insurer is:

$$\begin{aligned} &\max_{\vec{\theta}_k, L} \{ \frac{1}{I_{solvent}} \sum_{i \in solvent} (\sqrt[20]{(E_{20'}^{i} - E_1)/E_1} - 1) \\ &\qquad - \varphi \times \sigma(\sqrt[20]{(E_{20'}^{i} - E_1)/E_1} - 1 \,|\, i \in solvent) - \gamma \times \Pr(\text{ruin}) \}, \\ &\text{s.t.} \\ &\quad s_k + re_k + ai_k + b_k = 1 \\ &\quad 0 \le \vec{\theta}_k \le 1 \end{aligned} \qquad (22)$$

where $I_{solvent}$ is the number of simulated paths in which no insolvency occurs, $\varphi = 0.5$ is a constant chosen by the insurer to reflect risk aversion, $\gamma = 10$ is to reflect the relative importance of ruin probability. Ruin probability is measured as number of insolvent paths divided by the number of simulated paths.

Simulation Optimization for the Problem

Our problem can be regarded as a discrete-time stochastic optimization problem, and particle swarm optimization (PSO) is the recommended tool for obtaining a promising answer (Kendall and Su, 2005; and Lu et al., 2006). PSO is a population-based co-operative process first proposed by Eberhart and Kennedy (1995), inspired by flocks of birds and shoals of fish. It has been used with enormous success across a wide range of applications. Poli (2008) found around eleven publications in which PSO has been used for financial risk early warning, investment decision-making, option pricing, and investment portfolio selection.

We use PSO to solve the asset allocation problem for two reasons. One is that PSO has been recognized for being as good as genetic algorithm (GA) and evolution strategy (ES) in solving high-dimensional and non-linear functions. The other is that PSO is more applicable than other optimization algorithms (i.e. PSO has been less explored and offers more potential operations resources). Details of PSO, including its formulation, algorithm, and effectiveness, are provided in the Appendix.

RESULTS

Promising Asset Allocation and Leverage

In the promising solution obtained from our simulation, leverage level is 20 and asset allocations vary across periods. The objective value is 24.63% with 7 insolvencies occurring in the simulation. There is no consistency in the composition of allocations, in terms of risky (stock, real estate and alternative investment) and risk-free (default-free zero coupon bonds) assets in re-allocations for different periods. In first and sixteenth periods, weights of fix income securities are higher than weights of risky assets but it is the opposite in sixth and eleventh periods. Allocations in the first and the last period are relatively conservative; weights of fix income securities are 70.75% and 67.62%, respectively. In sixth and eleventh periods, the insurer adjusts its portfolio to have more risky assets, to improve the objective value; the ratio of risky assets to fix income securities goes up to 2.43. Besides, weights of fix income securities after all re-allocations are higher than 29% because of intermediation and investment strategies being subject to penalty for volatility of equity and insolvency norms.[10]

[Insert Table 1 Here]

In Figure 1, we observe that for risky assets, weights of stock index plus real estate index are higher than alternative investment. In addition, the promising asset allocations show that equities play an important role in the investment portfolio in each period. After each re-allocation, weights of stocks are above 13%. However, weights of real estate index and alternative investment vary across periods. Weights of real estate experience significant change in the sixth period (rising from 4.78% to 28.15%), as well as the sixteenth period (declining from 36.61% to 3.43%). The rationale is that the price of real estate follows a jump diffusion process with the average number of jumps being 0.1 per year. Alternative investment is not the first choice of investment in any of the periods because of its high risk. Even in sixth and eleventh periods, to improve the objective value, the insurer prefers stocks and real estate over alternative investment since risks of real estate (0.18) and stock (0.25) are smaller than alternative investment (0.5). When determining the composition of risky assets, the insurer would be more concerned about the risk than the return of each asset.

[Insert Figure 1 Here]

Comparison of Leverage

[10] We record the asset allocations and leverage level before PSO converges in Appendix (Table 6). The leverage is same as the promising case, 20, and asset allocations are similar to the promising case except the last period. However, in this case, there is more insolvency with a higher volatility.

We set two leverage levels, 12 and 16, and find the corresponding promising asset allocations. Especially, we examine the finite risk hypothesis of capital and risk in insurance literature. Promising asset allocations for leverages of 16 and 12 are shown in Tables 2 and 3. Compared to Table 1, the results show that when leverage increases, the insurer needs to hold more fix income securities in the first period. In other words, the finite risk hypothesis in insurance literature holds conditionally in a multi-period asset allocation.

[Insert Tables 2 and 3 Here]

When leverage is 12, the insurer holds more fix income securities in the first, sixth, and eleventh periods. But in sixteenth period and thereafter, the insurer invests in risky assets to the fullest extent, to increase the objective value. When leverage is 16, the insurer holds more fix income securities only in the first and the sixth periods. From the eleventh period and thereafter, the insurer increases weights of risky assets to increase the objective value. Asset allocations in these two scenarios are different from the promising allocation because leverage strategies in the simulation are not optimal. Besides, volatility of annual rate of return of equity is decreasing as the leverage increases. Insolvencies in simulation paths exhibit no consistent correlation with leverage. Objective value increases as leverage increases.

In Figure 2, where leverage = 16, we observe that weights of risky assets increase gradually from the sixth period onwards. It reveals that the simulation optimization mechanism leads the insurer to improve the objective value by increasing weights of risky assets. This phenomenon is also shown for leverage = 12 in Figure 3.

[Insert Figures 2 and 3 Here]

In Figures 2 and 3, we find that the stock index is important for the life insurer. Weights of stock index in each re-allocation are higher than 10.2%. Real estate index is also favorable for the insurer to improve the objective value. Weights of real estate index are higher than 11.37% in all periods. Besides, weights of stock index plus real estate index dominate weights of alternative investment. Even if leverage does not reach the promising level, the insurer is concerned more about the risk than the return while choosing risky assets.

CONCLUSIONS

This study presents a company-wide simulation model and optimization algorithm for analyzing asset allocation and leverage strategies for a life insurer selling traditional policies. The model allows the insurer to compare different leverage strategies to determine how to construct a promising asset allocation. The promising asset allocation and leverage strategies are derived from numerical calculations that consider leverage as an internal factor in asset allocation. Also, our results demonstrate how to compare different leverage strategies for specific promising asset allocations.

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TABLES AND FIGURES

Table 1: Promising Asset Allocation and Leverage

Leverage = 20
Objective Value = 0.2463
Simulation Paths Insolvencies = 7
Volatility of Annual Rate of Return on Equity = 0.0514

	Asset Weights (%)			
Asset Class / Time	1-5	6-10	11-15	16-20
(A) Stock	13.76	16.28	29.23	15.13
(B) Real Estate	4.78	28.15	36.61	3.43
(C) Alternative Investment	10.71	12.35	5.09	13.82
(D) Default-Free Zero-Coupon-Bonds	70.75	43.22	29.07	67.62
Total Risky Assets = (A)+(B)+(C)	29.25	56.78	70.93	32.38
Total Fix Income Securities = (D)	70.75	43.22	29.07	67.62

Table 2: Asset Allocation; Given Leverage = 16

Leverage = 16
Objective Value = 0.2390
Simulation Paths Insolvencies = 8
Volatility of Annual Rate of Return on Equity = 0.0543

	Asset Weights (%)			
Asset Class / Time	1-5	6-10	11-15	16-20
(A) Stock	10.20	21.03	35.68	18.25
(B) Real Estate	11.37	23.24	18.93	35.55
(C) Alternative Investment	16.79	5.23	10.46	44.09
(D) Default-Free Zero-Coupon-Bonds	61.64	50.50	34.92	2.10
Total Risky Assets = (A)+(B)+(C)	38.36	49.50	65.08	97.90
Total Fix Income Securities = (D)	61.64	50.50	34.92	2.10

Table 3: Asset Allocation; Given Leverage = 12

Leverage = 12
Objective Value = 0.2211
Simulation Paths Insolvencies = 5
Volatility of Annual Rate of Return on Equity = 0.0652

	Asset Weights (%)			
Asset Class / Time	1-5	6-10	11-15	16-20
(A) Stock	12.63	12.63	19.92	71.82
(B) Real Estate	12.63	16.46	12.55	12.47
(C) Alternative Investment	16.61	1.33	7.57	15.71
(D) Default-Free Zero-Coupon-Bonds	58.13	69.58	59.95	0.00
Total Risky Assets = (A)+(B)+(C)	41.87	30.42	40.05	100.00
Total Fix Income Securities = (D)	58.13	69.58	59.95	0.00

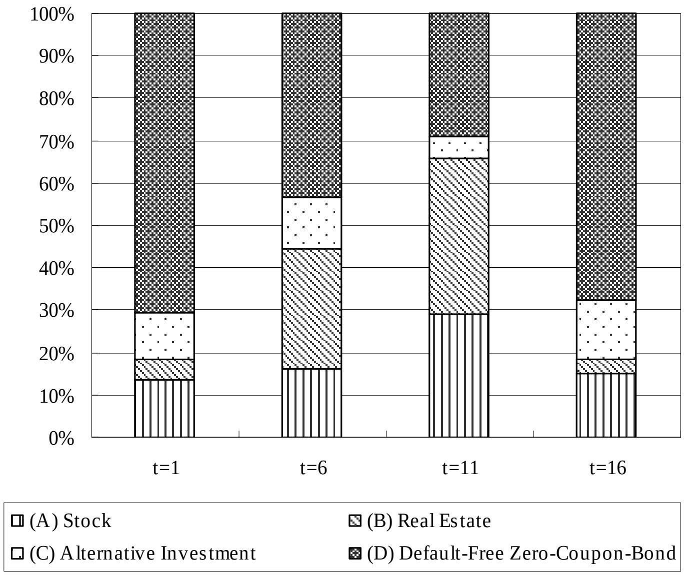

Figure 1: Composition of Promising Assets after Each Re-allocation under Optimal Leverage

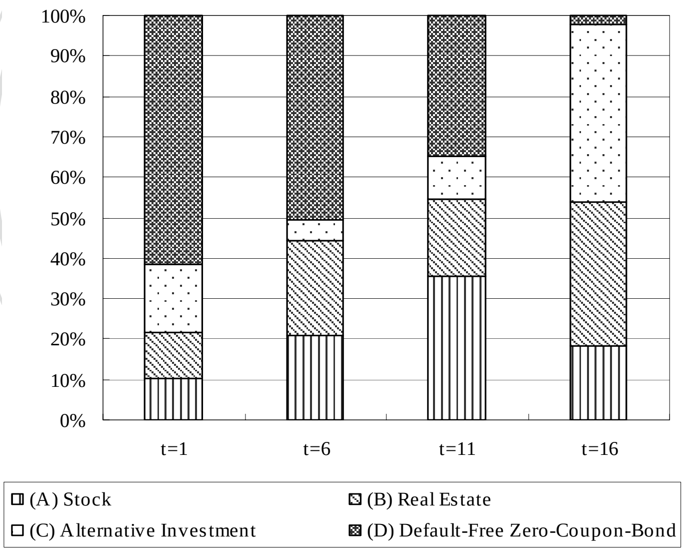

Figure 2: Assets Composition after Each Re-allocation under Leverage = 16

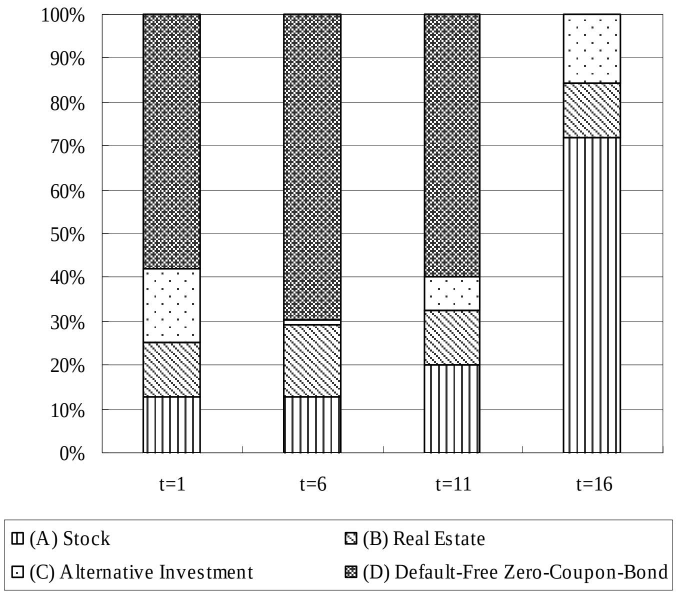

Figure 3: Assets Composition after Each Re-allocation under Leverage = 12

APPENDICES

Particle Swarm Optimization

In PSO, the particles are placed in the search space of some problem or function, and each evaluates the fitness at its current location. Each particle then determines its movement through the search space by combining some aspect of the history of its own fitness values with those of one or more members of the swarm, and then moving through the search space with a velocity determined by the locations and processed fitness values of other members, along with some random perturbations. Members of the swarm that a particle can interact with are called its social neighborhood. Social neighborhoods of all particles together form a PSO social network.

Take a maximizing *N*-dimensional function *f* for example. Each particle is *N*-dimensional, and is a potential optimum of *f*. Each particle has a memory of the best solution that is found, called its *personal best*. A particle flies through the search space with a velocity which is dynamically adjusted according to its personal best and the best solution found by a neighborhood of particles.

This is, thus, a sharing of information. Particles profit from discoveries and previous experiences of other particles during the exploration and search for higher objective function values. The first, called global best (*gbest*), connects all particles in the population to one another. The second, called local best (*lbest*), creates a neighborhood for each individual comprising it and its k nearest neighborhoods in the population.

Formulation

Let *i* indicate a particle's index in the swarm. Then $S = \{p_1, p_2, \cdots, p_s\}$ is a swarm of *s* particles. Each particle has a current position $p_i = (p_{i1}, p_{i2}, \cdots p_{iN})^T$ and flies through the *N*-dimensional search space $\Re^N$ with current velocity $v_i = (v_{i1,} v_{i2}, \cdots, v_{iN})^T$ which is dynamically adjusted according to its own previous best solution $x_i = (x_{i1}, x_{i2}, \cdots, x_{iN})^T$ and the current best solution $\tilde{x}_i$ of the entire swarm (*gbest*) or the particle's neighborhood (*lbest*).

At iteration time *t* of the PSO, the velocity and particle updates are specified separately for each dimension *j* of the velocity and particle vectors. A particle P_i will interact and move according to the follow equations:

$$v_{ij}^{t+1} = \omega_t v_{ij}^{t} + \varphi_1 R_1^t (x_{ij}^t - p_{ij}^t) + \varphi_2 R_2^t (\tilde{x}_{ij}^t - p_{ij}^t) \quad \text{(a1)}$$

$$\omega_t = \omega_{\max} - \frac{t}{t_{\max}} (\omega_{\max} - \omega_{\min}) \quad \text{(a2)}$$

$$p_{ij}^{t+1} = v_{ij}^{t+1} + p_{ij}^{t} \quad \text{(a3)}$$

where R_1 and R_2 are two independent variables uniformly distributed in [0,1], ω is a constant known as the inertia weight which determines the speed of convergence,

$\omega_{\max} = 0.9$ and $\omega_{\min} = 0.4$ have been shown to give good conversion, φ_1 and φ_2 are two constants known as the acceleration coefficients, and $0 \le \varphi_1, \varphi_2 \le 2$, which control the relative proportion of cognition and social interaction in the swarm (Shi and Eberhart, 1998). Values $t_{\max}$ and t indicate the maximum and current iteration numbers and we set $t_{\max}$ to be 1500.

Algorithm

The standard PSO algorithm to maximize function $f : \Re^N \to \Re$ is presented below:

1. Set the iteration number t to be zero, and initialize swarm S of N-dimensional particles p_i^0; each component p_{ij}^0 is randomly initialized to a value in the initial domain of the swarm, an interval $[p_{\min}, p_{\max}]$. Since the particles are already randomly distributed, velocities of particles are initialized to the zero vector 0^T.
2. Evaluate performance $f(p_i^t)$ of each particle.
3. Compare the personal best of each particle to its current performance, and set x_i^t to be the better performance for $x_i^t = \begin{cases} x_i^{t-1}, \text{ if } f(p_i^t) \le f(x_i^{t-1}) \\ p_i^t, \text{ if } f(p_i^t) > f(x_i^{t-1}) \end{cases}$.
4. Set the global best $\tilde{x}^t \in \{x_1^t, x_2^t, \cdots, x_s^t \mid f(\tilde{x}^t)\} = \max\{f(x_1^t), f(x_2^t), \cdots, f(x_s^t)\}$ to the position of the particle with the best performance within the entire swarm. When a local best PSO is implemented, set the neighborhood best $\tilde{x}_{Q_i}{}^t \in \{Q_i \mid f(\tilde{x}_{Q_i}{}^t) = \max\{f(x_j{}^t)\}$, $\forall x_j \in Q_i = \{x_{i-k}{}^t, \cdots, x_i^t, \cdots, x_{i+k}{}^t\}$, k is the number of nearest neighborhoods.
5. Change the velocity vector for each particle according to equation (a1).
6. Let $t = t + 1$.
7. Go to Step 2, and repeat until convergence or $t = t_{\max} = 1500$.

Effectiveness of PSO

Effectiveness of PSO has been recognized to be more efficient than other algorithms in solving complex non-linear and multi-modal functions with multi-variables (Huang, 2009). Complex functions in Huang (2009) originate from Schwefel (1981), Yao and Liu (1996), and Vesterstrom and Thomsen (2004). Appendix Table 1 presents these functions.

Table A1: High Dimension Complex Functions

Function list (n=50)	Constrains	Minimal value	Remark
$f_1(\vec{x}) = \frac{1}{4000}\left(\sum_{i=0}^{n-1} x_i^2\right) - \left(\prod_{i=0}^{n-1}\cos\left(\frac{x_i}{\sqrt{i+1}}\right)\right)+1$	$-600 \le x_i \le 600$	$f_1(\vec{0}) = 0$	Griewangk's Problem
$f_2(\vec{x}) = \sum_{i=0}^{n-1}\left(\left\lfloor x_i + \frac{1}{2}\right\rfloor\right)^2$	$-100 \le x_i \le 100$	$f_2(\vec{p}) = 0$; $-0.5 \le p \le 0.5$	Step Function
$f_3(\vec{x}) = \sum_{i=0}^{n-1} -x_i \sin\left(\sqrt{\lvert x_i \rvert}\right)$	$-500 \le x_i \le 500$	$f_3(\overrightarrow{420.97}) = -20,949.14$	Schewefel's Problem
$f_4(\vec{x}) = \sum_{i=0}^{n-1}(x_i^2 - 10\cos(2\pi x_i) + 10)$	$-5.12 \le x_i \le 512$	$f_4(\vec{0}) = 0$	Rastrigin's Function
$f_5(\vec{x}) = -20exp\left(-0.2\sqrt{\frac{1}{n}\sum_{i=0}^{n-1} x_i^2}\right) - exp\left(\frac{1}{n}\sum_{i=0}^{n-1}\cos(2\pi x_i)\right) + 20 + e$	$-32 \le x_i \le 32$	$f_5(\vec{0}) = 0$	Ackley's Function

Table A2: Notations and Values of Asset Models' Parameters

Description	Notation	Value
CIR Interest Rate Model		
Mean reverting speed	κ	0.25
Long term interest rate	μ	0.04
Volatility of interest rate	σ_r	0.03
Interest Rate Adjusted Geometric Brownian Model		
Risk premium	π_s	0.07
Volatility of the stock return	σ_s	0.25
Real Estate Model		
Expected return of real estate	μ_{RE}	0.055
Volatility of return of the real estate	σ_{RE}	0.18
Average jumps in one year	λ	0.1
Average jump size as proportion of the real estate index	β	Uniform(-0.5,0.5)
Alternative Investments		
Expected return of high-return and high risk investments	$\mu_{\chi^{hh}}$	0.15
Volatility of high-return –high-risk investments	$\sigma_{\chi^{hh}}$	0.5
Correlation Matrix		
Specific correlation matrix after Cholesky decomposition	R	$\begin{array}{c} \\ \varepsilon_S \\ \varepsilon_{RE} \\ \varepsilon_{x^{hh}} \\ \varepsilon_r \end{array} \begin{array}{cccc} \varepsilon_S & \varepsilon_{RE} & \varepsilon_{x^{hh}} & \varepsilon_r \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0.6 & 0 & 0.8 & 0 \\ -0.4 & -0.5774 & -0.575 & 0.4196 \end{bmatrix}$

Table A3: Actuarial Assumption of Twenty-Year Term Life Insurance

Insured's Age	Mortality Rate of age *a*	At the Beginning of Policy Year	Surrender Value	Commission Rate	Fixed Expense	Variable Cost Rate
a	$q_a^{(d)term}$	k	$B_{s,k}^{term}$	$L_{cm,k}^{term}$	λ_k^{term}	L_{vcost}^{term}
30	0.0009790	1	N/A	62.40%	420	0.001
31	0.0010055	2	N/A	22%	126	0.001
32	0.0010481	3	1,359	14.6%	126	0.001
33	0.0011075	4	2,758	8.0%	126	0.001
34	0.0011826	5	4,169	8.0%	126	0.001
35	0.0012712	6	5,566	8.0%	126	0.001
36	0.0013711	7	6,923	8.0%	126	0.001
37	0.0014807	8	8,212	8.0%	126	0.001
38	0.0015989	9	9,410	8.0%	126	0.001
39	0.0017291	10	10,491	8.0%	126	0.001
40	0.0018749	11	11,422	5.0%	126	0.001
41	0.0020407	12	11,981	5.0%	126	0.001
42	0.0022297	13	12,284	5.0%	126	0.001
43	0.0024446	14	12,282	5.0%	126	0.001
44	0.0026795	15	11,918	5.0%	126	0.001
45	0.0029268	16	11,141	5.0%	126	0.001
46	0.0031784	17	9,910	5.0%	126	0.001
47	0.0034268	18	8,198	5.0%	126	0.001
48	0.0036671	19	5,986	5.0%	126	0.001
49	0.0039091	20	3,263	5.0%	126	0.001
50	N/A	20*	N/A	N/A	N/A	N/A

1. The death benefit is $1,000,000. The policy is issued to a 30 year-old male, and the annual premium the insured is expected to pay at the beginning of each surviving year is $4,200 under the policy crediting rate of 4%.

2. We assume a fixed rate of surrenders in each policy year at 5% for term life insurance.

3. Notation 20* is used to denote the end of policy year 20.

4. Policies surrendered at the beginning of the first, second, and last policy year have no surrender value. Neither mortality nor expenses apply when these policies mature. We denote all these values as N/A.

5. The variable cost is assumed to be 0.1%.

Table A4: Actuarial Assumption of Twenty-Year Endowment

Insured's Age	Mortality Rate of age *a*	At the Beginning of Policy Year	Surrender Value	Commission Rate	Fixed Expense	Variable Cost Rate
a	$q_a^{(d)ed}$	k	$B_{s,k}^{ed}$	$L_{cm,k}^{ed}$	λ_k^{ed}	L_{vcost}^{ed}
30	0.0009790	1	N/A	62.40%	4,530	0.001
31	0.0010055	2	8,161	27.00%	1,359	0.001
32	0.0010481	3	39789	20.60%	1,359	0.001
33	0.0011075	4	73,767	14.00%	1,359	0.001
34	0.0011826	5	110,192	13.00%	1,359	0.001
35	0.0012712	6	149,173	12.00%	1,359	0.001
36	0.0013711	7	190,831	10.00%	1,359	0.001
37	0.0014807	8	235,294	10.00%	1,359	0.001
38	0.0015989	9	282,707	10.00%	1,359	0.001
39	0.0017291	10	333,223	10.00%	1,359	0.001
40	0.0018749	11	387,004	7.00%	1,359	0.001
41	0.0020407	12	437,655	7.00%	1,359	0.001
42	0.0022297	13	490,342	7.00%	1,359	0.001
43	0.0024446	14	545,163	7.00%	1,359	0.001
44	0.0026795	15	602,227	7.00%	1,359	0.001
45	0.0029268	16	661,664	7.00%	1,359	0.001
46	0.0031784	17	723,620	7.00%	1,359	0.001
47	0.0034268	18	788,259	7.00%	1,359	0.001
48	0.0036671	19	855,760	7.00%	1,359	0.001
49	0.0039091	20	926,314	7.00%	1,359	0.001
50	N/A	20*	1,000,000	N/A	N/A	N/A

1. The death benefit and survival benefit is $1,000,000. The policy is issued to a 30 year-old male, and the annual premium payable at the beginning of each surviving year is $45,300 under the policy crediting rate of 4%.

2. We assume a fixed surrender rate in each policy year at 7% level for endowment.

3. Notation 20* is used to denote the end of policy year 20.

4. A policy surrendered at the beginning of the first policy year has no surrender value. Neither mortality nor expenses apply when the policy matures. We denote all these values as N/A.

5. The variable cost is assumed to be 0.1%.

Table A5: Actuarial Assumption of Twenty-Year Pure Endowment

Insured's Age	Mortality Rate of age *a*	At the Beginning of Policy Year	Surrender Value	Commission Rate	Fixed Expense	Variable Cost Rate
a	$q_a^{(d)\,ped}$	k	$B_{s,k}^{ped}$	$L_{cm,k}^{ped}$	λ_k^{ped}	L_{vcost}^{ped}
30	0.0009790	1	N/A	72.96%	3,570	0.001
31	0.0010055	2	27,707	28.00%	1,071	0.001
32	0.0010481	3	57,552	21.60%	1,071	0.001
33	0.0011075	4	89,652	15.00%	1,071	0.001
34	0.0011826	5	124,130	14.00%	1,071	0.001
35	0.0012712	6	161,121	13.00%	1,071	0.001
36	0.0013711	7	200,769	6.00%	1,071	0.001
37	0.0014807	8	243,227	6.00%	1,071	0.001
38	0.0015989	9	288,660	6.00%	1,071	0.001
39	0.0017291	10	337,247	6.00%	1,071	0.001
40	0.0018749	11	389,178	5.00%	1,071	0.001
41	0.0020407	12	438,092	5.00%	1,071	0.001
42	0.0022297	13	489,259	5.00%	1,071	0.001
43	0.0024446	14	542,825	5.00%	1,071	0.001
44	0.0026795	15	598,957	5.00%	1,071	0.001
45	0.0029268	16	657,831	5.00%	1,071	0.001
46	0.0031784	17	719,633	5.00%	1,071	0.001
47	0.0034268	18	784,554	5.00%	1,071	0.001
48	0.0036671	19	852,785	5.00%	1,071	0.001
49	0.0039091	20	924,525	5.00%	1,071	0.001
50	N/A	20*	1,000,000	N/A	N/A	N/A

1. The survival benefit is $1,000,000. The policy is issued to a 30 year-old male, and the annual premium payable at the beginning of each surviving year is $35,700 under the policy crediting rate of 4%.

2. We assume a fixed surrender rate in each policy year at 7% level for pure endowment.

3. Notation 20* is used to denote the end of policy year 20.

4. A policy surrendered at the beginning of the first policy year has no surrender value. Neither mortality nor expenses apply when the policy matures. We denote all these values as N/A.

5. The variable cost is assumed to be 0.1%.

Table A6: Promising Asset Allocation and Leverage Ratio before PSO Converges

Leverage Ratio = 20
Objective Value = 0.2443
Simulation Paths Insolvencies = 20
Volatility of Annual Rate of Return on Equity = 0.07215

	Asset Weights (%)			
Asset Class / Time	1-5	6-10	11-15	16-20
(A) Stock	11.16	16.23	61.35	9.94
(B) Real Estate	3.83	29.14	9.64	2.83
(C) Alternative Investment	16.87	12.37	3.94	39.37
(D) Risk-Free Zero-Coupon –Bonds	68.15	42.25	25.08	47.85
Total Risky Assets = (A)+(B)+(C)	31.85	57.75	74.92	52.15
Total Fix Income Securities = (D)	68.15	42.25	25.08	47.85