國立政治大學應用數學系

碩士學位論文

Some Topics on Edge Connectivity and Edge Degrees

關於邊連通數和邊度數的問題

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Contents

Abstract

In this thesis, we classify some graphs into locally coconnected graphs or locally connected graphs, compute the number of its edge cuts of size $2k-1$ and $2k$ in a Harary graph, and show some properties of the number of vertices of degree 1 when the graph has the maximum difference of minimum edge degree and minimum vertex degree.

keywords: Locally Connected Graphs; Edge Cuts; Restricted Edge Connectivity.

中文摘要

在這篇論文中，我們根據局部連通和局部補連通性質將圖分類，計算在 Harary 圖裡大小爲 $2k-1$ 和 $2k$ 邊切集的個數，和證明當圖形有最大的最小邊度數和最小點度數差，一些關於度數爲 1 的點個數性質。

關鍵詞：局部連通圖；邊切集；限制邊連通數。

1 Introduction

It is convenient to travel among cites by airplane, ship, or train. Every traffic system is just a graph consisting of stations and routes. Nevertheless, if the system sometimes gets paralyzed at certain lines or certain locations because of earthquakes, tsunamis, or bad weathers, then we have to spend more time on traveling. Now, we put our attention on the connectivity among stations and the relationships of lines and stops.

We follow [4] for notations in graph theory and define a graph $G(V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. It is very hard to study the connectivity of an arbitrary graph. Therefore, in chapter 2, we reduce our focus on the neighborhood of a vertex in a graph and introduce the definitions of locally connected graphs and locally coconnected graphs. In chapter 3, we study the number of the edge cut of a fixed size in k-regular graphs. Ou Jianping and Fuji Zhang provide some new ideas for counting it in [2]. They introduce optimal graphs and super restricted edge connected graphs in order to show the number of the edge cuts of sizes less than $2k-1$. We extend the size of the edge cuts to $2k-1$ and $2k$, and get the number of the different edge cuts for specific Harary graphs. In chapter 4, we study the difference of the minimum edge degree and the minimum vertex degree of a graph. When the graph has the maximum difference, we compute the upper bound of the number of vertices with degree 1 and the lower bound of the number of edges in these graphs. Finally, we bring up some open problems and further directions of research.

2 Some Properties of Locally Connected Graphs and Locally Coconnected Graphs

To determine the connectivity of any graph is actually difficult, so we reduce the problem to study the connectivity of the neighborhood of a vertex.

Definition 2.1. The graph G is called *locally connected* if the neighborhood $N_G(v)$ of every vertex v in G is connected.

Definition 2.2. A vertex v of a graph G is called *locally coconnected* if the neighborhood $N_G(v)$ of v in the complement $\bar{G}$ of G is connected. A graph G is called *locally coconnected* if every vertex in G is locally coconnected.

A locally connected city is very important to its neighbor zones. Let us see some examples of locally conconnected graph.

Theorem 2.3. *Every triangle-free graph is locally coconnected.*

Proof. Let G be a triangle-free graph. For every vertex v of G, let $N(v)$ denote the neighborhood of v in G. $N(v)$ is an independent set. Then G is locally coconnected. □

Corollary 2.4. *Every bipartite graph is locally coconnected.*

Proof. Since any bipartite graph G is triangle-free, G is locally coconnected by Theorem 2.3. □

A *paw* is defined to be a triangle with an additional vertex adjacent to one of the vertices of the triangle and the graph is drawn in Figure 1.

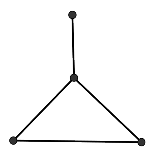

Figure 1: Paw.

Definition 2.5. A *chord* of a cycle C is an edge not in C whose endpoints lie in C. A *chordless cycle* in G is a cycle of length at least 4 in G that has no chord. A graph G is *chordal* if it is simple and has no chordless cycle.

Theorem 2.6. *Let G be a chordal graph with minimum degree $\delta(G) \geq 2$ and it contains no paw. Then G is a locally connected graph.*

Proof. Let x be any vertex of G. Because the degree $d(x)$ of x is greater than or equal to 2, we have at least two vertices in $N(x)$. If any two vertices are adjacent to each other, then G is locally connected. So, we may assume there exist two vertices u and v in $N(x)$ such that u is not adjacent to v and consider the following two cases.

Case 1: All u, v-paths contain x. Since the degree of each vertex in G is greater than or equal to 2 and G is chordal, x is contained in a cycle C_3. Then it forms a paw, which contradicts our assumption.

Case 2: There is a path u, v-path P that does not contain x. For any vertex w in P, it is a neighbor of x because G has no C_n, $n \geq 4$. If there is a vertex y in $N(x)$ and y does not belong to P, then the triangle x, u, w and edge xy form a paw in Figure 2, which contradicts to our assumption. Hence, $N(x)$ is connected.

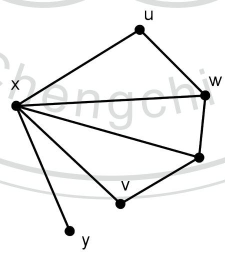

Figure 2: The graph shows the relationship of the neighbors of the vertex x in G.

□

Definition 2.7. Three vertices v_1, v_2, v_3 in a graph G form an *asteroidal triple* or *AT* of G if, for $i, j \in \{1, 2, 3\}$ and $i < j$, there is a path from v_i to v_j which avoids using any vertex in the *closed neighborhood* $N[v_k] = \{v_k\} \cup N(v_k)$.

An easy way to verify this for vertices v_1, v_2, v_3 in a graph G is to examine whether v_1, v_2 are connected in $G - N[v_3]$, v_2, v_3 are connected in $G - N[v_1]$,and v_1, v_3 are connected in $G - N[v_2]$. Ekkehard G. Köhler shows an important theorem about the AT-free graph in [3].

Lemma 2.8. *A graph G is AT-free if and only if it does not contain any of the graphs in Figure 3 as induced subgraph.*

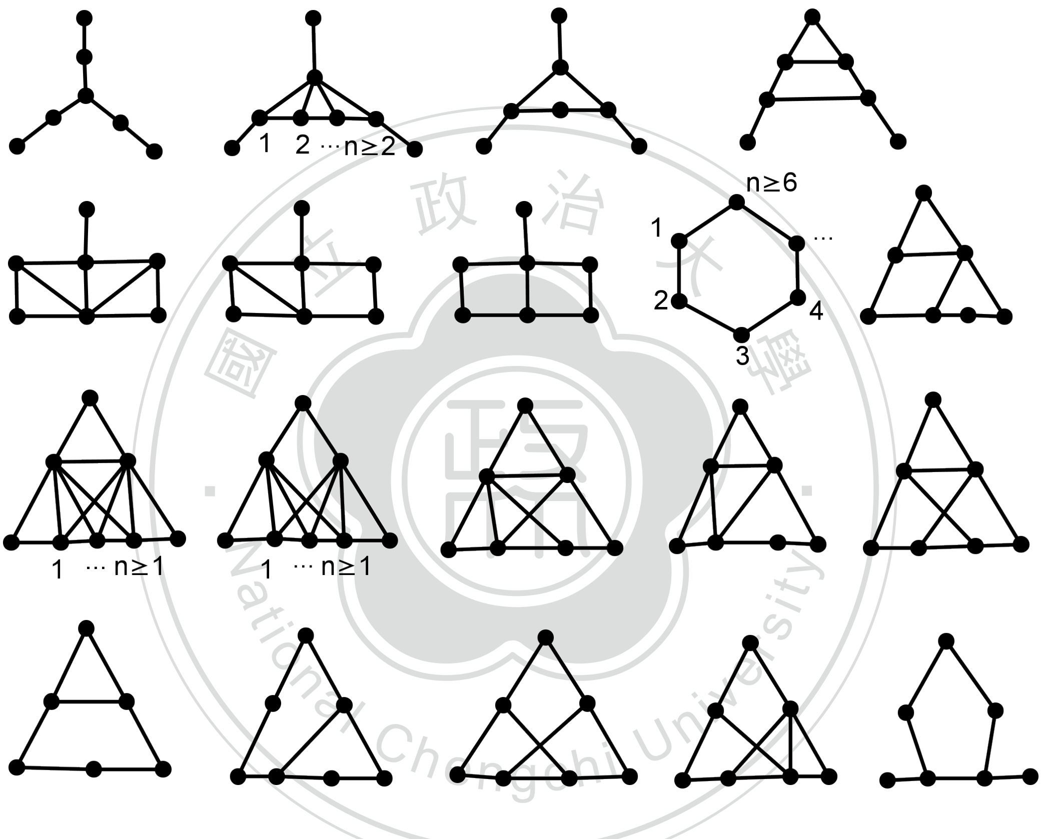

Figure 3: The structure of forbidden subgraphs of asteroidal triple-free graphs.

Theorem 2.9. *Let G be a graph without $C_n, n \geq 5$ and $K_{1,3}$ as an induced subgraph. G has AT and $\delta(G) \geq 2$. Then $\bar{G}$ is a locally coconnected graph.*

Proof. Let G be a graph with AT. Then G has induced subgraphs in Figure 3 and we have to check the assumptions in them. The graphs in Figure 4 contain $C_n, n \geq 5$ as an induded subgraph, the graph in Figure 5 have an induced subgraph $K_{1,3}$, and there exist some vertices with degree 1 in Figure 6. The only graphs which satisfy the conditions are in Figure 7 and the complement of them in Figure 8. It is obvious that every vertex in Figure 8 is locally coconnected.

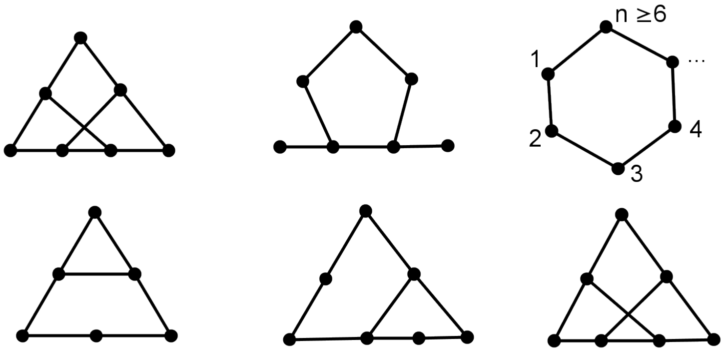

Figure 4: The graphs have AT and a induced subgraph $C_n, n \geq 5$.

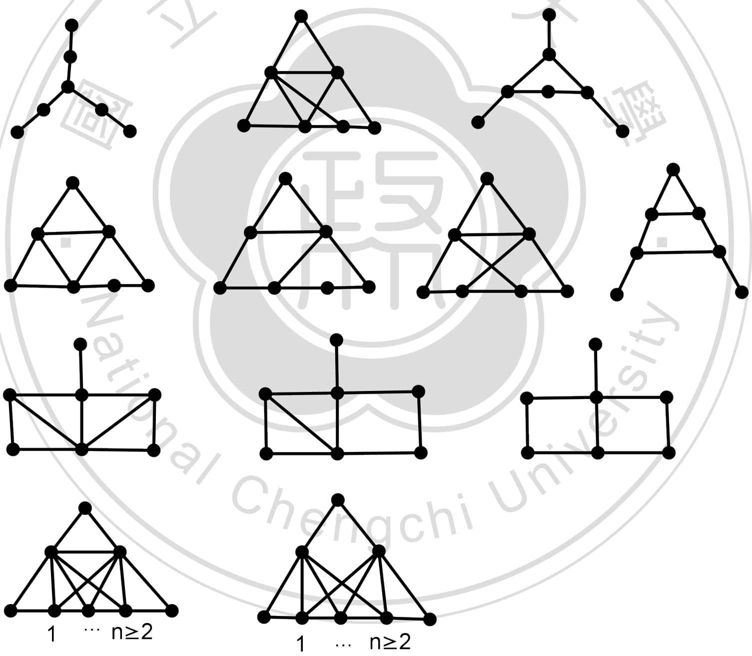

Figure 5: The graphs have AT and a induced subgraph $K_{1,3}$.

□

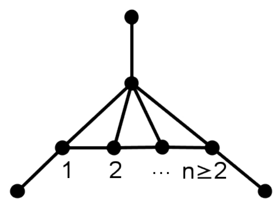

Figure 6: The graphs have AT and some vertices with degree 1.

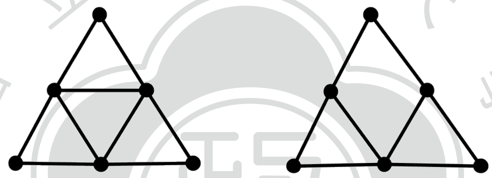

Figure 7: The graphs have AT without $C_n, n \geq 5$ and $K_{1,3}$ as an induced subgraph and $\delta(G) \geq 2$.

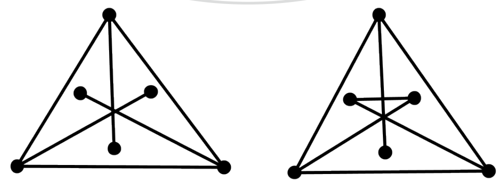

Figure 8: The graphs are the complement of graphs which have AT without $C_n, n \geq 5$ and $K_{1,3}$ as an induced subgraph and $\delta(G) \geq 2$.

3 The Number of Edge Cuts of k-regular Graphs

First, we give some condition to an edge cut of a graph G and observe the relationship between the conditional edge connectivity and the minimal edge degree. Abdol-Hossein Esfahanian and S. Louis Hakimi studied the conditional edge connectivity in [1]. In this chapter, we call a disconnected graph, a triangle, or a star *trivial* and all other graphs *non-trivial*.

Definition 3.1. Let G be a non-trivial simple graph and $S \subseteq E(G)$. The set S is called a *restricted edge cut* of G if $G - S$ is disconnected and each component has at least two vertices. The *restricted edge connectivity* $\lambda'(G)$ of G, is defined as the minimum size of all restricted edge cuts of G.

Definition 3.2. Let G be a simple graph. For an edge $e = xy$ in G, we denote $\xi(e) = d(x) + d(y) - 2$ and call the $\xi(G) = min\{\xi(e)|e \in E(G)\}$ as *minimum edge degree* of G.

Obviously, for any non-trivial graph G, $\lambda'(G)$ exists and satisfies the following inequality

$$\lambda'(G) \leq \xi(G).$$

Definition 3.3. Let G be a non-trivial simple graph. We say G is *optimal* if the equality $\lambda'(G) = \xi(G)$ holds; otherwise G is *non-optimal.* We call G is *super restricted edge connected* if $G - S$ has a component consisting of an isolated edge for every restricted edge cut S of the minimum size in G.

A graph G is called *k-regular* if every vertex of G has degree k. In the following statements, the graph G is simple if not specially stated. Jun-Ming Xu and Ke-Li Xu had some research for optimal and non-optimal graphs in [5]. It is clearly to see that super restricted edge connected graphs are optimal, but the converse is not true. Ou Jianping and Fuji Zhang showed that a sufficient condition for k-regular graphs to be optimal and super restricted edge connected.

Lemma 3.4. *[2] Let G be a k-regular connected graph of order n.*

(i) *If $k > \dfrac{n}{2}$, then G is optimal.*

(ii) *If* $k > \dfrac{n}{2} + 1$*, then* G *is super restricted edge connected.*

When we study the system reliability, we often focus on the model whose nodes never fail but lines have a breakdown independently of each other with equal probability. Suppose every station has the same number of branches and the system is very dense, then we describe it as a k-regular graph with $k > \dfrac{n}{2} + 1$. After some edges are broken such that the graph decomposes into more than one component, we call the edge set as an *edge cut*. When the size of an edge cut is not more than $2k-2$, we only consider the component which is a single vertex or an isolated edge by Lemma 3.4. In a graph G, we use the notation C_h to be the number of its edge cuts of size h in G.

Theorem 3.5. *[2] Let* G *be a connected* k*-regular graph of order* n*. If* $k > \dfrac{n}{2} + 1$*, then*

$$C_h = \begin{cases} 0 & , \text{ if } h < k \\ n & , \text{ if } h = k \\ n\begin{pmatrix} \dfrac{nk}{2} - k \\ h-k \end{pmatrix} & , \text{ if } k < h < 2k-2 \\ n\begin{pmatrix} \dfrac{nk}{2} - k \\ k-2 \end{pmatrix} + \dfrac{nk}{2} & , \text{ if } h = 2k-2. \end{cases}$$

In [4], given $k < n$, the construction of *Harary graphs* $H_{k,n}$ is to place n vertices uniformly around a circle and label the vertices by the integers modulo n, each vertex is adjacent to the nearest $\dfrac{k}{2}$ vertices in clockwise and counterclockwise directions if k is even, and each vertex is adjacent to the nearest $\dfrac{k-1}{2}$ vertices in each direction and to the diametrically opposite vertex if k is odd and n is even. In these two cases, $H_{k,n}$ is a k-regular graph. When k and n are both odd, the structure of $H_{k,n}$ is $H_{k-1,n}$ adding the edges $i \leftrightarrow i + \dfrac{n-1}{2}$ for $0 \leq i \leq \dfrac{n-1}{2}$. It indicates that every vertex has degree k except one vertex with degree $k-1$.

It is convenient to call an edge xy in $H_{k,n}$ as *diagonal edge* if $|x-y| = \dfrac{n}{2}$. In addition, given $n, r \in \mathbb{Z}$, we define the notation $\begin{pmatrix} n \\ r \end{pmatrix}$ to be zero if $r < 0$. In the following statement, we compute the number of triangles in $H_{k,n}$.

Lemma 3.6. *Let $T(k,n)$ be the number of triangles in $H_{k,n}$ and $t_{\frac{n}{2}}$ be the number of triangles containing a diagonal edge.*

(i) *If k is even, then $T(k,n) = n\begin{pmatrix} \frac{k}{2} \\ 2 \end{pmatrix} + t_0$, where*

$$t_0 = \frac{n}{3}[\begin{pmatrix} n-1 \\ n-3 \end{pmatrix} - 3\begin{pmatrix} n-\frac{k}{2}-1 \\ n-\frac{k}{2}-3 \end{pmatrix} + 3\begin{pmatrix} n-k-1 \\ n-k-3 \end{pmatrix} - \begin{pmatrix} n-\frac{3}{2}k-1 \\ n-\frac{3}{2}k-3 \end{pmatrix}].$$

(ii) *If k is odd and n is even, then $T(k,n) = n\begin{pmatrix} \frac{k-1}{2} \\ 2 \end{pmatrix} + t_0 + t_{\frac{n}{2}}$, where*

$$t_0 = \frac{n}{3}[\begin{pmatrix} n-1 \\ n-3 \end{pmatrix} - 3\begin{pmatrix} n-\frac{k-1}{2}-1 \\ n-\frac{k-1}{2}-3 \end{pmatrix} + 3\begin{pmatrix} n-k \\ n-k-2 \end{pmatrix} - \begin{pmatrix} n-\frac{3}{2}k \\ n-\frac{3}{2}k-2 \end{pmatrix}]$$

and

$$t_{\frac{n}{2}} = n[\begin{pmatrix} \frac{n}{2}-1 \\ \frac{n}{2}-2 \end{pmatrix} - 2\begin{pmatrix} \frac{n}{2}-\frac{k-1}{2}-1 \\ \frac{n}{2}-\frac{k-1}{2}-2 \end{pmatrix} + \begin{pmatrix} \frac{n}{2}-k \\ \frac{n}{2}-k-1 \end{pmatrix}].$$

Proof. **(i)** Let k be even. We name the distance d between two vertices x, y in $H_{k,n}$ as the following

$$d = \text{minimum}\{|x-y|, n-|x-y|\}.$$

For any three vertices in $H_{k,n}$, every vertex is incident to two edges, say two hands, of this triangle. Hence, we consider the following two cases. If two hands of a vertex x are incident to neighbors of x in the clockwise direction, then there is an edge adjacent two neighbors of x since the distance of any two neighbors of x in the same direction is less than $\frac{k}{2}$. It is similar for the counterclockwise direction, so each vertex provides $\begin{pmatrix} \frac{k}{2} \\ 2 \end{pmatrix} \times 2$ triangles in Figure 9.

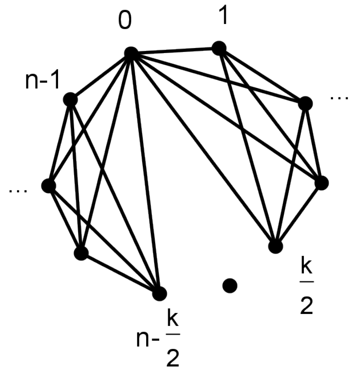

Figure 9: The graph shows that each vertex in $H_{k,n}$ has triangles in the same direction of hands.

In addition, every triangle is counted two times, so there are

$$\frac{n\begin{pmatrix} \frac{k}{2} \\ 2 \end{pmatrix} \times 2}{2} = n\begin{pmatrix} \frac{k}{2} \\ 2 \end{pmatrix}$$

triangles. If two hands of a vertex are in the different directions, then we have the following model

$$x_1 + x_2 + x_3 = n$$
$$1 \leq x_1, x_2, x_3 \leq \frac{k}{2},$$

where x_1, x_2, and x_3 are the distances between every pair of vertices of triangles in $H_{k,n}$. We solve the model by using the generating function

$$\begin{aligned} g(x) &= (x + x^2 + ... + x^{\frac{k}{2}})^3 \\ &= x^3(1 + x + ... + x^{\frac{k}{2}-1})^3 \\ &= x^3(\frac{1 - x^{\frac{k}{2}}}{1-x})^3 \\ &= x^3(\frac{1}{1-x})^3(1 - x^{\frac{k}{2}})^3. \end{aligned}$$

Moreover, we have the polynomial expansions

$$(\frac{1}{1-x})^3 = 1 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} x + \begin{pmatrix} 4 \\ 2 \end{pmatrix} x^2 + ... + \begin{pmatrix} r+2 \\ r \end{pmatrix} x^r + ...$$

and

$$(1-x^{\frac{k}{2}})^3 = 1 - 3x^{\frac{k}{2}} + 3x^k - x^{\frac{3}{2}k}.$$

The coefficient c_{n-3} of x^{n-3} in $g(x)$ is

$$\binom{n-1}{n-3} - 3\binom{n-\frac{k}{2}-1}{n-\frac{k}{2}-3} + 3\binom{n-k-1}{n-k-3} - \binom{n-\frac{3}{2}k-1}{n-\frac{3}{2}k-3}.$$

Since every triangle is counted three times, we get $t_0 = \dfrac{n}{3} \times c_{n-3}$ triangles.

(ii) Let k be odd and n be even. By similar argument, we consider three cases for each vertex, two hands are in the same direction, two hands are in different directions, or the triangle has a diagonal edge. If two hands of a vertex are in the same direction, then each vertex provides $\binom{\frac{k-1}{2}}{2} \times 2$ triangles. In addition, every triangle is counted two times, so there are

$$\frac{n\binom{\frac{k-1}{2}}{2} \times 2}{2} = n\binom{\frac{k-1}{2}}{2}$$

triangles. If two hands of a vertex are in the different directions, then we have the following model

$$x_1 + x_2 + x_3 = n$$
$$1 \le x_1, x_2, x_3 \le \frac{k-1}{2},$$

where x_1, x_2, and x_3 are the distances between every pair of vertices of triangles in $H_{k,n}$. We solve the model by using the generating function

$$\begin{aligned} g(x) &= (x + x^2 + ... + x^{\frac{k-1}{2}})^3 \\ &= x^3(1 + x + ... + x^{\frac{k-1}{2}-1})^3 \\ &= x^3\left(\frac{1-x^{\frac{k-1}{2}}}{1-x}\right)^3 \\ &= x^3\left(\frac{1}{1-x}\right)^3(1-x^{\frac{k-1}{2}})^3. \end{aligned}$$

The coefficient c_{n-3} of x^{n-3} in $g(x)$ is

$$\binom{n-1}{n-3} - 3\binom{n-\frac{k-1}{2}-1}{n-\frac{k-1}{2}-3} + 3\binom{n-k}{n-k-2} - \binom{n-\frac{3}{2}k}{n-\frac{3}{2}k-2}.$$

Since every triangle is counted three times, we get $t_0 = \dfrac{n}{3} \times c_{n-3}$ triangles. If the triangle contains a diagonal edge xy in the Figure 10, then we find a common neighbor z of x and y to form a triangle and yields the following model

$$x_1 + x_2 = \frac{n}{2}$$
$$1 \leq x_1, x_2 \leq \frac{k-1}{2},$$

where x_1 and x_2 are the distances between every pair of vertices of triangles in $H_{k,n}$.

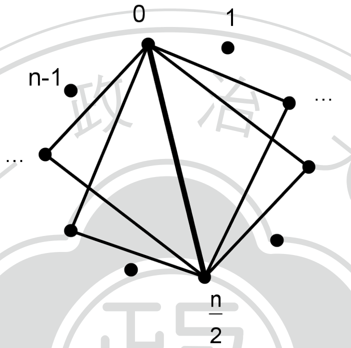

Figure 10: The graph shows the triangle with a diagonal edge in $H_{k,n}$.

We solve the model by using the generating function

$$\begin{aligned} h(x) &= (x + x^2 + ... + x^{\frac{k-1}{2}})^2 \\ &= x^2(1 + x + ... + x^{\frac{k-1}{2}-1})^2 \\ &= x^2\left(\frac{1 - x^{\frac{k-1}{2}}}{1-x}\right)^2 \\ &= x^2\left(\frac{1}{1-x}\right)^2 (1 - x^{\frac{k-1}{2}})^2. \end{aligned}$$

Moreover, we have the polynomial expansion

$$\left(\frac{1}{1-x}\right)^2 = 1 + \binom{2}{1} x + \binom{3}{2} x^2 + ... + \binom{r+1}{r} x^r + ...$$

The coefficient $c_{\frac{n}{2}-2}$ of $x^{\frac{n}{2}-2}$ in $h(x)$ is

$$\binom{\frac{n}{2}-1}{\frac{n}{2}-2} - 2\binom{\frac{n}{2} - \frac{k-1}{2} - 1}{\frac{n}{2} - \frac{k-1}{2} - 2} + \binom{\frac{n}{2} - k}{\frac{n}{2} - k - 1}.$$

Since x and y have a corresponding common neighbor in the other direction and there are $\frac{n}{2}$ diagonal edges, we get $t_{\frac{n}{2}} = 2 \times \frac{n}{2} \times c_{\frac{n}{2}-2} = n \times c_{\frac{n}{2}-2}$ triangles.

□

Now, we extend Theorem 3.5 to $h = 2k-1$ and $h = 2k$, and the k-regular graph is restricted to a Harary graph. When cutting more edges in a graph, a smaller component can be a single vertex, an isolated edge, a triangle, or K_4. We need Lemma 3.6 to compute the number of triangles in these cases and the number of triangles that are counted twice.

Theorem 3.7. *Let G be a Harary graph $H_{k,n}$ with $k > \frac{n}{2}+1$ and $h = 2k-1$. If k or n is even, then*

$$C_h = \begin{cases} n\begin{pmatrix} \frac{nk}{2}-k \\ k-1 \end{pmatrix} - \frac{nk}{2} + \frac{nk}{2}\begin{pmatrix} \frac{nk}{2}-2k+1 \\ 1 \end{pmatrix} & \text{, if } G \text{ is not } H_{5,6} \\ 6\begin{pmatrix} 10 \\ 4 \end{pmatrix} - 15 + 15\begin{pmatrix} 6 \\ 1 \end{pmatrix} + \frac{20}{2} & \text{, if } G \text{ is } H_{5,6}. \end{cases}$$

Proof. Let S be an edge cut of $H_{k,n}$ and $G-S$ has a component with p vertices, where $p \leq \lfloor \frac{n}{2} \rfloor$. Since k or n is even, $H_{k,n}$ is a k-regular graph and it implies that there are $\frac{nk}{2}$ edges. If p is 1, then S is a disjoint union of two edge sets. One consists of k edges incident to a single vertex chosen from $H_{k,n}$ and the other contains $k-1$ edges chosen arbitrarily from the remaining $\frac{nk}{2}-k$ edges of $H_{k,n}$. However, when we cut all incident edges of two vertices which are adjacent to each other, see Figure 11, this edge cut is counted two times. Hence, there are $n\begin{pmatrix} \frac{nk}{2}-k \\ k-1 \end{pmatrix} - \frac{nk}{2}$ ways.

If p is 2, then S is composed of $2k-2$ edges incident to a specific edge chosen from $\frac{nk}{2}$ edges in $H_{k,n}$ and one edge in the remaining edges. As a result, there are $\frac{nk}{2}\begin{pmatrix} \frac{nk}{2}-2k+1 \\ 1 \end{pmatrix}$ ways. If p is 3, then $H_{k,n}$ has at least six vertices and we have four following cases about this three vertices of $H_{k,n}$ in Figure 12, Figure 13, Figure 14, Figure 15.

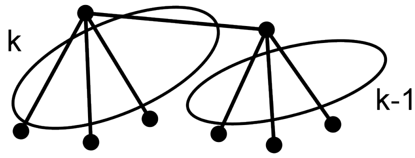

Figure 11: Two adjacent vertices in $H_{k,n}$ have $2k-1$ incident edges.

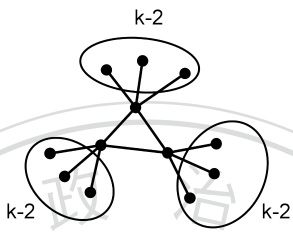

Figure 12: Three vertices x, y, z of $H_{k,n}$ are adjacent to each other and there are additional $3k-6$ edges which are incident to x, y, z.

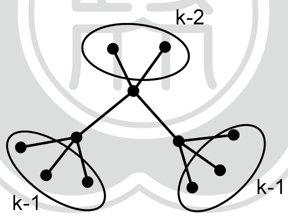

Figure 13: The graph of three vertices x, y, z of $H_{k,n}$ forms a path and there are additional $3k-4$ edges which are incident to x, y, z.

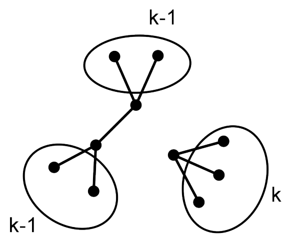

Figure 14: Two of three vertices x, y, z of $H_{k,n}$ are adjacent and there are additional $3k-2$ edges which are incident to x, y, z.

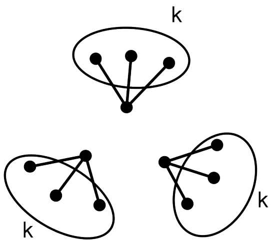

Figure 15: Three vertices x, y, z of $H_{k,n}$ form an independent set and there are additional $3k$ edges which are incident to x, y, z.

In Figure 12, we have

$$3k - 6 \leq 2k - 1$$
$$k \leq 5.$$

Since $n > k > \frac{n}{2} + 1$ and k or n is even, the remaining case is $k = 5$, $n = 6$ and $h = 2k - 1 = 9 = 3k - 6$. The graph $H_{5,6}$ is a complete graph and has $\binom{6}{3} = 20$ triangles. However, when we choose three vertices from $H_{5,6}$, the remaining three vertices are fixed. As a result, if G is $H_{5,6}$, then

$$C_h = 6\binom{10}{4} - 15 + 15\binom{6}{1} + \frac{20}{2}.$$

In Figure 13, we have

$$3k - 4 \leq 2k - 1$$
$$k \leq 3.$$

It is impossible because $n > k > \frac{n}{2} + 1$. In Figure 14, we have

$$3k - 2 \leq 2k - 1$$
$$k \leq 1.$$

It is impossible because $k > \frac{n}{2} + 1$. In Figure 15, we have

$$3k \leq 2k - 1$$
$$k \leq -1.$$

It is impossible. If $p > 3$, then we have

$$pk - 2\binom{p}{2} \le 2k - 1$$

$$pk - 2 \times \frac{p(p-1)}{2} \le 2k - 1$$
$$(p-2)k \le p(p-1) - 1$$
$$k \le p + 1 + \frac{1}{p-2}$$
$$\frac{n}{2} + 1 < k \le p + 1$$
$$n < 2p$$
$$n - p \le p - 1 \le (n-p) - 1,$$

which is a contradiction. Therefore, $G - S$ has a smaller component which is either a single vertex or an isolated edge when G is not $H_{5,6}$.

□

Theorem 3.8. *Let G be a Harary graph $H_{k,n}$ with $k > \frac{n}{2} + 1$ and $h = 2k$. If k or n is even, then the following statements are true.*

(i) *If G is not $H_{5,6}$, $H_{6,7}$, $H_{6,8}$, or $H_{6,9}$, then*

$$C_h = n\binom{\frac{nk}{2} - k}{k} - \frac{nk}{2}\binom{\frac{nk}{2} - 2k + 1}{1} - (\binom{n}{2} - \frac{nk}{2}) + \frac{nk}{2}\binom{\frac{nk}{2} - 2k + 1}{2}.$$

(ii) *If G is $H_{5,6}$, then*

$$C_h = 6\binom{10}{5} - 15\binom{6}{1} - (\binom{6}{2} - 15) + 15\binom{6}{2} + 60.$$

(iii) *If G is $H_{6,7}$, then*

$$C_h = 7\binom{15}{6} - 21\binom{10}{1} - (\binom{7}{2} - 21) + 21\binom{10}{2} + 35.$$

(iv) *If G is $H_{6,8}$, then*

$$C_h = 8\begin{pmatrix} 18 \\ 6 \end{pmatrix} - 24\begin{pmatrix} 13 \\ 1 \end{pmatrix} - (\begin{pmatrix} 8 \\ 2 \end{pmatrix} - 24) + 24\begin{pmatrix} 13 \\ 2 \end{pmatrix} + 32 + 4.$$

(v) *If G is $H_{6,9}$, then*

$$C_h = 9\begin{pmatrix} 21 \\ 6 \end{pmatrix} - 27\begin{pmatrix} 16 \\ 1 \end{pmatrix} - (\begin{pmatrix} 9 \\ 2 \end{pmatrix} - 27) + 27\begin{pmatrix} 16 \\ 2 \end{pmatrix} + 30 + 9.$$

Proof. Let S be an edge cut of $H_{k,n}$ and $G - S$ has a component with p vertices, where $p \le \lfloor \frac{n}{2} \rfloor$. Since k or n is even, $H_{k,n}$ is a k-regular graph and it implies that there are $\frac{nk}{2}$ edges.

If p is 1, then S is a disjoint union of two edge sets. One consists of k edges incident to a vertex chosen from $H_{k,n}$ and the other contains k edges chosen arbitrarily from the remaining $\frac{nk}{2} - k$ edges of $H_{k,n}$. However, there are two cases in which some triangles are counted two times. One is cutting all incident edges of two vertices which are adjacent to each other plus an extra edge, and the other is cutting all incident edges of two vertices which are not adjacent to each other. Hence, there are $n\begin{pmatrix} \frac{nk}{2} - k \\ k \end{pmatrix} - \frac{nk}{2}\begin{pmatrix} \frac{nk}{2} - 2k + 1 \\ 1 \end{pmatrix} - (\begin{pmatrix} n \\ 2 \end{pmatrix} - \frac{nk}{2})$ ways.

If p is 2, then S is composed of $2k - 2$ edges incident to a specific edge chosen from $\frac{nk}{2}$ edges in $H_{k,n}$ and two edges in the remaining edges. As a result, there are $\frac{nk}{2}\begin{pmatrix} \frac{nk}{2} - 2k + 1 \\ 2 \end{pmatrix}$ ways.

By an argument similar to the one of Theorem 3.7, if p is 3, then $H_{k,n}$ has at least six vertices and we only need to consider the case S contains $3k - 6$ edges incident to a triangle in $H_{k,n}$. It yields the following inequalities

$$3k - 6 \le 2k$$

$$k \le 6.$$

When $k = 6$, $h = 2k = 12 = 3k - 6$ and we have three cases $H_{6,7}$, $H_{6,8}$ and $H_{6,9}$ since $n > k > \frac{n}{2} + 1$. By Lemma 3.6, there are $T(6,7) = 35$, $T(6,8) = 32$, and

$T(6,9) = 30$ triangles, respectively. When $k = 5$, $h = 2k = 10 = (3k-6)+1$ and we get the case $H_{5,6}$ since $n > k > \frac{n}{2}+1$ and k or n is even. As we pick a triangle in $H_{5,6}$, the remaining three vertices are fixed. Hence, there are $\frac{T(5,6)}{2} \times \binom{6}{1} = 60$ ways.

If p is 4, then $H_{k,n}$ has at least eight vertices and we consider the case in Figure 16. We have

$$4k - 12 \leq 2k$$

$$k \leq 6.$$

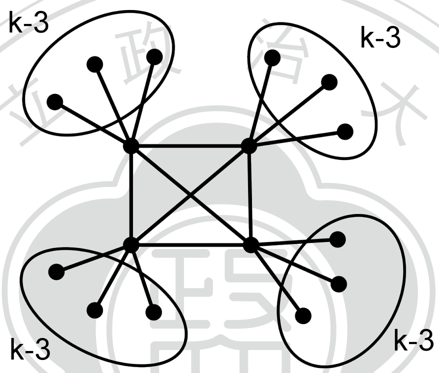

Figure 16: Four vertices x, y, z, w of $H_{k,n}$ are adjacent to each other and there are additional $4k-12$ edges which are incident to x, y, z, w.

In Figure 17, we have

$$4k - 10 \leq 2k$$

$$k \leq 5.$$

It is impossible because $n > k > \frac{n}{2} + 1$ and $n \geq 8$. The other cases are also impossible. It implies remaining cases are $H_{6,8}$ and $H_{6,9}$ and yields $h = 2k = 12 = 4k - 12$. If G is $H_{6,8}$, then the number of subgraphs isomorphic to K_4 is 8. When we choose four vertices from $H_{6,8}$, then remaining four vertices are fixed. Hence, there are $\frac{8}{2} = 4$ ways. If G is $H_{6,9}$, then the number of subgraphs isomorphic to K_4 is 9.

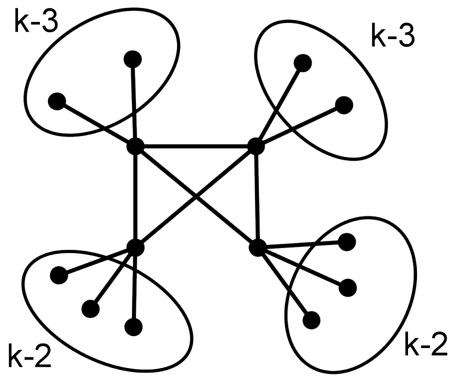

Figure 17: Four vertices x, y, z, w of $H_{k,n}$ are adjacent to each other except one pair vertices and there are additional $4k-10$ edges which are incident to x, y, z, w.

If $p > 4$, then we have

$$pk - 2\begin{pmatrix} p \\ 2 \end{pmatrix} \leq 2k$$

$$pk - 2 \times \frac{p(p-1)}{2} \leq 2k$$

$$(p-2)k \leq p(p-1)$$

$$k \leq p + 1 + \frac{2}{p-2}$$

$$\frac{n}{2} + 1 < k \leq p + 1$$

$$n < 2p$$

$$n - p \leq p - 1 \leq (n-p) - 1,$$

which is a contradiction. Therefore, $G - S$ has a smaller component which is either a single vertex or an isolated edge when G is not $H_{5,6}$, $H_{6,7}$, $H_{6,8}$, or $H_{6,9}$. If G is $H_{5,6}$, then

$$C_h = 6\begin{pmatrix} 10 \\ 5 \end{pmatrix} - 15\begin{pmatrix} 6 \\ 1 \end{pmatrix} - (\begin{pmatrix} 6 \\ 2 \end{pmatrix} - 15) + 15\begin{pmatrix} 6 \\ 2 \end{pmatrix} + 60.$$

If G is $H_{6,7}$, then

$$C_h = 7\begin{pmatrix} 15 \\ 6 \end{pmatrix} - 21\begin{pmatrix} 10 \\ 1 \end{pmatrix} - (\begin{pmatrix} 7 \\ 2 \end{pmatrix} - 21) + 21\begin{pmatrix} 10 \\ 2 \end{pmatrix} + 35.$$

If G is $H_{6,8}$, then

$$C_h = 8\binom{18}{6} - 24\binom{13}{1} - (\binom{8}{2} - 24) + 24\binom{13}{2} + 32 + 4.$$

If G is $H_{6,9}$, then

$$C_h = 9\binom{21}{6} - 27\binom{16}{1} - (\binom{9}{2} - 27) + 27\binom{16}{2} + 30 + 9.$$

□

After cutting a specific number of edge from a k-regular graph, we observe the remaining components.

Theorem 3.9. *Let G be a k-regular connected graph of order n. We obtain two components A and B by removing $\xi(G)$ edges from G and $1 < |A| \leq |B|$.*

(i) *If n is odd and $k = \frac{n+1}{2}$, then $|A| = 2$ or $|A| + 1 = |B| > 3$. In particular, A is a clique.*

(ii) *If n is even and $k = \frac{n}{2} + 1$, then $|A| = 2$ or $|A| = |B| > 2$. In particular, if $|A| > 2$,then A and B are cliques.*

Proof. **(i)** If $|A| = 2$, then A is K_2 and we are done. So, we consider the case $|A| > 2$. Since $k = \frac{n+1}{2} > \frac{n}{2}$, G is optimal by Lemma 3.4. That is $\lambda'(G) = \xi(G) = 2k - 2 = n - 1$. After we cut $n - 1$ edges in G, it forms two components A and B and $|A| \leq \frac{n-1}{2}$. In addition, G is a k-regular graph, so every vertex in A has at least $k - (|A| - 1) = \frac{n+1}{2} - |A| + 1$ neighbors which belong to B in G.

Assume $|A| < \frac{n-1}{2}$. Since $|A| > 2$, multiply both sides by $|A| - 2$ and we obtain

$$(|A| - 2)|A| < (|A| - 2)\frac{n-1}{2}$$

$$(|A| - 2)|A| < |A| \times \frac{n-1}{2} - (n - 1)$$

$$|A|(\frac{n-1}{2} - (|A| - 2)) > n - 1$$

$$|A|(\frac{n+1}{2} - |A| + 1) > n - 1.$$

It means that there are more than $n-1$ edges between A and B in G, which is a contradiction to the fact that A and B are disconnected after removing $n-1$ edges in G. So, we get $|A| = \frac{n-1}{2}$ and $|B| = |A| + 1$.

If A is not a clique, then every vertex in A has more than $\frac{n+1}{2} - |A| + 1$ neighbors in B. We obtain more than

$$|A|(\frac{n+1}{2} - |A| + 1) = \frac{n-1}{2}(\frac{n+1}{2} - \frac{n-1}{2} + 1) = n - 1$$

edges between A and B in G which contradicts to our assumption. Hence, A is a clique.

(ii) If $|A| = 2$, then we are done. So, we consider the case $|A| > 2$. Since $k = \frac{n}{2} + 1 > \frac{n}{2}$, G is optimal by Lemma 3.4. That is $\lambda'(G) = \xi(G) = 2k - 2 = n$. After we cut n edges in G, it forms two components A and B and $|A| \leq \frac{n}{2}$. In addition, G is a k-regular graph, so every vertex in A has at least $k - (|A| - 1) = (\frac{n}{2} + 1) - |A| + 1 = \frac{n}{2} - |A| + 2$ neighbors which belong to B in G.

Assume $|A| < \frac{n}{2}$. Since $|A| > 2$, multiply both sides by $|A| - 2$ and we obtain

$$(|A| - 2)|A| < (|A| - 2)\frac{n}{2}$$

$$(|A| - 2)|A| < |A| \times \frac{n}{2} - n$$

$$|A|(\frac{n}{2} - (|A| - 2)) > n$$

$$|A|(\frac{n}{2} - |A| + 2) > n.$$

It means that there are more than n edges between A and B in G, which is a contradiction to the fact that A and B are disconnected after removing n edges in G. So, we get $|A| = \frac{n}{2}$ and $|B| = |A|$.

If A is not a clique, then every vertex in A has more than $\frac{n}{2} - |A| + 2$ neighbors in B. We obtain more than

$$|A|(\frac{n}{2} - |A| + 2) = \frac{n}{2}(\frac{n}{2} - \frac{n}{2} + 2) = n$$

edges between A and B which contradicts to our assumption. Hence, A is a clique and there are $\frac{n^2}{8} - \frac{n}{4}$ edges in A. We need to n edges in order to separate A and B. We can compute the number of edges in B by

$$|E(G)| - |E(A)| - n = \frac{nk}{2} - (\frac{n^2}{8} - \frac{n}{4}) - n = \frac{n^2}{8} - \frac{n}{4}.$$

So, B is also a clique.

□

4 The Maximum Difference of the Minimum Edge Degree and the Minimum Vertex Degree of a Graph

In this chapter, we study the difference of the minimum edge degree $\xi(G)$ and the minimum vertex degree $\delta(G)$ of a graph G, say $m'' = \xi(G) - \delta(G)$. For example, every path of length at least 2 has $m'' = 1 - 1 = 0$ and the complete graph of order n , $n > 1$, has $m'' = 2(n-1) - 2 - (n-1) = n - 3$.

Theorem 4.1. *Let G be a simple graph without isolated vertices. Then the maximum of m'' is $n - 3$.*

Proof. Let uv be an edge of G with minimum edge degree $\xi(G) = d(u) + d(v) - 2$ and w be a vertex of G with minimum vertex degree $\delta(G) = d(w)$. Since G has no isolated vertices, we can find a vertex z in G adjacent to w. Then

$$m'' = (d(u)+d(v)-2)-\delta(G) \leq (d(w)+d(z)-2)-\delta(G) \leq \delta(G)+(n-1)-2-\delta(G) = n-3.$$

In addition, the complete graph of order n, $n > 1$, has $m'' = n-3$. So, the maximum of m'' is $n - 3$. □

Lemma 4.2. *Let G be a simple graph of order n without isolated vertices and has $m'' = n - 3$. If the vertex v in G has minimum vertex degree, then for any neighbor u of v has degree $n - 1$.*

Proof. Since G is a simple graph, $d(u) \leq n - 1$. The minimum edge degree $\xi(G)$ is less than or equal to $d(v) + d(u) - 2$ and $m'' = \xi(G) - \delta(G) = \xi(G) - d(v) = n - 3$. We have $d(u) \geq n - 1$. Hence, the degree of u is $n - 1$. □

In a simple graph of order n without isolated vertices and $m'' = n - 3$, the vertex with minimum degree is adjacent to the vertex with maximum degree and it is a connected graph.

Theorem 4.3. *Let G be a simple graph of order n without isolated vertices and has $m'' = n - 3$. If G is not a star, then $p \leq \lceil \frac{n-3}{2} \rceil$, where p is the number of vertices with degree 1.*

Proof. Let v be a vertex in G with minimum degree and u be a neighbor of v. By Lemma 4.2, the degree of u is $n-1$, that is, u is adjacent to every vertex in G except itself. Let

$$S = \{x \in V(G) | d(x) = 1\} \textit{ and } G' = G - S - \{u\}.$$

Because G is not a star, G' has at least an edge wz in Figure 18.

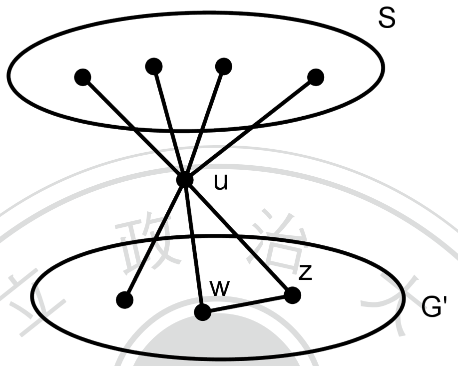

Figure 18: The graph shows the relationship between the maximum degree u and the subgraph G' in G.

Suppose $p = |S| > \lceil \frac{n-3}{2} \rceil$.

Case 1: If n is odd, then $p \geq \frac{n-3}{2} + 1 = \frac{n}{2} - \frac{1}{2}$ and $|V(G')| \leq n - (\frac{n}{2} - \frac{1}{2}) - 1 = \frac{n}{2} - \frac{1}{2}$. Then we have the inequality,

$$\xi(G) \leq d(w) + d(z) - 2 \leq (\frac{n}{2} - \frac{1}{2} - 1 + 1) + (\frac{n}{2} - \frac{1}{2} - 1 + 1) - 2 = n - 3.$$

It contradicts to $\xi(G) - \delta(G) = n - 3$.

Case 2: If n is even, then $p \geq \frac{n-3}{2} + \frac{3}{2} = \frac{n}{2}$ and $|V(G')| \leq n - \frac{n}{2} - 1 = \frac{n}{2} - 1$. Then we have the inequality,

$$\xi(G) \leq d(w) + d(z) - 2 \leq (\frac{n}{2} - 1 - 1 + 1) + (\frac{n}{2} - 1 - 1 + 1) - 2 = n - 4.$$

It contradicts to $\xi(G) - \delta(G) = n - 3$.

Hence, $p \leq \lceil \frac{n-3}{2} \rceil$. □

If G is a star with order n, then $m'' = (n-2)-1 = n-3$. However, the number of vertices with degree 1 is $n-1$ which is greater than $\lceil \frac{n-3}{2} \rceil$. In the following discussion, we use the same notation, $S = \{x \in V(G) | d(x) = 1\}$ and $G' = G - S - \{u\}$.

Theorem 4.4. *Let G be a simple graph of order n without isolated vertices and has $m'' = n-3$.*

(i) *If n is even and $p = |S| = \lceil \frac{n-3}{2} \rceil$, then G' is $K_{\frac{n}{2}}$.*

(ii) *If n is odd and $p = |S| = \lceil \frac{n-3}{2} \rceil$, then G' is $K_{\frac{n}{2}+\frac{1}{2}}$ or $K_{\frac{n}{2}+\frac{1}{2}} - e$.*

Proof. **(i)** Suppose n is even, $p = \frac{n}{2} - 1$ and $|V(G')| = n - (\frac{n}{2} - 1) - 1 = \frac{n}{2}$. If the order of G' is 2, then G' is K_2. Suppose G' has at least 3 vertices and there exist two vertices w, z in G' such that w is not adjacent to z, for any edge yz in G' , we have

$$\xi(G) \leq d(y) + d(z) - 2 \leq (\frac{n}{2} - 1 + 1) + (\frac{n}{2} - 2 + 1) - 2 = n - 3.$$

It contradicts to $\xi(G) - \delta(G) = n - 3$.

(ii) Suppose n is odd, $p = \frac{n}{2} - \frac{3}{2}$ and $|V(G')| = n - (\frac{n}{2} - \frac{3}{2}) - 1 = \frac{n}{2} + \frac{1}{2}$. If the order of G' is 2 or 3, then G' is K_2 , K_3 or $K_3 - e$, respectively. Suppose G' has at least 4 vertices and there are two edges e_1, e_2 in $\bar{G}'$, we have the following two cases.

Case 1: Assume e_1, e_2 have the same endpoint w in G'. Because $d(w) > 1$, there exists a vertex z which is adjacent to w in G as the following Figure 19. We have

$$\xi(G) \leq d(w) + d(z) - 2 \leq (\frac{n}{2} + \frac{1}{2} - 3 + 1) + (\frac{n}{2} + \frac{1}{2} - 1 + 1) - 2 = n - 3.$$

It contradicts to $\xi(G) - \delta(G) = n - 3$.

Case 2: Assume e_1, e_2 have different endpoints in G', say $e_1 = xy$ and $e_2 = wz$. If xz, xw, yz, or yw are edges in $\bar{G}'$, then it is back to Case 1. Hence, we consider x, y, z, w are pairwise adjacent to each other except x, y and w, z as in Figure 20, then

$$\xi(G) \le d(x) + d(z) - 2 \le (\frac{n}{2} + \frac{1}{2} - 2 + 1) + (\frac{n}{2} + \frac{1}{2} - 2 + 1) - 2 = n - 3.$$

It contradicts to $\xi(G) - \delta(G) = n - 3$.

Therefore, G' is $K_{\frac{n}{2}+\frac{1}{2}}$ or $K_{\frac{n}{2}+\frac{1}{2}} - e$.

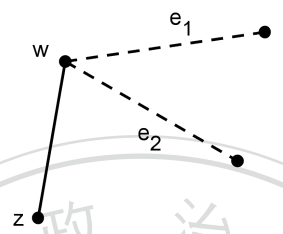

Figure 19: The edges e_1, e_2 in G' have the same endpoint.

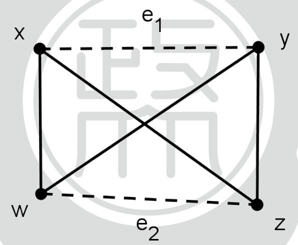

Figure 20: The edge e_1, e_2 in G' have different endpoints.

∎

A star with order n has $n - 1$ edges. Now, we discuss the lower bound of number of edges when the graph of order n has $m'' = n - 3$.

Theorem 4.5. *Let G be a simple graph of order n without isolated vertices and has $m'' = n - 3$. If G is not a star, then G has at least $2n - 4$ edges.*

Proof. **Case 1:** If S is not an empty set, then the minimum vertex degree $\delta(G)$ is 1 and $\xi(G) = n - 2$. Now, we compute the number of edges in G'. Since G is not a star, there is a vertex x in G'. Without loss of generality, let x be the vertex with maximum vertex degree $d_{G'}(x) = d$ in G', $1 \le d \le n-3$. If d is 1, then $\xi(G)$ is 2. Because $\xi(G) - \delta(G)$ is $n-3$, it implies that n is 4 and G is a paw. A paw has 4 edges and we are done. It remains to prove the case $d \ge 2$. For a neighbor y of x in G', we have

$$n - 2 = \xi(G) \le d(x) + d(y) - 2$$

$$d(x) + d(y) \ge n.$$

It implies $d_{G'}(x) + d_{G'}(y) \ge n-2$ and $d_{G'}(y) \ge n-d-2$. Thus, the number of edges in G' is at least

$$\frac{d_{G'}(x) + \sum_{y \in N_{G'}(x)} d_{G'}(y)}{2} = \frac{d + d(n-d-2)}{2}.$$

It implies that there exist at least $n-3$ edges in G'. Therefore, we get $|E(G)| = |E(G')| + d(u) \ge (n-3) + (n-1) = 2n-4$. The bound is best possible since the equality holds for the graph G in which $|S| = 1$ and maximum vertex degree in G' is $n-3$ as in the Figure 21.

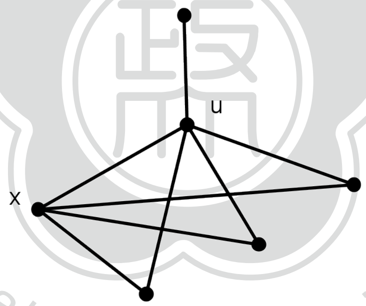

Figure 21: The graph shows that there are $n-3$ edges in G'.

Case 2: If S is an empty set, then the degree of every vertex in G is greater than 1. Suppose the vertex x in G has minimum vertex degree d, $2 \le d \le n-1$. For a neighbor y of x in G, the degree of y is $n-1$ by Lemma 4.2. Thus, the number of edges in G is at least

$$\frac{d(x) + \sum_{y \in N(x)} d(y) + \sum_{z \in G - N(x) - \{x\}} d(z)}{2} = \frac{d + d(n-1) + (n-d-1)d}{2}.$$

Hence, we get $|E(G)| \ge 2n-4$.

□

5 Open Problems and Further Directions of Studies

In this article, we prove some graphs are locally connected or locally coconnected, compute the number of edge cuts with size $2k-1$ and $2k$ in a Harary graph $H_{k,n}$, and show that the upper bound of the number of vertices with degree 1 and the lower bound of the number of edges in the graph with maximum difference of minimum edge degree and minimum vertex degree. There are still some open problems for future studies.

1. In Chapter 2, we have known some classes of locally connected graphs or locally coconnected graphs. Furthermore,

   a. We would like to characterize more graphs with these properties.

   b. We would like to study the relationship of a locally connected graph and a connected graph.

2. When we enlarge the size of an edge cut, it is more difficult to predict the remaining graph.

   a. We would like to find out the method of counting the number of K_s's with $s \geq 4$ in a Harary graph and count the number of edge cuts with larger size.

   b. We would like to count the number of edge cuts with a larger size in other k-regular graphs.

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