國立政治大學統計學系

碩士學位論文

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二元損失管制圖之設計

Design of the Bivariate Loss Control Chart

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中華民國一百年六月

Acknowledgment

回想兩年來的日子，歷經前所未有的充實生活，無論是在知識、生活上皆有所成長。今天能夠與大家分享這兩年的成果與喜悅，一切都要感謝幫助、鼓勵及支持我的人。

論文能如期順利的完成，首先要感謝的是我的指導老師　楊素芬教授。兩年來，老師辛苦的指導與訓練，不僅僅讓我精進統計品管的知識，同時，老師也鼓勵我們要學以致用，將品管方法落實於生活。這段學習過程，我更懂得對於任何事都該負責任且保有積極的態度去面對。

感謝口試委員鄭惟孝、曾勝滄及黃榮臣教授，對於本論文仔細的審查檢閱並給予精闢的見解，使得本論文能盡善盡美。

感謝陪伴我的同學及朋友，特別是鄭鈞遠、歐家玲和謝至芬，這段日子我們一起分享快樂與痛苦，感謝你們豐富我的研究所生活。

最後，我要感謝我最親愛的家人，溫暖的家始終是我的避風港。有你們的支持，我的人生才是完整的。謝謝你們。

本研究承蒙行政院國家科學委員會，計畫 NSC98-2118-M-004-005-MY2 補助及國立政治大學商學院研究團對補助，謹此致謝。

呂雨築　　謹致

中華民國一百年六月

ABSTRACT

A single Bivariate loss chart to monitor both the mean vector and the covariance matrix of a process with two correlated quality characteristics is proposed. Unlike existing multivariate charts, our proposed control chart is based on bivariate average loss function. With this feature, we could monitor the average loss of the product. It was shown that the proposed loss chart could detect small changes in process parameters quickly. We compared the performance of the proposed chart with some multivariate charts, like Max Bivariate chart, Max CUSUM chart, MEWMA chart, EWMA M-chart, $|S|$ chart and EWMA V-chart. Our proposed chart performs rather well when monitoring both the mean vector and the covariance matrix simultaneously. An example is given to illustrate the proposed chart.

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Chapter 1. Introduction

1.1 The Importance of the Process Control

Dr. Walter Shewhart first introduced the control charts in early 1920, and people started to use these charts to monitor the process. The X-bar chart and R (or S) chart were widely used to control process mean and variability for variables data. Because Shewhart charts are difficult to detect small shifts, Page developed the cumulative sum (CUSUM) chart in 1954 and Robert brought us the exponentially weighted moving average (EWMA) chart in 1959. These are mainly used for variables data. Later on some statisticians proposed the adaptive charts to improve the performance of the traditional Shewhart charts. Normally samples taken at fixed sampling interval, some studies had shown that control charts with variable sample sizes (VSS), and/or variable sampling intervals (VSI) perform better than the traditional charts. Reynold et al. (1988) and Chengalur et al. (1989) proposed VSI $\overline{X}$ chart to monitor process mean. Costa (1999) introduced the VSI $\overline{X}\,\&\,R$ charts to monitor both the process mean and the variability. Prabhu et al. (1994) proposed VSSI $\overline{X}$ chart. More and more newly proposed charts dealing with more complex situations, such as, to monitor both the process mean and variability simultaneously with several correlated quality characteristics.

1.2 Research Problem

Two charts are alwaus used to monitor the mean vector and the covariance matrix simultaneously. What we would like to do is to propose a single chart to achieve the same goal. Only a few single multivariate control charts existed to monitor the process mean vector and covariance matrix at the same time. Some showed that they are insensitive to detect small shifts. We take this into consideration when design the new chart.

1.3 Research Purpose

In this project, we would propose a Bivariate Loss (BL) control chart. The BL chart is based on the average loss of two quality characteristics. With bivariate loss function, we can combine the mean vector and the covariance matrix into a single statistic, which could be used to monitor the process mean vector and covariance matrix simultaneously. Many multivariate schemes that involved the target vector assumed it equals the process mean. However, our proposed chart would allow the mean vector to shift from the target vector when the process is in-control.

1.4 Literature Review

It is common to see that quality characteristics are correlated in some products/processes. Woodall and Montgomery (1999) and Stoumbos *et al*. (2000) pointed out the importance of the research of the multivariate control charts. To monitor such quality characteristics simultaneously, there existed a few control charts. Hotelling (1947) proposed the Hotelling-T^2 chart to monitor the process mean vector, but it is insensitive to small and moderate shifts. There were other issues in monitoring process mean vector. Jackson (1959) transformed the original correlated variables into principal components and used these new orthogonal variables to construct control charts. However, it's hard to get interpretation of out-of-control signals unless the principal components have meaning.

To improve the power of small shifts, there existed several multivariate CUSUM and multivariate EWMA charts. Woodall and Ncube (1985) suggested using p univariate CUSUM charts to monitor process mean for each of the p quality characteristics, and studying the ARL performance of the minimum of the p univaraite CUSUM run lengths using covariance matrix $\sum$. But, in practice, usually the results of the ARL performance were not discussed.

Healy (1987) showed that a multivariate CUSUM chart is more effective in detecting a shift in the mean vector in one specified direction. However, this procedure may not be effective in detecting the process mean vector shift in an unanticipated direction. Hawkins (1991) proposed an extension to Healy's results. It considered several specified directions of interest and showed that this improved procedure could be more effective than that of Woodall and Ncube (1985).

Crosier (1988) proposed two multivariate CUSUM charts that performs better. The statistic

$$C_i = \{(S_{i-1} + X_i)'\Sigma^{-1}(S_{i-1} + X_i)\}^{1/2},$$

and

$$S_i = 0, \quad if\ C_i \le k_1,$$

$$= (S_{i-1} + X_i)(1 - k_1 / C_i), \quad if\ C_i > k_1$$

$i = 1,2,\cdots$, where $S_0 = 0$ and $k_1 > 0$. This MCUSUM chart signals when

$$Y_i = \{S_i' \Sigma^{-1} S_i\}^{1/2} > h_2,$$

where $h_2 > 0$. The other chart performed not as well, because the directions between observations and the target vector are different, this increased the value of the statistics that gave false signals when the process was in-control. Pignatiello and Runger (1990) also proposed two multivariate CUSUM charts. In Crosier (1998) and Pignatiello and Runger (1990), they found the unnecessarily frequent out-of-control signals increased because of the observations in varying directions from target vector.

Mohebbi and Hayre (1989) proposed a MCUSUM chart based on loss function to detect shifts from the target value. The statistics for $n \geq 1$

$$T_n = \max(0, T_{n-1} + U_n - k), \text{ k} > 0$$

and

$$U = c'X + X'AX - trace(AV)$$

Let $E(U) = L(\mu)$.

$$L(\mu) = c'\mu + \mu' A\mu,$$

where $c' = (c_1, c_2, \cdots c_p)$ and A is a positive matrix of p dimensions. The chart signals when $T_n \geq h\tau_0$, where h > 0 and $\tau_0^2 = Var(U|\mu = 0) = c'Vc + 2trace[(AV)^2]$. The advantage of this chart is that through using $L(\mu)$ one could detect one-sided or two-sided shifts in the process mean. On the other hand, when the process mean had shifted, τ_0^2 will be increased and ARL reduced. Through τ_0^2, we may get some information. The weakness for this chart is that sometimes it could not specifically identify which variable caused alarm.

Qiu and Hawkins (2001) provided the rank-based multivariate CUSUM chart. The CUSUM chart is distribution free. It could detect shifts in all directions but not for the components of the shift in the mean vector are all the same. They also suggested that the shift with equal components can be detected by another univariate CUSUM chart.

Lowry *et al*. (1992) proposed the Multivariate EWMA (MEWMA) chart. It gave guidelines for designing an easily implemented multivariate procedure. The performance is better than Crosier's (1988) MCUSUM chart.

Tsui and Woodall (1993) suggested a MLEWMA chart based on a loss function, which is similar to Lowry's MEWMA procedure. The statistics are

$$Y_i = (\frac{2-r}{r}) Z_i' A Z_i \, , i = 1, 2, \cdots,$$

and

$$Z_i = rX_i + (1-r) Z_{i-1} \, , i = 1, 2, \cdots,$$

where $Z_0 = 0$ and $0 < r \le 1$. An out-of-control signal occurs when $Y_i > h_1$ and $h_1 > 0$. The chart performs better than Mohebbi and Hayre's (1989) MCUSUM chart. But, in some cases, MLEWMA chart isn't as good as Lowry's MEWMA procedure. Recently, Aparisi and Haro (2001) proposed the Hotelling's T^2 chart with variable sampling intervals (VSI T^2 chart) and Chen and Hsieh (2007) suggested Hotelling's T^2 chart with variable sample size and control limit (VSSC T^2 chart) to increase the efficiency in detecting the process changes. Chen and Hsieh pointed out that it's more convenient for administrating VSSC T^2 chart than VSI T^2 chart, since adaptive changes in sampling intervals increases the complexity in using VSI T^2 chart. The VSSC T^2 chart may provide a good option for quick response to small shifts in a multivariate process.

In 2010, Mahmoud and Zahran proposed a multivariate extension of the adaptive exponentially weighted moving average (AEWMA) control chart, which could be viewed as a smooth combination of a MEWMA chart proposed by Lowry *et al.* (1992) and a χ^2-chart. This procedure could detect both small and large shifts in the process mean vector effectively. It showed that its advantage over the MEWMA chart is that it could detect optimally both small and large shift. The MEWMA chart is designed to detect either a small or a large shift, but not both.

Above methods are mainly proposed to detect the process mean vector. However, monitoring process covariance matrix is also important. Alt (1985) used $|S|$ chart to monitor the process variability, but it is insensitive to small shifts. Alt and Bedewi

(1986) proposed two charts to detect the covariance matrix. One is based on the likelihood ratio principle and another uses the sample generalized variance which is sometimes taken as a measure of dispersion of multivariate processes. Tang and Barnett (1996a, b) proposed a chart based on independent statistics resulting from the decomposition of the covariance matrix. They also indicated that since the procedures do not depend on prior estimated of the process covariance matrix, it is suitable for short-run manufacturing environments. Chan and Zhang (2001) proposed two CUSUM charts. One is via the projection pursuit method and another chart is based on likelihood ratio. The former chart can be used in short-run environment. Yeh *et al*. (2003) proposed an EWMA V-chart to monitor small shifts in the process variability. Also, there are some related issues of multivariate processes presented in Yeh *et al*. (2004, 2005). Hawkins and Maboudou-Tchao (2008) provided multivariate exponentially weighted moving covariance matrix chart (MEC chart) to monitor the stability of the covariance matrix of the process. Costa and Machado (2008) provided a control chart (VMAX chart) based on VMAX statistic to control the covariance matrix of multivariate processes. The points plotted on the chart correspond to the maximum of the sample variances of the p quality characteristics. They also pointed that the VMAX chart is faster detection of the process changes than $|S|$ chart and is better at identifying the out-of-control variables.

Generally, the process mean vector and covariance matrix may change simultaneously during the monitoring period. Few methods are proposed to use two charts to monitor the process mean vector and variability. The traditional combination is the χ^2 chart and $|S|$ chart. Yeh *et al*. (2003) pointed out that the combined MEWMA and EWMA V-charts could detect shifts in both the process mean and the process covariance matrix better than EWMA M-chart (Yeh *et al*. (2003)) and EWMA

V-chart. Reynolds and Cho (2006) proposed a combination of MEWMA charts based on sample means and on the sum of the squared deviation from target. Hawkins and Maboudou-Tchao (2008) combined MEWMA chart and MEC chart (MAC chart).

As of a single chart to monitor shifts in both the process mean vector and the covariance matrix for multivariate quality characteristics. Liu (1995) proposed several charts (r, Q, S charts.) based on the concept of data depth. They are constructed by nonparametric method, thus, they are distributed free. Also, Liu pointed out that these charts could be visualized and interpreted easily as the univariate X, $\overline{X}$, and CUSUM charts. Spiring and Chen (1998) proposed the univariate and multivariate MSE chart. Yeh and Lin (2002) proposed box chart. The box-chart uses the probability integral transformation to get two independently and identically distributed uniform distributions. Therefore, when the process is out-of-control, the corresponding shifts in the mean vector and/or the covariance matrix could be easily investigate. Cheng and Thaga (2005) proposed a multivariate Max-CUSUM chart, Cheng and Xie (2005) suggested a multivariate Max-MEWMA chart. Also, Cheng and Thaga (2005) pointed out that the Max-CUSUM chart performs better than the Max-MEWMA chart in detecting small shifts in the process mean and the covariance matrix. For bivariate case, Khoo (2005) proposed a Max Bivariate chart by combining T^2 chart and $|S|$ chart, but it's slow to react to the small process shifts. For above methods, the quality characteristics followed a bivariate normal distribution and the mean vector is equal to the target vector when process is in-control. Also, they only allow the mean vector or covariance matrix shift the same scales. Our proposed control chart allows more flexibility.

In 2010, Zhang, Li, and Wang proposed a new single chart which integrates the exponentially weighted moving average (EWMA) procedure with the generalized likelihood ratio (GLR) test for jointly monitoring both the process mean vector and the covariance matrix. But, they pointed out that it might not be well handled in practice.

1.5 Proposed Method and Structure

To monitor the process mean and the covariance matrix of the process with two correlated quality characteristics, we propose a few new charts: a Bivariate Loss control chart (BL chart), an optimal BL chart, a variable sampline sizes and sampling intervals (VSSI) Bivariate Loss chart (VSSI BL chart), and an optimal VSSI BL chart. For these charts, we use the statistic: bivariate average loss which is based on James and Stein (1961) bivariate loss function. We will discuss how to construct these charts and show their numeric studies in the next chapter. Later, we compare BL chart with Max Bivariate chart (Khoo (2005)), MSE chart (Spiring and Cheng(1998)), Max-CUSUM chart(Cheng and Thaga (2005)), Max-MEWMA chart (Cheng and Xie (2005)) and EWMA V-chart (Yeh et al. (2003)) using 'average run length' (ARL), 'average time to signal' (ATS) as performance metrics. We will use an example to illustrate how to use BL chart and optimal VSSI BL chart.

Chapter 2. The Bivariate Loss Control Chart

2.1 Design of the Bivariate Loss Chart

Let (Y_1, Y_2) be the measurements of two quality characteristics, and assume that

(1) when process is in-control,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim BN(\mu_0 = \begin{pmatrix} \mu_{y_1} \\ \mu_{y_2} \end{pmatrix}, \Sigma_0 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}), \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \mu_{y_1} - \delta_5 \sigma_1 \\ \mu_{y_2} - \delta_6 \sigma_2 \end{pmatrix},$$

where $\sigma_{12} = \rho_0 \sigma_1 \sigma_2$ is the covariance of (Y_1, Y_2) and ρ_0 is the coefficient of correlation and $-1 \le \rho_0 \le 1$. δ_5 and δ_6 are target shifts, in order to simplify the processes for the following study we set $\delta_5, \delta_6 \ge 0$. Without loss of generality $\delta_5, \delta_6 < 0$ is acceptable.

(2) when the process is out-of-control,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim BN(\mu_1 = \begin{pmatrix} \mu_{y_1} + \delta_1 \\ \mu_{y_2} + \delta_2 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} \delta_3^2 \sigma_1^2 & \rho_1 \delta_3 \delta_4 \sigma_1 \sigma_2 \\ \rho_1 \delta_3 \delta_4 \sigma_1 \sigma_2 & \delta_4^2 \sigma_2^2 \end{pmatrix}), \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \mu_{y_1} - \delta_5 \sigma_1 \\ \mu_{y_2} - \delta_6 \sigma_2 \end{pmatrix},$$

where both δ_1 and δ_2 are the mean shift and $\delta_1, \delta_2 \ne 0$; δ_3^2, δ_4^2, are shift of the variance and $\delta_3, \delta_4 \ne 1$; and ρ_1 is the coefficient correlation, $-1 \le \rho_1 \le 1$.

2.2 Average Bivariate Loss and its Distribution

Use the following bivariate loss function to measure the loss per unit product:

$$L(Y_1, Y_2) = K_{11}(Y_1 - T_1)^2 + K_{12}(Y_1 - T_1)(Y_2 - T_2) + K_{22}(Y_2 - T_2)^2 \quad (1)$$

where Y_1, Y_2 are quality characteristics, T_1 and T_2 are target values and K_{11}, K_{12} and K_{22} are constants.

$$E[L(Y_1,Y_2)] = K_{11}E[(Y_1-T_1)^2] + K_{12}E[(Y_1-T_1)(Y_2-T_2)] + K_{22}E[(Y_2-T_2)^2]$$

Taking m samples of size n each in a fixed interval. When $E(L(Y_1,Y_2))$ is unknown, we use the sampled average loss to estimate it:

$$\begin{aligned}
BL &= \frac{1}{n}\sum_{j=1}^{n}[K_{11}(Y_{1j}-T_1)^2 + K_{12}(Y_{1j}-T_1)(Y_{2j}-T_2) + K_{22}(Y_{2j}-T_2)^2] \qquad (2)\\
&= \frac{1}{n}K_{11}\sum_{j=1}^{n}(Y_{1j}-T_1)^2 + \frac{1}{n}K_{12}\sum_{j=1}^{n}(Y_{1j}-T_1)(Y_{2j}-T_2) + \frac{1}{n}K_{22}\sum_{j=1}^{n}(Y_{2j}-T_2)^2\\
&= \frac{1}{n}K_{11}\sum_{j=1}^{n}[(Y_{1j}-T_1)^2 + \frac{K_{12}}{K_{11}}(Y_{1j}-T_1)(Y_{2j}-T_2) + (\frac{K_{12}}{2K_{11}})^2(Y_{2j}-T_2)^2]\\
&\quad -\frac{1}{n}\frac{K_{12}^2}{4K_{11}}\sum_{j=1}^{n}(Y_{2j}-T_2)^2 + \frac{1}{n}K_{22}\sum_{j=1}^{n}(Y_{2j}-T_2)^2\\
&= \frac{1}{n}K_{11}\sum_{j=1}^{n}(Y_{1j}-T_1+\frac{K_{12}}{2K_{11}}(Y_{2j}-T_2))^2 + \frac{1}{n}(K_{22}-\frac{K_{12}^2}{4K_{11}})\sum_{j=1}^{n}(Y_{2j}-T_2)^2\\
&= R_1 + R_2
\end{aligned}$$

where

$$R_1 = \frac{1}{n}K_{11}\sum_{j=1}^{n}[Y_{1j}-T_1+\frac{K_{12}}{2K_{11}}(Y_{2j}-T_2)^2],$$

$$R_2 = \frac{1}{n}(K_{22}-\frac{K_{12}^2}{4K_{11}})\sum_{j=1}^{n}(Y_{2j}-T_2)^2.$$

Define BL statistics as

$$BL = R_1 + R_2. \qquad (3)$$

It may be proved that the BL statistic is an unbiased estimator of $E(L(Y_1,Y_2))$.

When the process is in-control

$$BL \sim \frac{1}{n}K_{11}(\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2\sigma_2^2 + \frac{K_{12}}{K_{11}}\sigma_{12})\chi^2_{n,\tau_{01}} + \frac{1}{n}(K_{22}-\frac{K_{12}^2}{4K_{11}})\sigma_2^2\chi^2_{n,\tau_{02}}, \qquad (4)$$

where $\chi^2_{n,\tau_{01}}$ is a non-central random variable with n degrees of freedom and a non-centrality parameter τ_{01}, here $\tau_{01} = \dfrac{n(\delta_5\sigma_1 + \frac{K_{12}}{2K_{11}}\delta_6\sigma_2)^2}{(\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2\sigma_2^2 + \frac{K_{12}}{K_{11}}\sigma_{12})}$; and $\chi^2_{n,\tau_{02}}$ is a non-central random variable with n degree of freedom and non-centrality parameter τ_{02}, here $\tau_{02} = \dfrac{n\delta_6^2\sigma_2^2}{\sigma_2^2} = n\delta_6^2$. See detail in Appendix A.

Hence

$$BL \sim A\chi^2_{n,\tau_{01}} + B\chi^2_{n,\tau_{02}}, \tag{5}$$

where $A = \dfrac{1}{n}K_{11}(\sigma_1^2 + (\dfrac{K_{12}}{2K_{11}})^2\sigma_2^2 + \dfrac{K_{12}}{K_{11}}\sigma_{12})$ and $B = \dfrac{1}{n}(K_{22} - \dfrac{K_{12}^2}{4K_{11}})\sigma_2^2$.

2.3 The Approximated Distribution of BL

The exact distribution of BL is not available. We would approximate the distribution of BL by the following steps:

Step 1. Given $BL \sim A\chi^2_{n,\tau_{01}} + B\chi^2_{n,\tau_{02}}$.

Step 2. Approximate the linear combination of two central χ^2 distributions by Patnaik's method (1949).

$$\chi^2_{n,\tau_{01}} / r_1 \overset{\bullet}{\sim} \chi^2_{v_1}$$

$$\chi^2_{n,\tau_{02}} / r_2 \overset{\bullet}{\sim} \chi^2_{v_2}$$

where $r_1 = 1 + \dfrac{\tau_{01}}{n+\tau_{01}}, v_1 = n + \dfrac{\tau_{01}^2}{n+2\tau_{01}}$ and

$$r_2 = 1 + \frac{\tau_{02}}{n+\tau_{02}}, v_2 = n + \frac{\tau_{02}^2}{n+2\tau_{02}}.$$

Thus, $A\chi^2_{n,\tau_{01}} \overset{\bullet}{\sim} Ar_1\chi^2_{v_1}$ and $B\chi^2_{n,\tau_{02}} \overset{\bullet}{\sim} Br_2\chi^2_{v_2}$.

Step 3. From Steps 1 and 2, we know that $r_1R_1 \overset{\bullet}{\sim} Ar_1\chi^2_{v_1}$ and $r_2R_2 \overset{\bullet}{\sim} Br_2\chi^2_{v_2}$.

Thus,

$$BL \overset{\bullet}{\sim} Ar_1\chi^2_{v_1} + Br_2\chi^2_{v_2} \tag{6}$$

Step 4. Use Moschopoulos and Canada's (1984) method to get approximated distribution of BL.

Let $Q = Ar_1\chi^2_{v_1} + Br_2\chi^2_{v_2}$.

Thus, $BL \overset{\bullet}{\sim} Q$. (7)

Step 5. From Moschopoulos and Canada (1984), the cumulative distribution function (c.d.f.) of Q is

$$F_Q(q) = P(Q \le q) = b_2 \sum_{j=0}^{\infty} a_j \int_0^q g_j(y)dy$$

where $b_2 = (c_1/c_2)^{m_2}$, $c_1 = Ar_1$, $c_2 = Br_2$ and $m_i = v_i/2, i = 1,2$.

Also, $a_j = A_j^{(2)} = A(c_2, j)$ with $A(c_i, j) = \dfrac{(m_i)_j(1-c_1/c_i)^j}{j!}$ and

g(y) is the pdf of a Gamma variable with shape parameter ($m_1 + m_2 + j$) and scale parameter $2c_1$. That is, $y \sim \Gamma(m_1 + m_2 + j, 2c_1)$.

The approximated c.d.f. of Q is compared with Imhof (1961), Farebrother (1984), and Liu (2009) (See Table 1.). Imhof's (1961) method shows the exact tail probability for a linear combination of chi-squares. From Table 1, Farebrother's (1984) method is more precise, but it's hard to calculate. Compare Liu's (2009) method and our method, we found that our method is close to the exact value when any one of the two weights is greater than 0.2.

Table 1. Compare Error% of Tail Probability P (W > w) with Imhof's Method

$c_1(n_1, \tau_1), c_2(n_2, \tau_2)$	w	Imhof	Fare.	Liu.	Error% (Liu)	Q	Error% (Q)
0.7(1,6),3(1,0.2)	**6**	0.591269	0.591269	0.567547	4.01%	0.583908	1.24%
	15	0.127068	0.127068	0.132639	4.38%	0.127036	0.03%
	20	0.052153	0.052153	0.056008	7.39%	0.053024	1.67%
	25	0.022100	0.022099	0.023224	5.09%	0.022698	2.71%
10(1,0.1),1(1,10)	**40**	0.114930	0.114930	0.122072	6.21%	0.115342	0.36%
	50	0.064546	0.064546	0.069020	6.93%	0.064661	0.18%
	60	0.037203	0.037203	0.039296	5.63%	0.03724	0.10%
	70	0.021772	0.021772	0.022482	3.26%	0.02181	0.17%
$1(1,1),0.6^4(1,7)$	**4**	0.249843	0.249843	0.254194	1.74%	0.241465	3.35%
	5	0.169313	0.169313	0.171716	1.42%	0.161445	4.65%
	6	0.115043	0.115043	0.115845	0.70%	0.10937	4.93%
	9	0.035785	0.035784	0.035350	1.22%	0.035563	0.62%
$1(2,0),0.8^6(1,8)$	**6**	0.215605	0.215605	0.217333	0.80%	0.427449	0.77%
	8	0.085359	0.085359	0.088209	3.34%	0.264386	1.21%
	10	0.032178	0.032178	0.033241	3.30%	0.16129	3.10%
	14	0.004395	0.004394	0.004111	6.46%	0.059459	5.05%

Note: Error% = (Exact tail probability – Approximated tail probability)/(Exact tail probability)•100%.

2.4 The Control Limits of the BL Chart

Construct a bivariate loss control chart as below:

Step 1. Give a fixed false alarm rate α .

Step 2. Let $b_2\sum_{j=0}^{\infty}a_j\int_0^{UCL}g_j(y)dy = 1-\alpha/2$ and $b_2\sum_{j=0}^{\infty}a_j\int_0^{LCL}g_j(y)dy = \alpha/2$ to get control limits.

Step 3. Since $F_Q(q)$ is the sum of an infinite sequence, we compute control limits by taking the first N terms. Thus, our control limits are approximate values.

$$b_2\sum_{j=0}^{N}a_j\int_0^{UCL}g_j(y)dy \doteq 1-\alpha/2, \tag{8}$$

$$b_2\sum_{j=0}^{N}a_j\int_0^{LCL}g_j(y)dy \doteq \alpha/2 \tag{9}$$

Hence the limits of the proposed Bivariate Loss Control Chart are

$$UCL \doteq F_Q^{-1}(1-\alpha/2)$$

$$CL \doteq F_Q^{-1}(p^*)$$

$$LCL \doteq F_Q^{-1}(\alpha/2)$$

Note that $p^* = b_2 \sum_{j=0}^{N} a_j \int_0^{CL} g_j(y)dy$.

Charting procedure:

Step 1. Specify target vector (T_1, T_2), mean vector μ_0, and covariance matrix Σ_0 for in-control process. Also, set K_{11}, K_{22} and K_{12}.

Step 2. If μ_0 is unknown, use the sample mean vector $\overline{\overline{Y}} = \begin{pmatrix} \overline{\overline{Y_1}} \\ \overline{\overline{Y_2}} \end{pmatrix}$ to estimate, where $\overline{\overline{Y_i}} = \sum_{k=1}^{m}\sum_{j=1}^{n} y_{ijk} / mn, i = 1,2$ and m is the number of subgroups and n is the sample size. If Σ_0 is unknown, use sample covariance matrix $S = \begin{pmatrix} \overline{s_1^2} & s_{12} \\ s_{12} & \overline{s_2^2} \end{pmatrix}$ to estimate it, where we use $\overline{s_1} = (\sum_{k=1}^{m} s_{1k} / c_4) / m$ to estimate σ_1, $\overline{s_2} = (\sum_{k=1}^{m} s_{2k} / c_4) / m$ to estimate σ_2 and use sample correlation to estimate ρ_0.

Step 3. For each subgroup calculate BL_i.

Step 4. Specify our ARL_0, the in-control ARL, and calculate control limits.

Step 5. Plot the sample points and control limits on the chart.

Step 6. If any points fall outside the control limits, i.e. $BL_i > UCL, BL_i < LCL$, then we have to stop the process and investigate the unusual causes.

2.5 The Out-of-control Approximate Distribution of BL

When process is out-of-control,

$$BL \sim \frac{1}{n} K_{11}(\delta_3^2\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2 \delta_4^2\sigma_2^2 + \frac{K_{12}}{K_{11}}\frac{\rho_1}{\rho_0}\delta_3\delta_4\sigma_{12})\chi^2_{n,\tau_{11}} + \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}})\delta_4^2\sigma_2^2\chi^2_{n,\tau_{12}}$$

with $\chi^2_{n,\tau_{11}}$ has a non-central chi-square distribution with n degrees of freedom, and non-centrality parameter $\tau_{11} = \dfrac{n((\delta_1+\delta_6)\sigma_1 + \frac{K_{12}}{2K_{11}}(\delta_2+\delta_7)\sigma_2)^2}{(\delta_3^2\sigma_1^2 + (\frac{K_{12}}{2K_{11}})\delta_4^2\sigma_2^2 + \frac{K_{12}}{K_{11}}\frac{\rho_1}{\rho_0}\delta_3\delta_4\sigma_{12})}$ and $\chi^2_{1,\tau_{12}}$ has a non-central chi-square distribution with n degrees of freedom, and non-centrality parameter $\tau_{12} = \dfrac{n((\delta_2+\delta_7)\sigma_2)^2}{\delta_4^2\sigma_2^2} = \dfrac{n((\delta_2+\delta_7))^2}{\delta_4^2}$.

Again, we could use the same procedure as in section 1.3 to find the out-of-control approximate distribution (Q^*) of BL (see 1.2). Thus, we can get that

$$BL \overset{\bullet}{\sim} Q^* = A' r_1' \chi^2_{v_1'} + B' r_2' \chi^2_{v_2'} \tag{10}$$

where $A' = \frac{1}{n} K_{11}(\delta_3^2\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2 \delta_4^2\sigma_2^2 + \frac{K_{12}}{K_{11}}\frac{\rho_1}{\rho_0}\delta_3\delta_4\sigma_{12})$, $B' = \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}})\delta_4^2\sigma_2^2$,

$r_1' = 1 + \frac{\tau_{11}}{n+\tau_{11}}, v_1' = n + \frac{\tau_{11}^2}{n+2\tau_{11}}$ and $r_2' = 1 + \frac{\tau_{12}}{n+\tau_{12}}, v_2' = n + \frac{\tau_{12}^2}{n+2\tau_{12}}$.

Also, by Moschopoulos and Canada (1984), we could calculate the c.d.f. of Q^*.

2.6 Performance Measurement of the BL Chart

We would use average run length (ARL) to measure the performance of the control charts. Define ARL_1 as the average run length to signal when process is out-of-control. We give a fixed false alarm rate α .

When the process is out-of-control, the power is

$$1-\beta = 1 - P(LCL < BL < UCL \mid BL \overset{\bullet}{\sim} Q^*)$$

$$= 1 - (F_{Q^*}(F_Q^{-1}(1-\alpha/2)) - F_{Q^*}(F_Q^{-1}(\alpha/2)))$$

where β is the type II error when the process is out-of-control and $F_{Q^*}(\bullet)$ is a the c.d.f. of Q^*.

Thus, $ARL_1 = \frac{1}{1-\beta}$

$$= \frac{1}{1-(F_{Q^*}(F_Q^{-1}(1-\alpha/2)) - F_{Q^*}(F_Q^{-1}(\alpha/2)))} \qquad (11)$$

2.7 Illustrating Example of the Bivariate Loss Chart

The example is taken from Yang, Lin and Hung (2009).

Example:

The thickness of the gold film on the surface of the terminals strongly affects the stability of signal transfer between the computer main hardware and its peripheral components. Also, a thin film will cause an unstable signal transfer and the terminals must be re-processed or discarded. Thus, we have to control the film thickness efficiently. 30 samples of size 4 are taken in a fixed interval from the process. For each sample, we measure two related thickness from two location of a film, that is, AP-2.8 and AN-1.3. The data are listed below:

Table 2. The data of film thickness

No.	AP-2.8	AN-1.3	No.	AP-2.8	AN-1.3	No.	AP-2.8	AN-1.3	No.	AP-2.8	AN-1.3	No.	AP-2.8	AN-1.3	No.	AP-2.8	AN-1.3
1	19.7	17.8	**6**	18.8	19	**11**	20.5	19.3	**16**	18.5	18.4	**21**	18.5	16.8	**26**	18.4	16.9
1	19.4	20.7	**6**	20.1	18.4	**11**	20.9	18.9	**16**	20.2	18	**21**	18.8	17	**26**	16.9	16.4
1	17.7	18.8	**6**	19.3	17.5	**11**	18.1	18.7	**16**	19.6	17.9	**21**	18.3	16.8	**26**	18.4	16.2
1	18.9	17.4	**6**	19.7	17.7	**11**	19.4	18.4	**16**	18.6	16.5	**21**	18.1	18.5	**26**	17.7	16.2
2	18.9	17.7	**7**	18.6	17.1	**12**	18.4	18.6	**17**	18.4	18	**22**	18.4	18.5	**27**	19.1	16.2
2	20.1	18.9	**7**	18.2	17.6	**12**	19.1	19.6	**17**	19.7	16.7	**22**	19.5	16.3	**27**	18.9	17.6
2	19.6	17.2	**7**	18.4	18.8	**12**	21.6	18.3	**17**	20	17	**22**	18.4	17.5	**27**	17.1	15.6
2	19.8	18.5	**7**	18.9	17.4	**12**	19.3	19.4	**17**	20.8	18.9	**22**	18.5	17.6	**27**	18.4	17.2
3	16.7	18.8	**8**	19.8	18.7	**13**	20	19.6	**18**	20.2	18	**23**	17.7	17.6	**28**	17.4	17.4
3	20.2	19.1	**8**	19.7	18.2	**13**	19	18.2	**18**	20.1	19.1	**23**	18.6	17	**28**	17.9	17.5
3	18.8	17.4	**8**	19.5	19.4	**13**	19.6	17.9	**18**	19.4	17.9	**23**	17.3	17.1	**28**	17.5	16.7
3	18.6	18.7	**8**	18.9	19.4	**13**	19.7	19.1	**18**	18.4	18.7	**23**	18.2	18.3	**28**	19.3	16.7
4	19.8	18.8	**9**	20.7	18.9	**14**	18.6	17.1	**19**	18.4	17.5	**24**	19.4	17.4	**29**	18.8	17.6
4	20	18.8	**9**	19.1	19.8	**14**	19.6	18.7	**19**	19.4	17.1	**24**	19.3	17.7	**29**	18.6	16.1
4	20.5	18.1	**9**	19.2	18.6	**14**	19.1	17.8	**19**	19.3	16.4	**24**	16.6	16.4	**29**	17.7	16.3
4	19.9	18.3	**9**	20.5	20.3	**14**	20.1	18.2	**19**	18.1	17.3	**24**	17.9	17	**29**	19	16.6
5	19.6	19.6	**10**	19.4	19.1	**15**	19.2	17.4	**20**	18.8	17.2	**25**	18.1	18.5	**30**	16	16.9
5	19.9	19.4	**10**	20.2	19.1	**15**	19.7	18.2	**20**	19.5	18.2	**25**	18.9	16.5	**30**	19.2	16.6
5	20.8	17.2	**10**	19.2	18.1	**15**	20.3	17.9	**20**	19.7	17.1	**25**	18.3	16.3	**30**	19.5	16.5
5	18.8	19.4	**10**	19.2	18.1	**15**	20	17.8	**20**	19.2	18.6	**25**	18.6	16.4	**30**	17.1	15.9

Next, in order to use BL chart to monitor process, we have to check if the data follow a normal distribution. Thus, we use chi-square plot (see Johnson (1992)) to check.

The procedure of plotting chi-square plot (or QQ-plot):

Step 1. Compute $d_j^2 = (y_j - \overline{y})' S^{-1} (y_j - \overline{y})$, $j = 1, 2, \cdots, 120$ and we rank d values from small to large. j means the jth observation of the sample.

Step 2. Graph the pairs $(d_{(j)}^2, \chi_2^2((j-1/2)/n))$, where $\chi_2^2((j-1/2)/n)$ is the 100(j-1/2)/n percentile of the chi-square distribution with 2 degrees of freedom.

Step 3. If any points fall near a line, we think they follow a bivariate normal distribution.

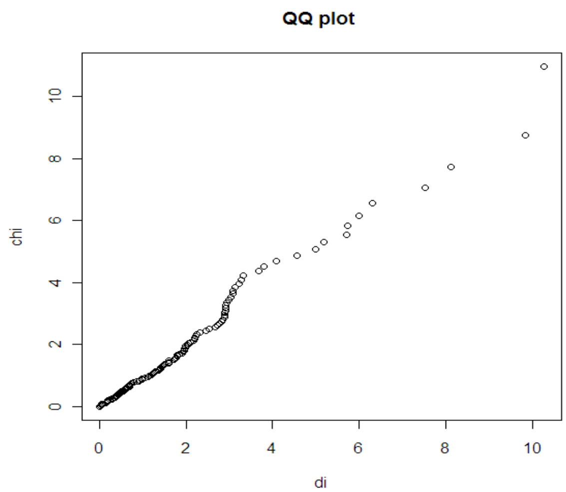

Figure 1. QQ-plot for the Film Thickness Data

From Figure 1, there are some points off the line. Let's do the test to make further check the assumption.

Use Generalized Shapiro-Wilk test to test the normality.

H_0 : The data follow a bivariate normal distribution

H_1 : The data do not follow a bivariate normal distribution

The result is as follows:

Generalized Shapiro-Wilk test for Multivariate Normality

MVW = 0.9847, p-value = 0.1486

The result shows that p-value = 0.1486 > $\alpha = 0.05$, hence we don't have enough evident to reject the hypothesis that the data follow a bivariate normal distribution.

We will now construct the BL chart to monitor the process.

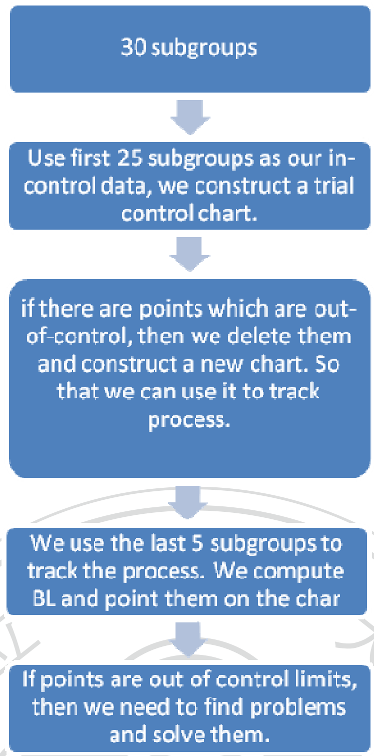

Figure 2. The Procedure for Constructing the BL Chart

Also, we could get the performance of the BL chart by using out-of-control points on phase 1 (which are deleted) as our out-of-control data, thus, we could get all shift scales. Then, we would calculate ARL_1.

Give $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \end{pmatrix}$, $K_{11} = 0.5, K_{12} = K_{22} = 1$, n = 4 and $c_4 = 0.9213$. Note that c_4 is S control chart factor.

Phase I

Since μ_0, Σ are unknown, we use $\bar{\bar{Y}} = \begin{pmatrix} 19.21 \\ 18.11 \end{pmatrix}$ to estimate μ_0 and use $S = \begin{pmatrix} 0.58 & 0.002 \\ 0.002 & 0.63 \end{pmatrix}$ to estimate Σ. Under $\alpha = 0.0027$, $\hat{\rho}_0 = 0.003$, we could calculate the limits of the BL chart using equation (8) and (9).

$$UCL = 4.593$$

$$LCL = 0.244$$

For each subgroup, we compute

$$BL_i = \frac{1}{n_1} K_{11} \sum_{j=1}^{n} (Y_{1ij} - T_1 + \frac{K_{12}}{2K_{11}} (Y_{2ij} - T_2))^2 + \frac{1}{n} (K_{22} - \frac{K_{12}^2}{4K_{11}}) \sum_{j=1}^{n} (Y_{2ij} - T_2)^2, i = 1, \ldots, 25$$

and plot them on the chart.

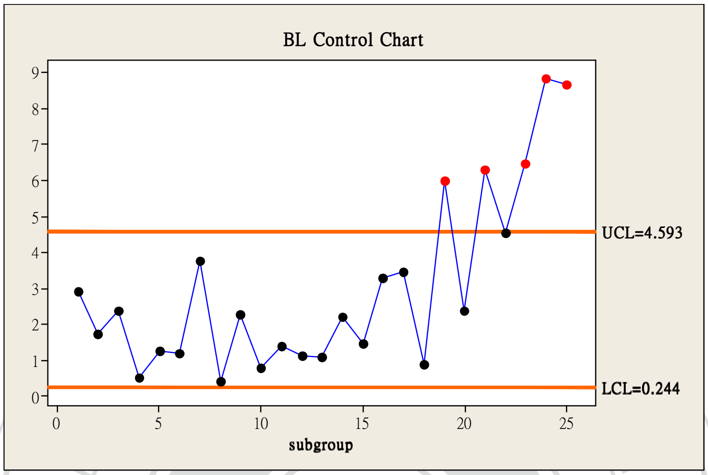

Figure 3. The Trial BL Control Chart (1)

As we could see from Figure 3, there are 5 points above the upper limit. Thus, we delete subgroups 19, 21, 23, 24, and 25. Then, we revise the data and construct a new control chart.

Now, we use new $\bar{\bar{Y}} = \begin{pmatrix} 19.42 \\ 18.34 \end{pmatrix}$ to estimate μ_0 and use new $S = \begin{pmatrix} 0.6 & 0.01 \\ 0.01 & 0.64 \end{pmatrix}$ to estimate Σ_0. Under $\alpha = 0.0027$, $\hat{\rho}_0 = 0.02$, the limits of the BL chart are:

$$UCL = 3.733$$

$$LCL = 0.157$$

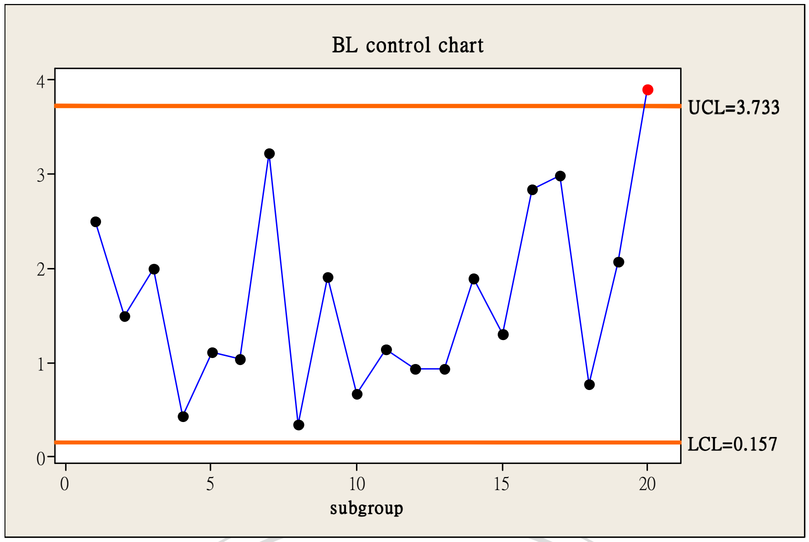

Figure 4. The Trial BL Control Chart (2)

Again, we compute BL for 24 subgroups and plot them on the chart.

From Figure 4, we found sample 20 is above the UCL. Thus, we delete subgroup 20 (the original subgroup 22) and revise the chart.

We use new $\bar{\bar{Y}} = \begin{pmatrix} 19.45 \\ 18.38 \end{pmatrix}$ to estimate μ_0 and use new $S = \begin{pmatrix} 0.62 & 0.04 \\ 0.04 & 0.62 \end{pmatrix}$ to estimate Σ_0. Under $\alpha = 0.0027$, $\hat{\rho}_0 = 0.06$, the new limits of the BL chart are now:

$$UCL = 3.726$$

$$LCL = 0.151$$

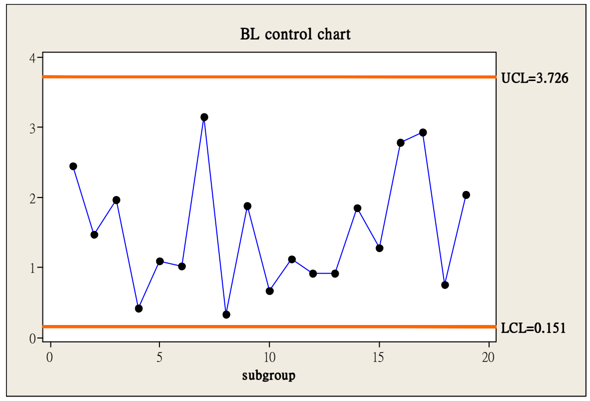

Figure 5. The BL Control Chart

Now from Figure 5, no points fall outside the control limits and the process seems to be in-control. We would use this chart to monitor the process.

To check the performance of the chart, we use subgroups 19, 21, 22, 23, 24, and 25 as our out-of-control data, then use these samples to estimate $\delta_1, \delta_2, \delta_3, \delta_4$ and ρ_1. We have $\bar{\bar{Y}}^* = \begin{pmatrix} 18.44 \\ 17.22 \end{pmatrix}$ to estimate μ_1 and $S^* = \begin{pmatrix} 0.45 & -0.16 \\ -0.16 & 0.64 \end{pmatrix}$ to estimate Σ_1.

We get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} \dfrac{18.44-19.45}{\sqrt{0.62}} \\ \dfrac{17.22-18.38}{\sqrt{0.62}} \end{pmatrix} = \begin{pmatrix} -1.28 \\ -1.47 \end{pmatrix}$, $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{0.45}{0.62}} \\ \sqrt{\dfrac{0.64}{0.62}} \end{pmatrix} = \begin{pmatrix} 0.85 \\ 1.02 \end{pmatrix}$,

$\begin{pmatrix} \delta_5 \\ \delta_6 \end{pmatrix} = \begin{pmatrix} 0.57 \\ -0.79 \end{pmatrix}$ and $\rho_1 = -0.3$. Thus, we could compute $ARL_1 = 1.22$ using equation (11).

Phase II

With the BL chart, we track the following process. Using last 5 subgroups to track, we compute BL for each subgroup and plot them on the chart to see if the process is out-of-control.

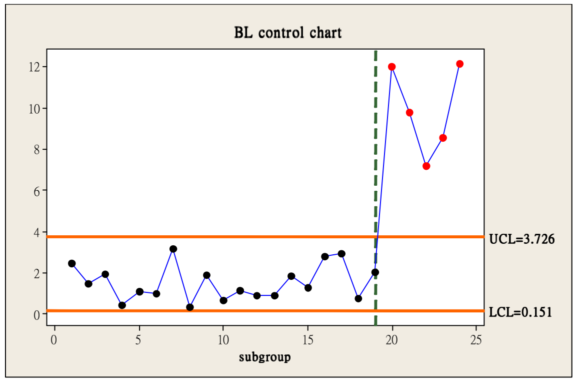

Figure 6. The BL Control Chart for Tracking the Process

From Figure 6, we found that the process is out-of-control since the last 5 subgroups were outside the upper control limit. We must find cause and rectify them if any. Since the BL chart couldn't tell what caused the signal, mean vector or covariance matrix. There are two methods to investigate. First, we can use in-control data to construct T^2 chart and $|S|$ chart separately, then we check points on both charts. So, we may find out if the signal(s) came from mean vector and/or covariance matrix. Second, we could compare both mean vector and covariance matrix with in-control process through observed data directly. Two methods are illustrated below.

(i) Use T^2 chart and $|S|$ chart to investigate.

For T^2 chart, the control limits are:

$$UCL_{T^2} = \chi^2_{2,1-\alpha/2} = 13.215$$

$$LCL_{T^2} = \chi^2_{2,\alpha/2} = 0.003$$

For $|S|$ chart, the control limits are:

$$UCL_{|S|} = \chi^2_{2n-4,1-\alpha/2} = 17.8$$

$$LCL_{|S|} = \chi^2_{2n-4,\alpha/2} = 0.11$$

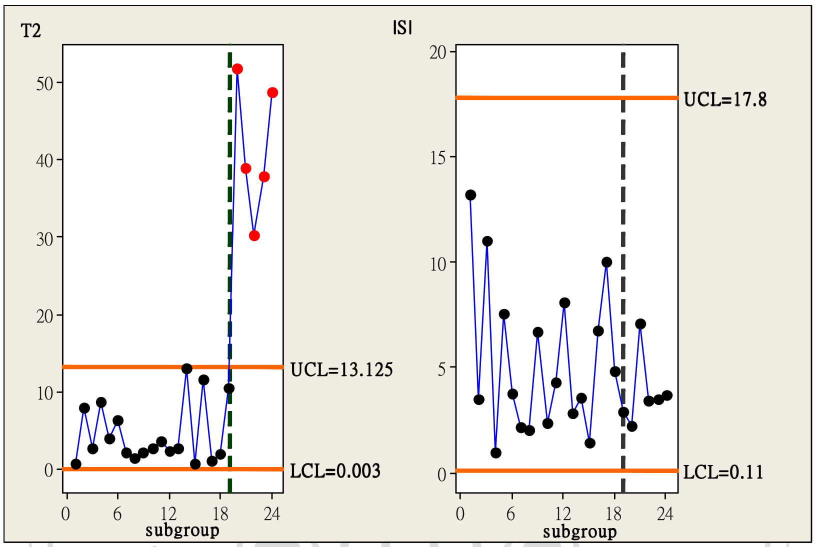

Figure 7. Hotelling's T^2 chart and $|S|$ chart

From Figure 7, first 19 subgroups are in-control data and the tracking data is from subgroup 20 to 24. No point falls outside the control limits for $|S|$ chart. But, we could see T^2 values of subgroup 20 to 24 are outside the control limits for T^2 chart. We found the mean vector has shifted from the target vector. For the subgroup 20, we could calculate the mean vector has shifted $\begin{pmatrix} -1.44\hat{\sigma}_1 \\ -3.32\hat{\sigma}_2 \end{pmatrix}$ from the target vector $\begin{pmatrix} 19 \\ 19 \end{pmatrix}$. Also we could get the mean vector of the subgroup 21 to 24 have shifted $\begin{pmatrix} -0.78\hat{\sigma}_1 \\ -3.03\hat{\sigma}_2 \end{pmatrix}$,

$\begin{pmatrix} -1.22\hat{\sigma}_1 \\ -2.49\hat{\sigma}_2 \end{pmatrix}$, $\begin{pmatrix} -0.6\hat{\sigma}_1 \\ -3.03\hat{\sigma}_2 \end{pmatrix}$ and $\begin{pmatrix} -1.32\hat{\sigma}_1 \\ -3.26\hat{\sigma}_2 \end{pmatrix}$ from the target vector.

(ii) We calculate mean vector and covariance matrix of subgroup 20 to 24 and compare them with in-control mean vector $\begin{pmatrix} 19.45 \\ 18.38 \end{pmatrix}$, the covariance matrix $\begin{pmatrix} 0.62 & 0.04 \\ 0.04 & 0.62 \end{pmatrix}$.

1. Subgroup 20: the mean vector is $\begin{pmatrix} 17.85 \\ 16.43 \end{pmatrix}$ and the covariance matrix is $\begin{pmatrix} 0.51 & 0.05 \\ 0.05 & 0.11 \end{pmatrix}$. And we get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -2.03 \\ -2.48 \end{pmatrix}$ and $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0.91 \\ 0.42 \end{pmatrix}$.
2. Subgroup 21: the mean vector is $\begin{pmatrix} 18.38 \\ 16.65 \end{pmatrix}$ and the covariance matrix is $\begin{pmatrix} 0.81 & 0.38 \\ 0.38 & 0.84 \end{pmatrix}$. And we get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -1.36 \\ -2.2 \end{pmatrix}$ and $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 1.14 \\ 1.16 \end{pmatrix}$.
3. Subgroup 22: the mean vector is $\begin{pmatrix} 18.03 \\ 17.08 \end{pmatrix}$ and the covariance matrix is $\begin{pmatrix} 0.77 & -0.13 \\ -0.13 & 0.19 \end{pmatrix}$. And we get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -1.8 \\ -1.65 \end{pmatrix}$ and $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 1.11 \\ 0.55 \end{pmatrix}$.
4. Subgroup 23: the mean vector is $\begin{pmatrix} 18.53 \\ 16.65 \end{pmatrix}$ and the covariance matrix is $\begin{pmatrix} 0.33 & 0.12 \\ 0.12 & 0.44 \end{pmatrix}$. And we get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -1.17 \\ -2.2 \end{pmatrix}$ and $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0.73 \\ 0.84 \end{pmatrix}$.
5. Subgroup 24: the mean vector is $\begin{pmatrix} 17.95 \\ 16.48 \end{pmatrix}$ and the covariance matrix is $\begin{pmatrix} 2.83 & -0.04 \\ -0.04 & 0.18 \end{pmatrix}$. And we get $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -1.91 \\ -2.41 \end{pmatrix}$ and $\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 2.14 \\ 0.54 \end{pmatrix}$.

As we could see that the mean vector has larger shifted than the shifts in the covariance matrix. In some subgroups, the variations decreased. We need to improve this situation.

Another useful approach is change point, detail of the apparach see (Hawkins et al. (2003), Hawkins and Zamba (2005) and Zamba (2006).

2.8 ARL_1 of the Bivariate Loss Chart

(1). The Bivariate Loss Chart with Specified Sample Size and Sampling Interval

To investigate the impact of various combinations of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6, ρ_1 and specified ρ_0 and K_{11} / K_{22} on ARL_1, the various levels of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6, ρ_1 and specified ρ_0 and K_{11} / K_{22} are listed in Table 3. Set $\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma_0 = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} -\delta_5 \\ -\delta_6 \end{pmatrix}$ and $K_{12} = K_{22} = 1$.

Table 3. Various levels of $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1)$, ρ_0 and K_{11} / K_{22}

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	ρ_0	K_{11}/K_{22}
0.5	0.5	1.5	1.5	0.5	0.5	0.1	0.1	0.5
1.5	1.5	2	2	1.5	1.5	0.5	0.5	1
2.5	2.5	2.5	2.5	2.5	2.5	0.8	0.8	2

The combinations of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 and ρ_1 are expressed under a specified (ρ_0, K_{11} / K_{22}). The ARL_1s are calculated based on the combinations of δ_1 to δ_6 and ρ_1 in L_{27} (3^{13}) table given n = 5, $\alpha = 0.0027$.

Table 4. The ARL_1 for Specified BL Chart Given n = 5, $\alpha = 0.0027$ and $\rho_0 = 0.1$

No.								$K_{11}/K_{22} = 0.5$	$K_{11}/K_{22} = 1$	$K_{11}/K_{22} = 2$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe.ARL$_1$**	**Spe.ARL$_1$**	**Spe.ARL$_1$**
1	0.5	0.5	1.5	1.5	0.5	0.5	0.1	1.51	1.54	1.56
2	0.5	0.5	2	2	1.5	1.5	0.5	1.02	1.15	1.02
3	0.5	0.5	2.5	2.5	2.5	2.5	0.8	1.00	1.00	1.00
4	0.5	1	1.5	2	1.5	2.5	0.8	1.00	1.00	1.01
5	0.5	1	2	2.5	2.5	0.5	0.1	1.00	1.00	1.00
6	0.5	1	2.5	1.5	0.5	1.5	0.5	1.00	1.00	1.00
7	0.5	2.5	1.5	2.5	2.5	1.5	0.5	1.00	1.00	1.00
8	0.5	2.5	2	1.5	1.5	2.5	0.8	1.00	1.00	1.00
9	0.5	2.5	2.5	2	2.5	0.5	0.1	1.00	1.00	1.00
10	1	1	2	2	2.5	2.5	0.5	1.00	1.00	1.00
11	1	1	2.5	2.5	0.5	0.5	0.8	1.00	1.00	1.00
12	1	1	1.5	1.5	1.5	1.5	0.1	1.00	1.00	1.00
13	1	2.5	2	2.5	0.5	1.5	0.1	1.00	1.00	1.00
14	1	2.5	2.5	1.5	1.5	2.5	0.5	1.00	1.00	1.00
15	1	2.5	1.5	2	2.5	0.5	0.8	1.00	1.00	1.00
16	1	0.5	2	1.5	1.5	0.5	0.8	1.00	1.00	1.00
17	1	0.5	2.5	2	2.5	1.5	0.1	1.00	1.00	1.00
18	1	0.5	1.5	2.5	0.5	2.5	0.5	1.00	1.00	1.00
19	2.5	2.5	2.5	2.5	1.5	1.5	0.8	1.00	1.00	1.00
20	2.5	2.5	1.5	1.5	2.5	2.5	0.1	1.00	1.00	1.00
21	2.5	2.5	2	2	0.5	0.5	0.5	1.00	1.00	1.00
22	2.5	0.5	2.5	1.5	2.5	0.5	0.5	1.00	1.00	1.00
23	2.5	0.5	1.5	2	0.5	1.5	0.8	1.00	1.00	1.00
24	2.5	0.5	2	2.5	1.5	2.5	0.1	1.00	1.00	1.00
25	2.5	1	2.5	2	0.5	2.5	0.1	1.00	1.00	1.00
26	2.5	1	1.5	2.5	1.5	0.5	0.5	1.00	1.00	1.00
27	2.5	1	2	1.5	2.5	1.5	0.8	1.00	1.00	1.00

Table 5. The ARL_1 for Specified BL Chart Given n = 5, $\alpha = 0.0027$ and $\rho_0 = 0.5$

No.								$K_{11}/K_{22}=0.5$	$K_{11}/K_{22}=1$	$K_{11}/K_{22}=2$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1
1	0.5	0.5	1.5	1.5	0.5	0.5	0.1	1.39	2.03	1.45
2	0.5	0.5	2	2	1.5	1.5	0.5	1.01	1.06	1.01
3	0.5	0.5	2.5	2.5	2.5	2.5	0.8	1.00	1.00	1.00
4	0.5	1	1.5	2	1.5	2.5	0.8	1.00	1.01	1.01
5	0.5	1	2	2.5	2.5	0.5	0.1	1.00	1.00	1.00
6	0.5	1	2.5	1.5	0.5	1.5	0.5	1.00	1.00	1.00
7	0.5	2.5	1.5	2.5	2.5	1.5	0.5	1.00	1.00	1.00
8	0.5	2.5	2	1.5	1.5	2.5	0.8	1.00	1.00	1.00
9	0.5	2.5	2.5	2	2.5	0.5	0.1	1.00	1.00	1.00
10	1	1	2	2	2.5	2.5	0.5	1.00	1.00	1.00
11	1	1	2.5	2.5	0.5	0.5	0.8	1.00	1.00	1.00
12	1	1	1.5	1.5	1.5	1.5	0.1	1.00	1.00	1.00
13	1	2.5	2	2.5	0.5	1.5	0.1	1.00	1.00	1.00
14	1	2.5	2.5	1.5	1.5	2.5	0.5	1.00	1.00	1.00
15	1	2.5	1.5	2	2.5	0.5	0.8	1.00	1.00	1.00
16	1	0.5	2	1.5	1.5	0.5	0.8	1.00	1.00	1.00
17	1	0.5	2.5	2	2.5	1.5	0.1	1.00	1.00	1.00
18	1	0.5	1.5	2.5	0.5	2.5	0.5	1.00	1.00	1.00
19	2.5	2.5	2.5	2.5	1.5	1.5	0.8	1.00	1.00	1.00
20	2.5	2.5	1.5	1.5	2.5	2.5	0.1	1.00	1.00	1.00
21	2.5	2.5	2	2	0.5	0.5	0.5	1.00	1.00	1.00
22	2.5	0.5	2.5	1.5	2.5	0.5	0.5	1.00	1.00	1.00
23	2.5	0.5	1.5	2	0.5	1.5	0.8	1.00	1.00	1.00
24	2.5	0.5	2	2.5	1.5	2.5	0.1	1.00	1.00	1.00
25	2.5	1	2.5	2	0.5	2.5	0.1	1.00	1.00	1.00
26	2.5	1	1.5	2.5	1.5	0.5	0.5	1.00	1.00	1.00
27	2.5	1	2	1.5	2.5	1.5	0.8	1.00	1.00	1.00

Table 6. The ARL_1 for Specified BL Chart Given n = 5, $\alpha = 0.0027$ and $\rho_0 = 0.8$

No.								$K_{11}/K_{22} = 0.5$	$K_{11}/K_{22} = 1$	$K_{11}/K_{22} = 2$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1
1	0.5	0.5	1.5	1.5	0.5	0.5	0.1	1.31	2.00	1.39
2	0.5	0.5	2	2	1.5	1.5	0.5	1.00	1.03	1.01
3	0.5	0.5	2.5	2.5	2.5	2.5	0.8	1.00	1.00	1.00
4	0.5	1	1.5	2	1.5	2.5	0.8	1.00	1.01	1.01
5	0.5	1	2	2.5	2.5	0.5	0.1	1.00	1.00	1.00
6	0.5	1	2.5	1.5	0.5	1.5	0.5	1.00	1.00	1.00
7	0.5	2.5	1.5	2.5	2.5	1.5	0.5	1.00	1.00	1.00
8	0.5	2.5	2	1.5	1.5	2.5	0.8	1.00	1.00	1.00
9	0.5	2.5	2.5	2	2.5	0.5	0.1	1.00	1.00	1.00
10	1	1	2	2	2.5	2.5	0.5	1.00	1.00	1.00
11	1	1	2.5	2.5	0.5	0.5	0.8	1.00	1.00	1.00
12	1	1	1.5	1.5	1.5	1.5	0.1	1.00	1.00	1.00
13	1	2.5	2	2.5	0.5	1.5	0.1	1.00	1.00	1.00
14	1	2.5	2.5	1.5	1.5	2.5	0.5	1.00	1.00	1.00
15	1	2.5	1.5	2	2.5	0.5	0.8	1.00	1.00	1.00
16	1	0.5	2	1.5	1.5	0.5	0.8	1.00	1.00	1.00
17	1	0.5	2.5	2	2.5	1.5	0.1	1.00	1.00	1.00
18	1	0.5	1.5	2.5	0.5	2.5	0.5	1.00	1.00	1.00
19	2.5	2.5	2.5	2.5	1.5	1.5	0.8	1.00	1.00	1.00
20	2.5	2.5	1.5	1.5	2.5	2.5	0.1	1.00	1.00	1.00
21	2.5	2.5	2	2	0.5	0.5	0.5	1.00	1.00	1.00
22	2.5	0.5	2.5	1.5	2.5	0.5	0.5	1.00	1.00	1.00
23	2.5	0.5	1.5	2	0.5	1.5	0.8	1.00	1.00	1.00
24	2.5	0.5	2	2.5	1.5	2.5	0.1	1.00	1.00	1.00
25	2.5	1	2.5	2	0.5	2.5	0.1	1.00	1.00	1.00
26	2.5	1	1.5	2.5	1.5	0.5	0.5	1.00	1.00	1.00
27	2.5	1	2	1.5	2.5	1.5	0.8	1.00	1.00	1.00

From Table 4 to Table 6, we found out that no matter what the correlation between the two quality characteristics is when the process is in-control, the specified BL chart is insensitive to larger process shifts. And the ratio of K_{11} / K_{22} hardly affects the performance, since their ARL_1 is near 1 for δ_1, δ_2 ≥ 0.5, δ_3, δ_4 ≥ 2, δ_6, δ_7 ≥ 1.5. Also, we found that the specified BL chart is insensitive to ρ_1 when the process has large shifted.

2. To investigate the performance of the BL chart for small shifts, we consider the following combinations of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 and ρ_1. Also, we calculate ARL_1 under specified K_{11} / K_{22} = 0.5, 1, 2, 4 and ρ_0 = 0.1, 0.5, 0.8 given n = 5 and $\alpha = 0.0027$.

Table 7. Various levels of $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1)$

δ_1	δ_2	δ_3	δ_4	δ_6	δ_7	ρ_1
0.1	0.1	1.1	1.1	0.1	0.1	0.1
0.5	0.5	1.5	1.5	0.5	0.5	0.5
1	1	2	2	1	1	0.8

Table 8. The ARL_1 for Specified BL Chart of Small Shifts Given n = 5, $\alpha = 0.0027$ and $\rho_0 = 0.5$

No.								$K_{11}/K_{22} = 0.5$	$K_{11}/K_{22} = 1$	$K_{11}/K_{22} = 2$	$K_{11}/K_{22} = 4$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe.ARL₁**	**Spe.ARL₁**	**Spe.ARL₁**	**Spe.ARL₁**
1	0.1	0.1	1.1	1.1	0.1	0.1	0.1	132.97	391.00	67.14	4.54
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	15.63	76.42	18.14	2.94
3	0.1	0.1	2	2	1	1	0.8	4.58	23.02	5.99	1.74
4	0.1	0.5	1.1	1.5	0.5	1	0.8	1.92	2.93	2.33	1.34
5	0.1	0.5	1.5	2	1	0.1	0.1	1.89	3.79	2.31	1.28
6	0.1	0.5	2	1.1	0.1	0.5	0.5	1.18	1.88	1.38	1.07
7	0.1	1	1.1	2	1	0.5	0.5	1.10	1.25	1.21	1.05
8	0.1	1	1.5	1.1	0.5	1	0.8	1.02	1.11	1.09	1.03
9	0.1	1	2	1.5	1	0.1	0.1	1.00	1.03	1.02	1.00
10	0.5	0.5	1.5	1.5	1	1	0.5	1.48	1.71	1.87	1.21
11	0.5	0.5	2	2	0.1	0.1	0.8	1.16	1.43	1.40	1.09
12	0.5	0.5	1.1	1.1	0.5	0.5	0.1	1.17	1.23	1.05	1.00
13	0.5	1	1.5	2	0.1	0.5	0.1	2.32	2.06	1.71	1.13
14	0.5	1	2	1.1	0.5	1	0.5	1.31	1.48	2.40	1.04
15	0.5	1	1.1	1.5	1	0.1	0.8	1.08	1.08	1.01	1.00
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	1.32	1.80	2.64	1.49
17	0.5	0.1	2	1.5	1	0.5	0.1	1.08	1.69	2.01	1.23
18	0.5	0.1	1.1	2	0.1	1	0.5	1.09	1.30	1.13	1.02
19	1	1	2	2	0.5	0.5	0.8	1.20	1.17	1.13	1.03
20	1	1	1.1	1.1	1	1	0.1	1.13	1.11	1.01	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.02	1.02	1.00	1.00
22	1	0.1	2	1.1	1	0.1	0.5	1.09	1.22	1.46	1.20
23	1	0.1	1.1	1.5	0.1	0.5	0.8	1.12	1.32	1.13	1.02
24	1	0.1	1.5	2	0.5	1	0.1	1.03	1.12	1.03	1.00
25	1	0.5	2	1.5	0.1	1	0.1	1.20	1.25	1.29	1.07
26	1	0.5	1.1	2	0.5	0.1	0.5	1.11	1.21	1.05	1.00
27	1	0.5	1.5	1.1	1	0.5	0.8	1.01	1.04	1.00	1.00

Table 8 shows that there are unreasonable ARL_1 with δ_1, δ_2 $= 0.1$, δ_3, δ_4 = 1.1, δ_5, δ_6 $= 0.1$, ρ_1 $= 0.1$ under $\rho_0 = 0.5$. We also did the analysis under ρ_0 = 0.1, 0.8 (See Appendix B, Table 49 and Table 50.). When $\rho_0 = 0.8$, we saw the same problem as $\rho_0 = 0.5$.

Next, we calculate ARL_1 under some combinations of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 and ρ_1. The results are shown in Table 9.

Table 9. Reasonable ARL_1 under Small Shifts of process Parameters

No.	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe. ARL_1**
1	0.1	0.1	1.05	1.05	0.1	0.1	0.1	506.56
2	0.1	0.1	1.1	1.1	0.1	0.1	0.1	391.00
3	0.1	0.1	1.2	1.1	0.1	0.1	0.1	195.36
4	0.1	0.1	1.1	1.2	0.1	0.1	0.1	260.52
5	0.1	0.1	1.2	1.2	0.1	0.1	0.1	125.36
6	0.1	0.1	1.1	1.1	0.1	0.1	0.2	284.08
7	0.1	0.1	1.2	1.2	0.1	0.1	0.2	86.54

From Table 9, we have

1. Change δ_3 from 1.1 to 1.2, then ARL_1 becomes reasonable.

2. Change δ_4 from 1.1 to 1.2, then ARL_1 becomes reasonable.

3. Change ρ_1 from 0.1 to 0.2, then ARL_1 becomes reasonable.

4. Compare to δ_4 and ρ_1, changing δ_3 has more effect on ARL_1.

We found that when the shift scales of $(\delta_3, \delta_4, \rho_1)$ is small, (δ_3, δ_4 = 1.1, ρ_1 = 0.1), ARL_1 gets bigger than ARL_0 = 370. To have a reasonable ARL_1, at least one δ_3,

$\delta_4 \geq 1.2$ or $\rho_1 \geq 0.2$. So, for the following ARL_1 analysis we will change $\delta_3 = 1.1$ to $\delta_3 = 1.2$.

We change $\delta_3 = 1.1$ to $\delta_3 = 1.2$ in Table 7 and calculate ARL_1 with specified K_{11}/K_{22} = 0.5, 1, 2, 4 and ρ_0 = 0.1, 0.5, 0.8 based on the various combinations of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 and ρ_1 (See Table 10.). The results are shown in Table 11- 13. Also, we drew the response graph of ARL_1 – bar (See Figure 8 – 10.) based on each table.

Table 10. Various levels of $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1)$

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1
0.1	0.1	1.2	1.1	0.1	0.1	0.1
0.5	0.5	1.5	1.5	0.5	0.5	0.5
1	1	2	2	1	1	0.8

Table 11. The ARL_1 for Specified BL Chart of Small Shifts Given n = 5,

$\alpha = 0.0027$ and $\rho_0 = 0.1$

No.								K_{11}/K_{22} = 0.5	K_{11}/K_{22} = 1	K_{11}/K_{22} = 2	K_{11}/K_{22} = 4
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe.ARL$_1$**	**Spe.ARL$_1$**	**Spe.ARL$_1$**	**Spe.ARL$_1$**
1	0.1	0.1	1.2	1.1	0.1	0.1	0.1	59.31	47.16	40.06	3.36
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	2.70	2.81	2.84	1.35
3	0.1	0.1	2	2	1	1	0.8	1.12	1.19	1.12	1.02
4	0.1	0.5	1.2	1.5	0.5	1	0.8	1.83	2.38	4.02	1.79
5	0.1	0.5	1.5	2	1	0.1	0.1	1.18	1.43	1.14	1.02
6	0.1	0.5	2	1.1	0.1	0.5	0.5	2.00	1.82	1.85	1.25
7	0.1	1	1.2	2	1	0.5	0.5	1.03	1.23	1.17	1.05
8	0.1	1	1.5	1.1	0.5	1	0.8	1.45	1.61	2.09	1.28
9	0.1	1	2	1.5	1	0.1	0.1	1.05	1.12	1.03	1.00
10	0.5	0.5	1.5	1.5	1	1	0.5	1.15	1.27	1.15	1.01
11	0.5	0.5	2	2	0.1	0.1	0.8	1.09	1.10	1.15	1.03
12	0.5	0.5	1.2	1.1	0.5	0.5	0.1	3.61	3.78	3.63	1.35
13	0.5	1	1.5	2	0.1	0.5	0.1	1.03	1.08	1.26	1.12
14	0.5	1	2	1.1	0.5	1	0.5	1.09	1.09	1.12	1.02
15	0.5	1	1.2	1.5	1	0.1	0.8	1.10	1.30	1.10	1.01
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	2.42	2.15	1.61	1.10
17	0.5	0.1	2	1.5	1	0.5	0.1	1.15	1.16	1.02	1.00
18	0.5	0.1	1.2	2	0.1	1	0.5	1.37	1.58	2.15	1.85
19	1	1	2	2	0.5	0.5	0.8	1.00	1.00	1.00	1.00
20	1	1	1.2	1.1	1	1	0.1	1.05	1.12	1.06	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.09	1.10	1.17	1.02
22	1	0.1	2	1.1	1	0.1	0.5	1.05	1.02	1.00	1.00
23	1	0.1	1.2	1.5	0.1	0.5	0.8	1.71	1.70	1.93	1.26
24	1	0.1	1.5	2	0.5	1	0.1	1.10	1.09	1.11	1.01
25	1	0.5	2	1.5	0.1	1	0.1	1.05	1.05	1.05	1.02
26	1	0.5	1.2	2	0.5	0.1	0.5	1.14	1.16	1.12	1.01
27	1	0.5	1.5	1.1	1	0.5	0.8	1.10	1.13	1.02	1.00

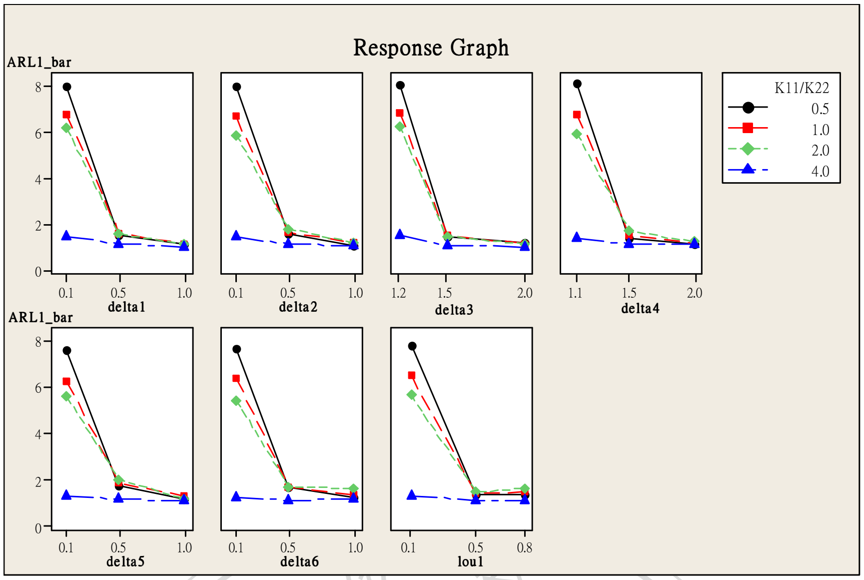

Figure 8. Response Graph of ARL_1-bar with $\rho_0 = 0.1$ Based on Table 11

Table 12. The ARL_1 for Specified BL Chart of Small Shifts Given n = 5,

$\alpha = 0.0027$ and $\rho_0 = 0.5$

No.								K_{11}/K_{22} = 0.5	K_{11}/K_{22} = 1	K_{11}/K_{22} = 2	K_{11}/K_{22} = 4
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe.ARL_1**	**Spe.ARL_1**	**Spe.ARL_1**	**Spe.ARL_1**
1	0.1	0.1	1.2	1.1	0.1	0.1	0.1	82.37	195.36	30.62	3.20
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	15.63	76.42	18.14	2.94
3	0.1	0.1	2	2	1	1	0.8	4.58	23.02	5.99	1.74
4	0.1	0.5	1.2	1.5	0.5	1	0.8	1.62	3.47	3.98	1.73
5	0.1	0.5	1.5	2	1	0.1	0.1	1.89	3.79	2.31	1.28
6	0.1	0.5	2	1.1	0.1	0.5	0.5	1.18	1.88	1.38	1.07
7	0.1	1	1.2	2	1	0.5	0.5	1.02	1.17	1.13	1.04
8	0.1	1	1.5	1.1	0.5	1	0.8	1.02	1.11	1.09	1.03
9	0.1	1	2	1.5	1	0.1	0.1	1.00	1.03	1.02	1.00
10	0.5	0.5	1.5	1.5	1	1	0.5	1.48	1.71	1.87	1.21
11	0.5	0.5	2	2	0.1	0.1	0.8	1.16	1.43	1.40	1.09
12	0.5	0.5	1.2	1.1	0.5	0.5	0.1	3.85	9.61	3.42	1.36
13	0.5	1	1.5	2	0.1	0.5	0.1	2.32	2.06	1.71	1.13
14	0.5	1	2	1.1	0.5	1	0.5	1.31	1.48	2.40	1.04
15	0.5	1	1.2	1.5	1	0.1	0.8	1.06	1.20	1.08	1.01
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	1.32	1.80	2.64	1.49
17	0.5	0.1	2	1.5	1	0.5	0.1	1.08	1.69	2.01	1.23
18	0.5	0.1	1.2	2	0.1	1	0.5	1.52	2.01	2.79	1.76
19	1	1	2	2	0.5	0.5	0.8	1.20	1.17	1.13	1.03
20	1	1	1.2	1.1	1	1	0.1	1.02	1.23	1.04	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.02	1.02	1.00	1.00
22	1	0.1	2	1.1	1	0.1	0.5	1.09	1.22	1.46	1.20
23	1	0.1	1.2	1.5	0.1	0.5	0.8	2.05	2.34	2.32	1.24
24	1	0.1	1.5	2	0.5	1	0.1	1.03	1.12	1.03	1.00
25	1	0.5	2	1.5	0.1	1	0.1	1.20	1.25	1.29	1.07
26	1	0.5	1.2	2	0.5	0.1	0.5	1.11	1.27	1.09	1.01
27	1	0.5	1.5	1.1	1	0.5	0.8	1.01	1.04	1.00	1.00

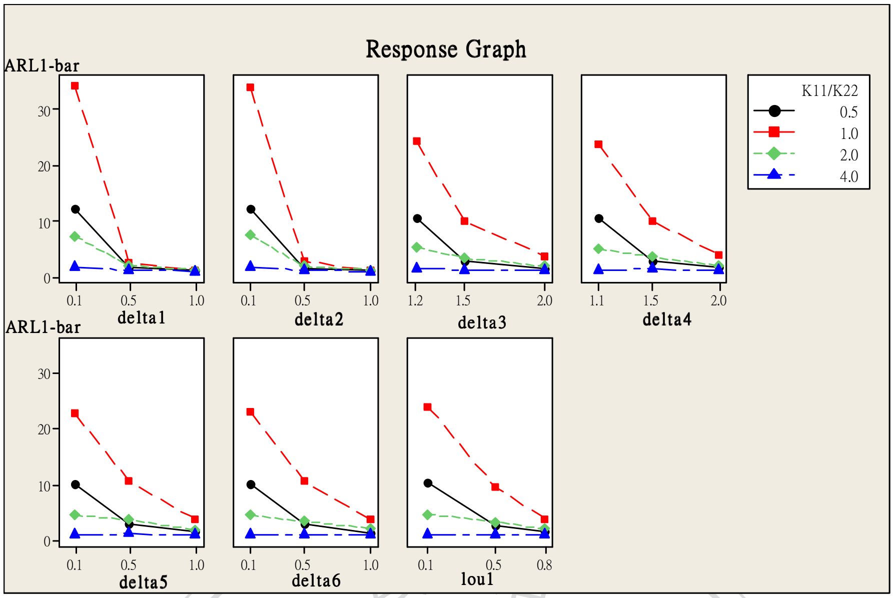

Figure 9. Response Graph of ARL_1-bar with $\rho_0 = 0.5$ Based on Table 12

Table 13. The ARL_1 for Specified BL Chart of Small Shifts Given n = 5,

$\alpha = 0.0027$ and $\rho_0 = 0.8$

No.								K_{11}/K_{22} = 0.5	K_{11}/K_{22} = 1	K_{11}/K_{22} = 2	K_{11}/K_{22} = 4
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1
1	0.1	0.1	1.2	1.1	0.1	0.1	0.1	52.50	254.84	24.46	3.11
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	11.87	71.89	15.11	2.87
3	0.1	0.1	2	2	1	1	0.8	3.65	14.64	5.09	1.70
4	0.1	0.5	1.2	1.5	0.5	1	0.8	1.47	4.30	3.48	1.69
5	0.1	0.5	1.5	2	1	0.1	0.1	1.72	3.70	2.15	1.27
6	0.1	0.5	2	1.1	0.1	0.5	0.5	1.13	1.64	1.32	1.07
7	0.1	1	1.2	2	1	0.5	0.5	1.01	1.13	1.12	1.04
8	0.1	1	1.5	1.1	0.5	1	0.8	1.02	1.11	1.08	1.02
9	0.1	1	2	1.5	1	0.1	0.1	1.00	1.02	1.02	1.00
10	0.5	0.5	1.5	1.5	1	1	0.5	1.38	2.08	1.78	1.20
11	0.5	0.5	2	2	0.1	0.1	0.8	1.12	1.54	1.34	1.08
12	0.5	0.5	1.2	1.1	0.5	0.5	0.1	3.19	9.25	3.08	1.35
13	0.5	1	1.5	2	0.1	0.5	0.1	2.05	2.59	1.63	1.12
14	0.5	1	2	1.1	0.5	1	0.5	1.23	1.79	1.30	1.04
15	0.5	1	1.2	1.5	1	0.1	0.8	1.05	1.16	1.07	1.01
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	1.24	2.30	2.43	1.46
17	0.5	0.1	2	1.5	1	0.5	0.1	1.06	1.97	1.82	1.21
18	0.5	0.1	1.2	2	0.1	1	0.5	1.40	2.49	3.36	1.71
19	1	1	2	2	0.5	0.5	0.8	1.16	1.24	1.17	1.03
20	1	1	1.2	1.1	1	1	0.1	1.01	1.14	1.03	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.01	1.01	1.00	1.00
22	1	0.1	2	1.1	1	0.1	0.5	1.06	1.34	1.61	1.18
23	1	0.1	1.2	1.5	0.1	0.5	0.8	1.84	3.11	2.17	1.23
24	1	0.1	1.5	2	0.5	1	0.1	1.02	1.08	1.02	1.00
25	1	0.5	2	1.5	0.1	1	0.1	1.14	1.42	1.40	1.07
26	1	0.5	1.2	2	0.5	0.1	0.5	1.08	1.22	1.08	1.01
27	1	0.5	1.5	1.1	1	0.5	0.8	1.01	1.03	1.00	1.00

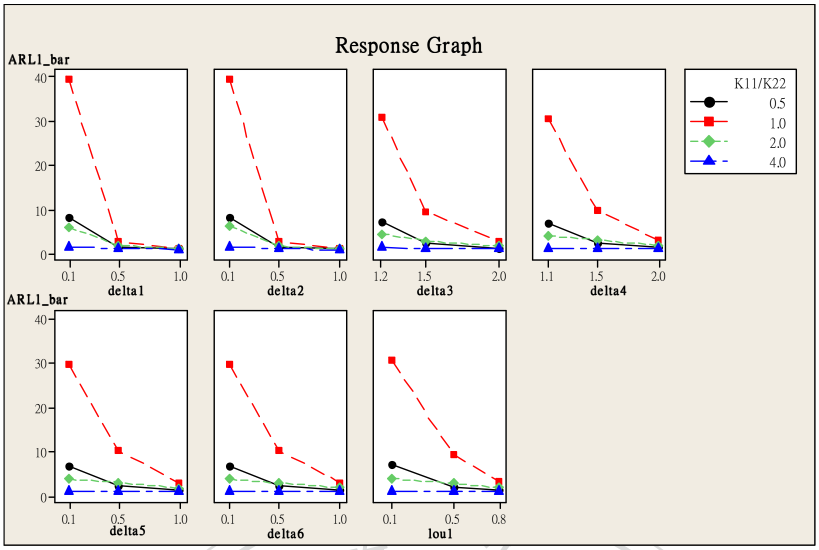

Figure 10. Response Graph of ARL$_1$-bar with $\rho_0 = 0.8$ Based on Table 13

From Table 11 – 13 and Figure 8 – 10, we found similar results. Firstly, under different ρ_0, the specified BL chart could detect the process shifts quickly for $K_{11}/K_{22} = 4$. Secondly, the specified BL chart has better performance when δ_1, δ_2 > 0.1, $\delta_3 > 1.2$, $\delta_4 > 1.1$ and $\rho_1 > 0.1$. Thirdly, the specified BL chart is insensitive to shift scales of Table 10 under $K_{11} / K_{22} = 4$. As we could see that for 27 combinations, the ARL$_1$ is small and not changing much. Through the response graphs Figure 8 to Figure 10 which are based on Table 11 to Table 13, we could see the effects of ((δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 , ρ_1)) on ARL$_1$-bar. When(δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 , ρ_1) increase, the ARL$_1$s decrease. There are some different results between $\rho_0 = 0.1$ and ρ_0 = 0.5, 0.8. For $\rho_0 = 0.1$, comparing performance under K_{11}/K_{22} = 0.5, 1, 2, 4, the ARL$_1$ is smaller when K_{11} / K_{22} is larger. But for ρ_0 = 0.5, 0.8, the specified BL chart has poorer performance when K_{11}/K_{22} = 1. For $\rho_0 = 0.1$, the specified BL chart is more sensitive when δ_1, δ_2, δ_5, δ_6 < 0.5, δ_3, δ_4 < 1.5 and $\rho_1 < 0.5$ under K_{11}/K_{22}= 0.5, 1, 2. But for ρ_0 = 0.5, 0.8, the ARL_1s decrease rapidly when δ_1, δ_2, δ_5, δ_6 < 0.5, δ_3, δ_4 < 1.5 and $\rho_1 < 0.5$ under K_{11}/K_{22} = 1 comparing with other K_{11}/K_{22}, especially when $\rho_0 = 0.8$. From the above results, we saw that the performance is affected by the ratio of K_{11}/K_{22} and ρ_0. To get better performance, we will choose the larger K_{11}/K_{22}.

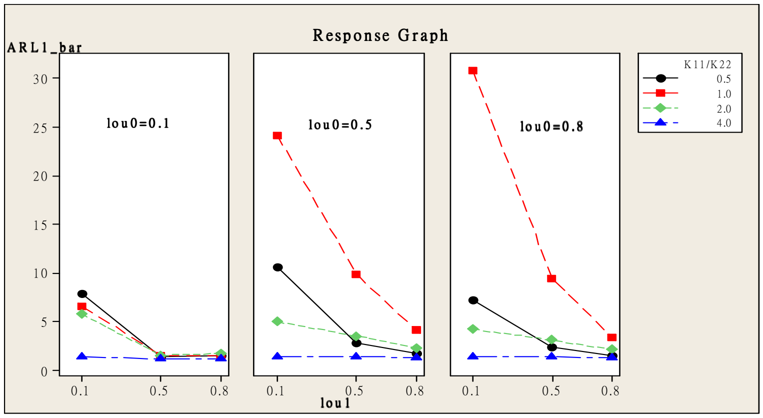

Figure 11. Response Graph of ARL_1-bar with $\rho_0 = 0.1, 0.5, 0.8$ Based on Table 11 – Table 13

Through Figure 11, we could see clearly the different effects of ρ_1 under different ρ_0 and K_{11}/K_{22}. First, unlike $\rho_0 = 0.5, 0.8$, under $\rho_0 = 0.1$, the larger the ratio of K_{11}/K_{22} is, the smaller the ARL_1 is. Second, under $K_{11}/K_{22} = 1$, the specified BL chart has better performance under $\rho_0 = 0.5$ than that under $\rho_0 = 0.8$. Also, the result shows that when correlation shifts to $\rho_1 = 0.1$, the specified BL chart detects quicker under $\rho_0 = 0.5$ than that under $\rho_0 = 0.8$. Comparing performance for different K_{11}/K_{22}, the specified BL chart has better performance under $K_{11}/K_{22} = 4$ whether $\rho_0 = 0.1, 0.5$ or $\rho_0 = 0.8$. The specified BL chart is more sensitive when ρ_1 shifts to lower correlation, which is $\rho_1 < 0.5$.

We also calculate ARL_1 based on combinations of (δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 , ρ_1)with specified ρ_0 = 0.5 and K_{11}/K_{22}= 0.5, 1, 2, 4 for smaller shift (See Appendix B, Table 51.). The specified BL chart detects quickly for smaller shift when $\delta_1 > 0.01$, $\delta_2 > 0.01$, $\delta_3 > 1.2$ and $\delta_4 > 1.1$. Compare to K_{11}/K_{22} = 0.5, 1, 2, the specified BL chart performs better under K_{11}/K_{22} = 4. From response graph (See Appendix B Figure.), we found that comparing with δ_1, δ_2, δ_5, δ_6, the specified BL chart is more sensitive to δ_3 , δ_4 , ρ_1 shift. When $\delta_3, \delta_4 \le 1.5$ and $\rho_1 \le 0.3$, the specified BL chart is more sensitive. When δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 , ρ_1 increase, the ARL_1 decreases.

2.9 The Bivariate Loss Chart with Optimal Sample Size and Sampling Interval

Define ATS_1 as the average time to signal when process is out-of-control. ATS_1 can be calculated by multiplying ARL_1 and sampling interval (h). Reasonably, the BL control chart should have a large ATS_0 and a small ATS_1.

To investigate the performance of the optimal BL chart of the small shift, we use the combinations of (δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 , ρ_1) (See Table 10,) with specified ρ_0 = 0.5 and K_{11}/ K_{22} = 0.5, 1, 2, 4. ATS_1^*s are calculated by finding the optimal (n^*, h^*) to minimize ATS_1^*. Note that we use PORT routine in R program to find the optimal solutions. With different K_{11}/K_{22}, we have 4 cases below:

Case 1: Let $K_{11}/K_{22} = 0.5$ and $\alpha = 0.0027$. (delte star)

$$\begin{aligned}&Minimize \quad ATS_1 = f(n,h)\\&s.t.\\&2 \le n \le 25\\&0.1 \le h < 4\end{aligned}$$

Case 2: Let $K_{11}/K_{22} = 1$ and $\alpha = 0.0027$.

$$\begin{aligned}&Minimize \quad ATS_1 = f(n,h)\\&s.t.\\&2 \le n \le 25\\&0.1 \le h < 4\end{aligned}$$

Case 3: Let $K_{11}/K_{22} = 2$ and $\alpha = 0.0027$.

$$\begin{aligned}&Minimize \quad ATS1 = f(n,h)\\&s.t.\\&2 \le n \le 25\\&0.1 \le h < 4\end{aligned}$$

Case 4: Let $K_{11}/K_{22} = 4$ and $\alpha = 0.0027$.

$$\begin{aligned}&Minimize \quad ATS_1 = f(n,h)\\&s.t.\\&2 \le n \le 25\\&0.1 \le h < 4\end{aligned}$$

Table 14. The Optimal (n^*, h^*, ATS_1^*) for BL Chart Given $\rho_0 = 0.1$ and $\alpha = 0.0027$

							$K_{11}/K_{22}=0.5$			$K_{11}/K_{22}=1$			$K_{11}/K_{22}=2$			$K_{11}/K_{22}=4$		
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*
0.1	0.1	1.2	1.1	0.1	0.1	0.1	15.00	0.10	0.10	25.00	0.10	0.16	25.00	0.10	0.10	25.00	0.10	0.10
0.1	0.1	1.5	1.5	0.5	0.5	0.5	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10
0.1	0.1	2	2	1	1	0.8	19.36	0.10	0.10	22.86	0.10	0.10	19.16	0.10	0.10	15.18	0.10	0.10
0.1	0.5	1.2	1.5	0.5	1	0.8	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10
0.1	0.5	1.5	2	1	0.1	0.1	19.71	0.10	0.10	21.82	0.10	0.10	19.15	0.10	0.10	15.85	0.10	0.10
0.1	0.5	2	1.1	0.1	0.5	0.5	15.00	0.10	0.10	11.04	0.10	0.12	8.68	0.10	0.13	24.61	0.10	0.10
0.1	1	1.2	2	1	0.5	0.5	14.54	0.10	0.10	19.20	0.10	0.10	18.63	0.10	0.10	16.54	0.10	0.10
0.1	1	1.5	1.1	0.5	1	0.8	15.00	0.10	0.10	7.83	0.10	0.13	9.37	0.10	0.13	21.18	0.10	0.10
0.1	1	2	1.5	1	0.1	0.1	15.53	0.10	0.10	17.85	0.10	0.10	15.85	0.10	0.10	8.00	0.10	0.10
0.5	0.5	1.5	1.5	1	1	0.5	17.07	0.10	0.10	21.02	0.10	0.10	17.18	0.10	0.10	13.64	0.10	0.10
0.5	0.5	2	2	0.1	0.1	0.8	21.15	0.10	0.10	25.00	0.10	0.10	20.83	0.10	0.10	17.27	0.10	0.10
0.5	0.5	1.2	1.1	0.5	0.5	0.1	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10	22.73	0.10	0.10
0.5	1	1.5	2	0.1	0.5	0.1	16.42	0.10	0.10	25.00	0.10	0.10	22.27	0.10	0.10	18.56	0.10	0.10
0.5	1	2	1.1	0.5	1	0.5	19.14	0.10	0.10	7.50	0.10	0.10	8.27	0.10	0.10	14.49	0.10	0.10
0.5	1	1.2	1.5	1	0.1	0.8	17.35	0.10	0.10	19.63	0.10	0.10	17.24	0.10	0.10	14.00	0.10	0.10
0.5	0.1	1.5	1.1	0.5	0.1	0.8	8.00	0.10	0.10	9.74	0.10	0.14	25.00	0.10	0.10	21.14	0.10	0.10
0.5	0.1	2	1.5	1	0.5	0.1	18.74	0.10	0.10	19.18	0.10	0.10	14.83	0.10	0.10	8.00	0.10	0.10
0.5	0.1	1.2	2	0.1	1	0.5	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10
1	1	2	2	0.5	0.5	0.8	8.00	0.10	0.10	8.00	0.10	0.10	8.00	0.10	0.10	8.00	0.10	0.10
1	1	1.2	1.1	1	1	0.1	12.74	0.10	0.10	16.00	0.10	0.10	13.11	0.10	0.10	8.00	0.10	0.10
1	1	1.5	1.5	0.1	0.1	0.5	19.22	0.10	0.10	23.38	0.10	0.10	19.19	0.10	0.10	15.46	0.10	0.10
1	0.1	2	1.1	1	0.1	0.5	16.47	0.10	0.10	14.88	0.10	0.10	8.00	0.10	0.10	8.00	0.10	0.10
1	0.1	1.2	1.5	0.1	0.5	0.8	25.00	0.10	0.10	25.00	0.10	0.10	25.00	0.10	0.10	21.67	0.10	0.10
1	0.1	1.5	2	0.5	1	0.1	16.94	0.10	0.10	20.98	0.10	0.10	16.83	0.10	0.10	12.97	0.10	0.10
1	0.5	2	1.5	0.1	1	0.1	17.41	0.10	0.10	25.00	0.10	0.10	19.63	0.10	0.10	14.72	0.10	0.10
1	0.5	1.2	2	0.5	0.1	0.5	19.49	0.10	0.10	21.42	0.10	0.10	17.37	0.10	0.10	13.67	0.10	0.10
1	0.5	1.5	1.1	1	0.5	0.8	17.32	0.10	0.10	17.72	0.10	0.10	13.47	0.10	0.10	8.00	0.10	0.10

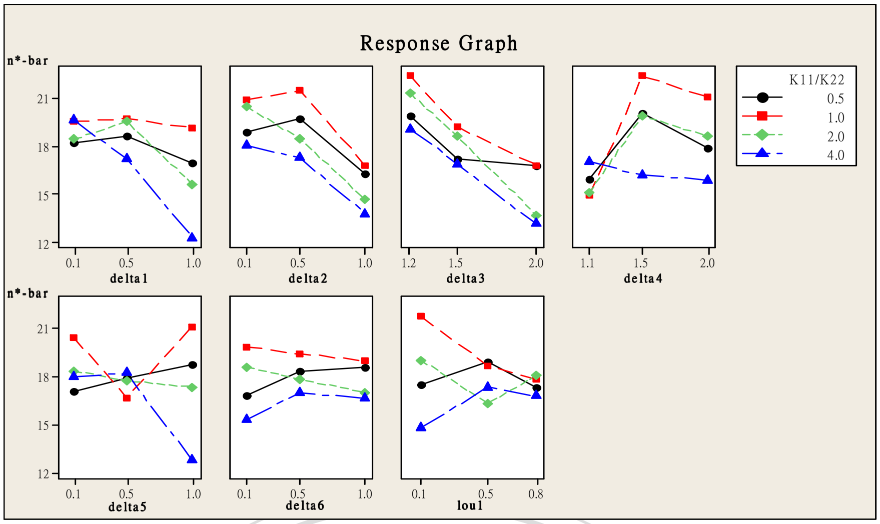

Figure 12. Response Graph for n*-bar with $\rho_0 = 0.1$ Based on Table 14

Table 15. The Optimal (n^*, h^*, ATS_1^*) for BL Chart Given $\rho_0 = 0.5$ and $\alpha = 0.0027$

							K_{11}/K_{22} = 0.5			K_{11}/K_{22} = 1			K_{11}/K_{22} = 2			K_{11}/K_{22} = 4		
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*	n^*	h^*	Opt.ATS_1^*
0.1	0.1	1.2	1.1	0.1	0.1	0.1	25.00	0.1	0.10	25.00	0.1	0.11	25.00	0.1	0.10	25.00	0.1	0.10
0.1	0.1	1.5	1.5	0.5	0.5	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.1	0.1	2	2	1	1	0.8	17.76	0.1	0.10	20.29	0.1	0.10	18.17	0.1	0.10	15.06	0.1	0.10
0.1	0.5	1.2	1.5	0.5	1	0.8	22.64	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.1	0.5	1.5	2	1	0.1	0.1	18.53	0.1	0.10	20.83	0.1	0.10	18.38	0.1	0.10	15.76	0.1	0.10
0.1	0.5	2	1.1	0.1	0.5	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	24.22	0.1	0.10
0.1	1	1.2	2	1	0.5	0.5	13.64	0.1	0.10	17.67	0.1	0.10	17.76	0.1	0.10	16.08	0.1	0.10
0.1	1	1.5	1.1	0.5	1	0.8	19.50	0.1	0.10	23.00	0.1	0.10	23.00	0.1	0.10	20.10	0.1	0.10
0.1	1	2	1.5	1	0.1	0.1	15.01	0.1	0.10	17.06	0.1	0.10	15.55	0.1	0.10	8.00	0.1	0.10
0.5	0.5	1.5	1.5	1	1	0.5	15.59	0.1	0.10	18.51	0.1	0.10	16.59	0.1	0.10	13.55	0.1	0.10
0.5	0.5	2	2	0.1	0.1	0.8	19.78	0.1	0.10	22.59	0.1	0.10	20.20	0.1	0.10	16.83	0.1	0.10
0.5	0.5	1.2	1.1	0.5	0.5	0.1	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	22.61	0.1	0.10
0.5	1	1.5	2	0.1	0.5	0.1	15.11	0.1	0.10	21.38	0.1	0.10	21.17	0.1	0.10	18.43	0.1	0.10
0.5	1	2	1.1	0.5	1	0.5	15.31	0.1	0.10	22.48	0.1	0.10	18.79	0.1	0.10	14.30	0.1	0.10
0.5	1	1.2	1.5	1	0.1	0.8	16.42	0.1	0.10	18.71	0.1	0.10	16.59	0.1	0.10	13.93	0.1	0.10
0.5	0.1	1.5	1.1	0.5	0.1	0.8	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	21.00	0.1	0.10
0.5	0.1	2	1.5	1	0.5	0.1	17.43	0.1	0.10	17.59	0.1	0.10	14.52	0.1	0.10	8.00	0.1	0.10
0.5	0.1	1.2	2	0.1	1	0.5	22.91	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
1	1	2	2	0.5	0.5	0.8	8.00	0.1	0.10	12.89	0.1	0.10	8.00	0.1	0.10	8.00	0.1	0.10
1	1	1.2	1.1	1	1	0.1	11.34	0.1	0.10	14.04	0.1	0.10	12.43	0.1	0.10	8.00	0.1	0.10
1	1	1.5	1.5	0.1	0.1	0.5	17.53	0.1	0.10	20.82	0.1	0.10	18.20	0.1	0.10	15.38	0.1	0.10
1	0.1	2	1.1	1	0.1	0.5	15.66	0.1	0.10	13.75	0.1	0.10	8.00	0.1	0.10	8.00	0.1	0.10
1	0.1	1.2	1.5	0.1	0.5	0.8	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	21.54	0.1	0.10
1	0.1	1.5	2	0.5	1	0.1	15.54	0.1	0.10	18.30	0.1	0.10	15.90	0.1	0.10	12.90	0.1	0.10
1	0.5	2	1.5	0.1	1	0.1	15.54	0.1	0.10	20.83	0.1	0.10	18.41	0.1	0.10	14.58	0.1	0.10
1	0.5	1.2	2	0.5	0.1	0.5	18.15	0.1	0.10	19.78	0.1	0.10	17.03	0.1	0.10	13.64	0.1	0.10
1	0.5	1.5	1.1	1	0.5	0.8	16.16	0.1	0.10	16.55	0.1	0.10	13.23	0.1	0.10	8.00	0.1	0.10

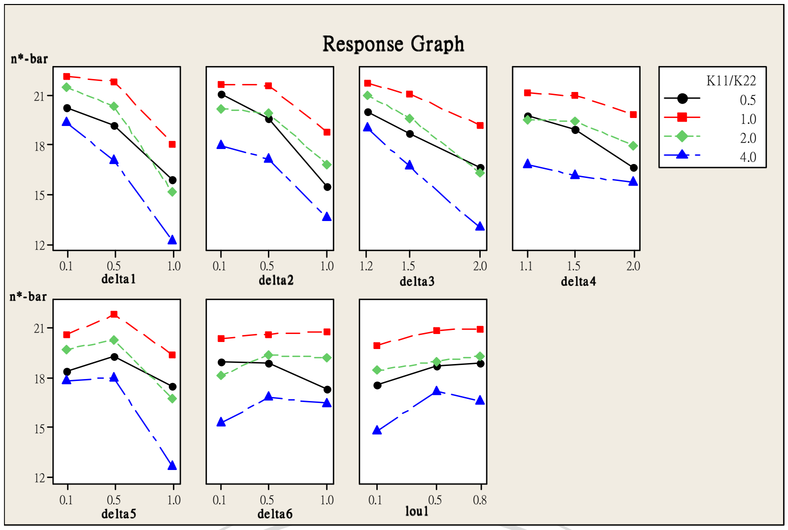

Figure 13. Response Graph for n^*-bar with $\rho_0 = 0.5$ Based on Table 15

Table 16. The Optimal (n^*, h^*, ATS_1^*) for BL Chart Given $\rho_0 = 0.8$ and $\alpha = 0.0027$

							K_{11}/K_{22} = 0.5			**K_{11}/K_{22} = 1**			**K_{11}/K_{22} = 2**			**K_{11}/K_{22} = 4**		
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$
0.1	0.1	1.2	1.1	0.1	0.1	0.1	25.00	0.1	0.10	25.00	0.1	0.11	25.00	0.1	0.10	23.00	0.1	0.10
0.1	0.1	1.5	1.5	0.5	0.5	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.1	0.1	2	2	1	1	0.8	16.97	0.1	0.10	19.61	0.1	0.10	17.89	0.1	0.10	14.98	0.1	0.10
0.1	0.5	1.2	1.5	0.5	1	0.8	21.60	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	24.89	0.1	0.10
0.1	0.5	1.5	2	1	0.1	0.1	18.27	0.1	0.10	20.02	0.1	0.10	18.20	0.1	0.10	15.44	0.1	0.10
0.1	0.5	2	1.1	0.1	0.5	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	24.14	0.1	0.10
0.1	1	1.2	2	1	0.5	0.5	13.45	0.1	0.10	17.17	0.1	0.10	17.52	0.1	0.10	16.01	0.1	0.10
0.1	1	1.5	1.1	0.5	1	0.8	18.46	0.1	0.10	24.38	0.1	0.10	23.91	0.1	0.10	19.91	0.1	0.10
0.1	1	2	1.5	1	0.1	0.1	14.49	0.1	0.10	16.42	0.1	0.10	15.41	0.1	0.10	8.00	0.1	0.10
0.5	0.5	1.5	1.5	1	1	0.5	14.94	0.1	0.10	17.82	0.1	0.10	15.99	0.1	0.10	13.49	0.1	0.10
0.5	0.5	2	2	0.1	0.1	0.8	18.97	0.1	0.10	21.89	0.1	0.10	19.92	0.1	0.10	16.79	0.1	0.10
0.5	0.5	1.2	1.1	0.5	0.5	0.1	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	22.56	0.1	0.10
0.5	1	1.5	2	0.1	0.5	0.1	14.48	0.1	0.10	20.34	0.1	0.10	20.73	0.1	0.10	18.35	0.1	0.10
0.5	1	2	1.1	0.5	1	0.5	14.65	0.1	0.10	18.96	0.1	0.10	17.52	0.1	0.10	14.21	0.1	0.10
0.5	1	1.2	1.5	1	0.1	0.8	16.22	0.1	0.10	17.92	0.1	0.10	16.44	0.1	0.10	13.89	0.1	0.10
0.5	0.1	1.5	1.1	0.5	0.1	0.8	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	20.94	0.1	0.10
0.5	0.1	2	1.5	1	0.5	0.1	16.76	0.1	0.10	17.14	0.1	0.10	14.37	0.1	0.10	8.00	0.1	0.10
0.5	0.1	1.2	2	0.1	1	0.5	21.80	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
1	1	2	2	0.5	0.5	0.8	8.00	0.1	0.10	8.00	0.1	0.10	8.00	0.1	0.10	8.00	0.1	0.10
1	1	1.2	1.1	1	1	0.1	8.00	0.1	0.10	13.28	0.1	0.10	12.03	0.1	0.10	8.00	0.1	0.10
1	1	1.5	1.5	0.1	0.1	0.5	17.16	0.1	0.10	19.64	0.1	0.10	17.92	0.1	0.10	15.01	0.1	0.10
1	0.1	2	1.1	1	0.1	0.5	15.43	0.1	0.10	13.56	0.1	0.10	8.00	0.1	0.10	8.00	0.1	0.10
1	0.1	1.2	1.5	0.1	0.5	0.8	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	21.46	0.1	0.10
1	0.1	1.5	2	0.5	1	0.1	14.91	0.1	0.10	17.58	0.1	0.10	15.62	0.1	0.10	12.86	0.1	0.10
1	0.5	2	1.5	0.1	1	0.1	14.85	0.1	0.10	19.28	0.1	0.10	17.55	0.1	0.10	14.50	0.1	0.10
1	0.5	1.2	2	0.5	0.1	0.5	17.85	0.1	0.10	19.28	0.1	0.10	16.53	0.1	0.10	13.62	0.1	0.10
1	0.5	1.5	1.1	1	0.5	0.8	15.88	0.1	0.10	15.83	0.1	0.10	12.85	0.1	0.10	8.00	0.1	0.10

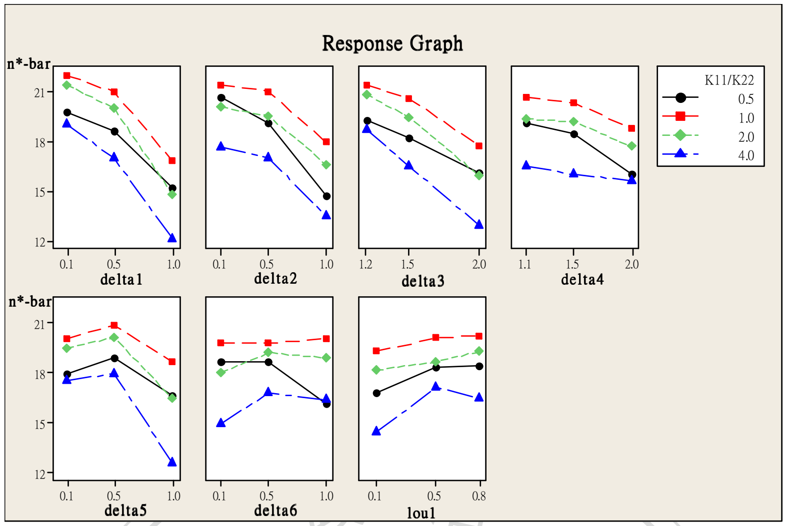

Figure 14. Response Graph for n*-bar with $\rho_0 = 0.8$ Based on Table 16

From Tables 14 – 16, we could see that n* changes and h* is fixed at 0.1. Also, we found that the ratio of K_{11}/K_{22} does not affect the performance too much, and whether $\rho_0 = 0.1$, 0.5 or 0.8, ATS_1 is near 0.1. We could say that the optimal BL chart is insensitive to these combinations of all process shifts.

From Figures 12 – 14, we observed that the response of n*-bar under $\rho_0 = 0.1$ is different from that of $\rho_0 = 0.5$ and 0.8. Under $\rho_0 = 0.1$, there is no fixed pattern when $K_{11}/K_{22} = 1$. But for $\rho_0 = 0.5$ and 0.8, the effects of all shifts on n*-bar have a fixed pattern. When $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ increase, n*-bar decreases. Under $K_{11}/K_{22} = 4$, the optimal BL chart takes smaller samples to detect process shifts. While $K_{11}/K_{22} = 1$, the optimal BL chart needs more samples.

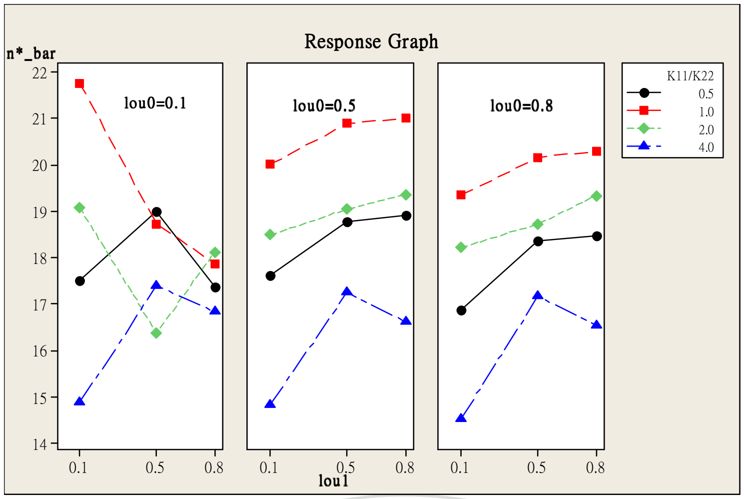

Figure 15. Response Graph for Comparing n*-bar under $\rho_0 = 0.1, 0.5, 0.8$

From Figure 15, we could see that the significant difference of responses of n^*-bar between $\rho_0 = 0.1$, $\rho_0 = 0.5$ and 0.8. Also, the pattern is more stable under $K_{11}/K_{22} = 4$.

We also investigated the performance of optimal BL chart of smaller shift (See Appendix B, Table 52 and Figure 21). Only n^* changes and the $ATS_1{}^*$ is near 0.1. When the process has smaller shifts, the optimal solution for n^* is quite large. Most of n^* are over 15. The performance is very good, but it requires more samples to detect process shifts. The response graph for n*-bar shows that the optimal BL chart needs less samples to detect the shift, except for $K_{11}/K_{22} = 1$.

2.10 ATS_1 Comparison of the Specified Bivariate Loss Chart and the Optimal Bivariate Loss Chart

(1). To compare the performance between two charts, we use 'time saved%' to measure. Define Saved%= $\frac{Spe.ATS_1 - Opt.ATS_1}{Spe.ATS_1} \times 100\%$, where Opt. ATS_1 is the ATS_1 of the optimal BL chart and Spe.ATS_1 is a ATS_1 of the specified BL chart. The Saved% shows how much time the optimal BL chart had saved in detecting the process shifts with an out-of-control process when comparing with the specified BL chart.

(2). Set sampling interval for the specified BL chart as 1, we could calculate Spe.ATS_1 based on Tables 11 – 13. Also, we calculate Saved% based on Tables 11 – 13 and Tables 14 – 16. The results are shown in Tables 17 – 19.

Table 17. Saved% of Optimal BL Chart and Specified BL Chart for Small Shifts with

$\rho_0 = 0.1$

No.	$K_{11}/K_{22} = 0.5$			$K_{11}/K_{22} = 1$			$K_{11}/K_{22} = 2$			$K_{11}/K_{22} = 4$		
	$Spe.ATS_1$	$Opt.ATS_1$	$Saved\%$	$Spe.ATS_1$	$Opt.ATS_1$	$Saved\%$	$Spe.ATS_1$	$Opt.ATS_1$	$Saved\%$	$Spe.ATS_1$	$Opt.ATS_1$	$Saved\%$
1	59.31	0.10	99.83%	47.16	0.16	99.65%	40.06	0.10	99.75%	3.36	0.10	97.02%
2	2.70	0.10	96.29%	2.81	0.10	96.45%	2.84	0.10	96.48%	1.35	0.10	92.58%
3	1.12	0.10	91.05%	1.19	0.10	91.56%	1.12	0.10	91.04%	1.02	0.10	90.16%
4	1.83	0.10	94.54%	2.38	0.10	95.80%	4.02	0.10	97.51%	1.79	0.10	94.42%
5	1.18	0.10	91.52%	1.43	0.10	93.02%	1.14	0.10	91.20%	1.02	0.10	90.22%
6	2.00	0.10	95.01%	1.82	0.12	93.39%	1.85	0.13	92.84%	1.25	0.10	92.00%
7	1.03	0.10	90.31%	1.23	0.10	91.86%	1.17	0.10	91.48%	1.05	0.10	90.44%
8	1.45	0.10	93.13%	1.61	0.13	92.04%	2.09	0.13	93.57%	1.28	0.10	92.16%
9	1.05	0.10	90.48%	1.12	0.10	91.10%	1.03	0.10	90.31%	1.00	0.10	90.04%
10	1.15	0.10	91.30%	1.27	0.10	92.13%	1.15	0.10	91.30%	1.01	0.10	90.13%
11	1.09	0.10	90.84%	1.10	0.10	90.89%	1.15	0.10	91.34%	1.03	0.10	90.32%
12	3.61	0.10	97.23%	3.78	0.10	97.35%	3.63	0.10	97.25%	1.35	0.10	92.58%
13	1.03	0.10	90.26%	1.08	0.10	90.74%	1.26	0.10	92.04%	1.12	0.10	91.05%
14	1.09	0.10	90.85%	1.09	0.10	90.59%	1.12	0.10	90.97%	1.02	0.10	90.20%
15	1.10	0.10	90.89%	1.30	0.10	92.29%	1.10	0.10	90.88%	1.01	0.10	90.13%
16	2.42	0.10	95.86%	2.15	0.14	93.50%	1.61	0.10	93.79%	1.10	0.10	90.89%
17	1.15	0.10	91.32%	1.16	0.10	91.37%	1.02	0.10	90.24%	1.00	0.10	90.01%
18	1.37	0.10	92.71%	1.58	0.10	93.66%	2.15	0.10	95.35%	1.85	0.10	94.61%
19	1.00	0.10	90.02%	1.00	0.10	90.02%	1.00	0.10	90.03%	1.00	0.10	90.00%
20	1.05	0.10	90.45%	1.12	0.10	91.09%	1.06	0.10	90.54%	1.00	0.10	90.02%
21	1.09	0.10	90.79%	1.10	0.10	90.91%	1.17	0.10	91.42%	1.02	0.10	90.24%
22	1.05	0.10	90.45%	1.02	0.10	90.23%	1.00	0.10	90.01%	1.00	0.10	90.00%
23	1.71	0.10	94.15%	1.70	0.10	94.10%	1.93	0.10	94.83%	1.26	0.10	92.06%
24	1.10	0.10	90.91%	1.09	0.10	90.81%	1.11	0.10	90.99%	1.01	0.10	90.11%
25	1.05	0.10	90.52%	1.05	0.10	90.45%	1.05	0.10	90.52%	1.02	0.10	90.22%
26	1.14	0.10	91.22%	1.16	0.10	91.39%	1.12	0.10	91.05%	1.01	0.10	90.12%
27	1.10	0.10	90.89%	1.13	0.10	91.11%	1.02	0.10	90.16%	1.00	0.10	90.00%

Table 18. Saved% of Optimal BL Chart and Specified BL Chart for Small Shifts with $\rho_0 = 0.5$

No.	$K_{11}/K_{22} = 0.5$			$K_{11}/K_{22} = 1$			$K_{11}/K_{22} = 2$			$K_{11}/K_{22} = 4$		
	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %
1	82.37	0.10	99.88%	195.36	0.11	99.94%	30.62	0.10	99.67%	3.20	0.10	96.88%
2	15.63	0.10	99.36%	76.42	0.10	99.87%	18.14	0.10	99.45%	2.94	0.10	96.60%
3	4.58	0.10	97.82%	23.02	0.10	99.57%	5.99	0.10	98.33%	1.74	0.10	94.25%
4	1.62	0.10	93.82%	3.47	0.10	97.12%	3.98	0.10	97.49%	1.73	0.10	94.21%
5	1.89	0.10	94.71%	3.79	0.10	97.36%	2.31	0.10	95.67%	1.28	0.10	92.19%
6	1.18	0.10	91.53%	1.88	0.10	94.66%	1.38	0.10	92.75%	1.07	0.10	90.65%
7	1.02	0.10	90.17%	1.17	0.10	91.48%	1.13	0.10	91.19%	1.04	0.10	90.40%
8	1.02	0.10	90.20%	1.11	0.10	90.99%	1.09	0.10	90.83%	1.03	0.10	90.29%
9	1.00	0.10	90.00%	1.03	0.10	90.29%	1.02	0.10	90.20%	1.00	0.10	90.00%
10	1.48	0.10	93.24%	1.71	0.10	94.15%	1.87	0.10	94.65%	1.21	0.10	91.74%
11	1.16	0.10	91.38%	1.43	0.10	93.01%	1.40	0.10	92.86%	1.09	0.10	90.83%
12	3.85	0.10	97.40%	9.61	0.10	98.96%	3.42	0.10	97.08%	1.36	0.10	92.66%
13	2.32	0.10	95.69%	2.06	0.10	95.15%	1.71	0.10	94.15%	1.13	0.10	91.15%
14	1.31	0.10	92.37%	1.48	0.10	93.24%	2.40	0.10	95.83%	1.04	0.10	90.38%
15	1.06	0.10	90.60%	1.20	0.10	91.65%	1.08	0.10	90.72%	1.01	0.10	90.12%
16	1.32	0.10	92.42%	1.80	0.10	94.44%	2.64	0.10	96.21%	1.49	0.10	93.29%
17	1.08	0.10	90.74%	1.69	0.10	94.08%	2.01	0.10	95.02%	1.23	0.10	91.87%
18	1.52	0.10	93.43%	2.01	0.10	95.01%	2.79	0.10	96.41%	1.76	0.10	94.32%
19	1.20	0.10	91.67%	1.17	0.10	91.45%	1.13	0.10	91.15%	1.03	0.10	90.29%
20	1.02	0.10	90.23%	1.23	0.10	91.86%	1.04	0.10	90.41%	1.00	0.10	90.02%
21	1.02	0.10	90.20%	1.02	0.10	90.20%	1.00	0.10	90.00%	1.00	0.10	90.00%
22	1.09	0.10	90.83%	1.22	0.10	91.80%	1.46	0.10	93.15%	1.20	0.10	91.67%
23	2.05	0.10	95.13%	2.34	0.10	95.73%	2.32	0.10	95.68%	1.24	0.10	91.93%
24	1.03	0.10	90.29%	1.12	0.10	91.07%	1.03	0.10	90.29%	1.00	0.10	90.00%
25	1.20	0.10	91.67%	1.25	0.10	92.00%	1.29	0.10	92.25%	1.07	0.10	90.65%
26	1.11	0.10	90.95%	1.27	0.10	92.13%	1.09	0.10	90.85%	1.01	0.10	90.11%
27	1.01	0.10	90.10%	1.04	0.10	90.38%	1.00	0.10	90.00%	1.00	0.10	90.00%

Table 19. Saved% of Optimal BL Chart and Specified BL Chart for Small Shifts with $\rho_0 = 0.8$

No.	$K_{11}/K_{22} = 0.5$			$K_{11}/K_{22} = 1$			$K_{11}/K_{22} = 2$			$K_{11}/K_{22} = 4$		
	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %
1	52.50	0.10	99.81%	254.84	0.10	99.96%	24.46	0.10	99.59%	3.11	0.10	96.76%
2	11.87	0.10	99.16%	71.89	0.10	99.86%	15.11	0.10	99.34%	2.87	0.10	96.52%
3	3.65	0.10	97.26%	14.64	0.10	99.32%	5.09	0.10	98.04%	1.70	0.10	94.12%
4	1.47	0.10	93.22%	4.30	0.10	97.67%	3.48	0.10	97.12%	1.69	0.10	94.08%
5	1.72	0.10	94.19%	3.70	0.10	97.30%	2.15	0.10	95.35%	1.27	0.10	92.13%
6	1.13	0.10	91.15%	1.64	0.10	93.90%	1.32	0.10	92.42%	1.07	0.10	90.65%
7	1.01	0.10	90.13%	1.13	0.10	91.16%	1.12	0.10	91.05%	1.04	0.10	90.38%
8	1.02	0.10	90.20%	1.11	0.10	90.99%	1.08	0.10	90.74%	1.02	0.10	90.20%
9	1.00	0.10	90.00%	1.02	0.10	90.20%	1.02	0.10	90.20%	1.00	0.10	90.00%
10	1.38	0.10	92.75%	2.08	0.10	95.19%	1.78	0.10	94.38%	1.20	0.10	91.67%
11	1.12	0.10	91.07%	1.54	0.10	93.51%	1.34	0.10	92.54%	1.08	0.10	90.74%
12	3.19	0.10	96.87%	9.25	0.10	98.92%	3.08	0.10	96.75%	1.35	0.10	92.58%
13	2.05	0.10	95.12%	2.59	0.10	96.14%	1.63	0.10	93.87%	1.12	0.10	91.07%
14	1.23	0.10	91.87%	1.79	0.10	94.41%	1.30	0.10	92.31%	1.04	0.10	90.38%
15	1.05	0.10	90.50%	1.16	0.10	91.38%	1.07	0.10	90.65%	1.01	0.10	90.11%
16	1.24	0.10	91.94%	2.30	0.10	95.65%	2.43	0.10	95.88%	1.46	0.10	93.15%
17	1.06	0.10	90.57%	1.97	0.10	94.92%	1.82	0.10	94.51%	1.21	0.10	91.74%
18	1.40	0.10	92.88%	2.49	0.10	95.99%	3.36	0.10	97.02%	1.71	0.10	94.14%
19	1.16	0.10	91.38%	1.24	0.10	91.94%	1.17	0.10	91.45%	1.03	0.10	90.29%
20	1.01	0.10	90.13%	1.14	0.10	91.21%	1.03	0.10	90.30%	1.00	0.10	90.01%
21	1.01	0.10	90.10%	1.01	0.10	90.10%	1.00	0.10	90.00%	1.00	0.10	90.00%
22	1.06	0.10	90.57%	1.34	0.10	92.54%	1.61	0.10	93.79%	1.18	0.10	91.53%
23	1.84	0.10	94.58%	3.11	0.10	96.78%	2.17	0.10	95.39%	1.23	0.10	91.84%
24	1.02	0.10	90.20%	1.08	0.10	90.74%	1.02	0.10	90.20%	1.00	0.10	90.00%
25	1.14	0.10	91.23%	1.42	0.10	92.96%	1.40	0.10	92.86%	1.07	0.10	90.65%
26	1.08	0.10	90.78%	1.22	0.10	91.80%	1.08	0.10	90.75%	1.01	0.10	90.10%
27	1.01	0.10	90.10%	1.03	0.10	90.29%	1.00	0.10	90.00%	1.00	0.10	90.00%

Tables 17 – 19 showed that the optimal BL chart gave ATS_1 saved% a minimum of 90% to detect process signal. But, the optimal BL chart requires more samples to monitor the process. The same result of detecting smaller shifts (See Appendix B, Table 53). In practice, we suggest using specified BL chart since it requires less samples to detect process signal and it is easy to implement.

Chapter 3. The VSSI Bivaraite Loss Control Chart

3.1 Design of the VSSI Bivariate Loss Chart

To detect signals quickly and improve the sensitivity in detecting the process small shifted, we add two warning limits. Thus, the control chart is divided into three main regions: the region between the warning limits is called central region, the region between the warning limit and upper control limit or the warning limit and lower control limit is warning region, and the region above upper control limit or below the lower control limit is called action region. We take different sampling rules of monitoring the process for each region. We have two types of sample size and sampling interval, i.e. (n_1, h_1) and (n_2, h_2), where $n_2 > n_0 > n_1$ and $h_1 > h_0 > h_2$, n_0 and h_0 are sample size and sampling interval for the fixed parameters (FP) BL chart. Note that at the first sampling we take sample of size n_1.

To monitor the process, we follow the rules below (see Figure 16):

(i) When a point falls within the central region, next time, adopt the long sampling interval and small sample size, (h_1, n_1).

(ii) When a point falls within warning region, next time, adopt the short sampling interval and large sample size, (h_2, n_2).

(iii)When a point falls in the action region we will stop the process and look for any special causes and rectify them.

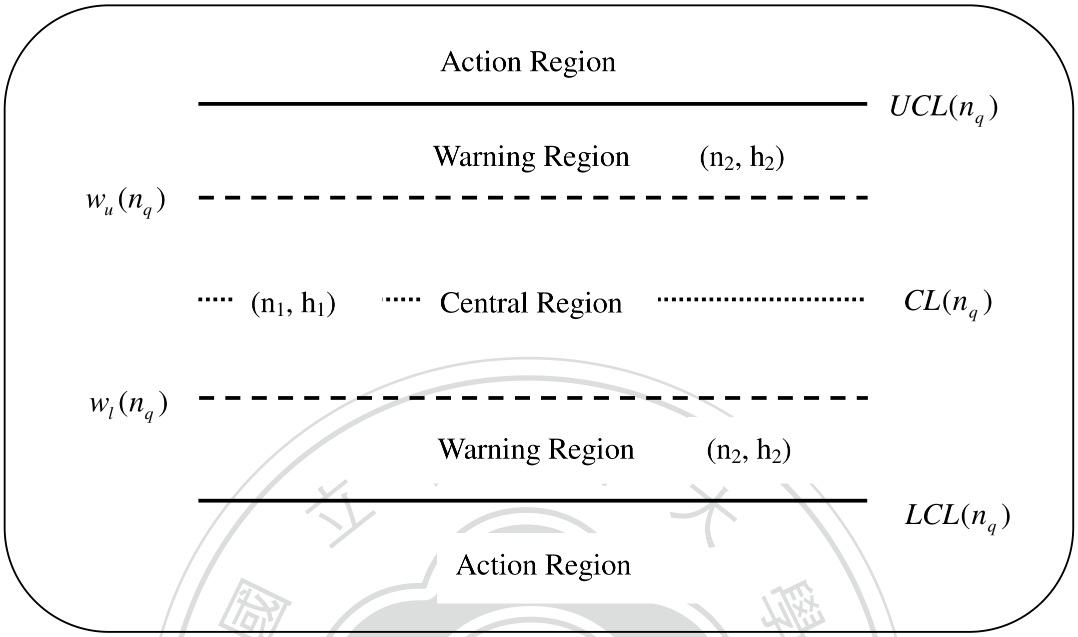

Figure 16. The VSSI BL chart

Note that the control limits depend on the variable sample size (n_q , q = 1, 2)

That is, we may express the UCL and LCL as $UCL(n_q)$ and $LCL(n_q)$.

3.2 The Approximate Distribution of BL

When process is in-control,

$$BL \overset{\bullet}{\sim} Q(n_q) = Ar_1(n_q)\chi^2_{v_1(n_q)} + Br_2(n_q)\chi^2_{v_2(n_q)}, \text{ q = 1, 2,} \qquad (12)$$

where

$$r_1(n_q) = 1 + \frac{\tau_{01}}{n_q + \tau_{01}}, v_1(n_q) = n_q + \frac{\tau_{01}^2}{n_q + 2\tau_{01}} \text{ and } r_2(n_q) = 1 + \frac{\tau_{02}}{n_q + \tau_{02}},$$

$$v_2(n_q) = n_q + \frac{\tau_{02}^2}{n_q + 2\tau_{02}}, q = 1,2.$$

3.3 The Control Limits of the VSSI BL Chart

Let false alarm rate of the VSSI BL chart is α

$$F_{Q(n_q)}(UCL(n_q)) = 1 - \alpha / 2 \tag{13}$$

$$F_{Q(n_q)}(LCL(n_q)) = \alpha / 2 \tag{14}$$

$$p^*(n_q) = F_{Q(n_q)}(CL(n_q)), q = 1,2 \tag{15}$$

The control limits of the VSSI BL chart are:

$$UCL(n_q) \doteq F^{-1}_{Q(n_q)}(1 - \alpha / 2)$$

$$CL(n_q) \doteq F^{-1}_{Q(n_q)}(p^*(n_q))$$

$$LCL(n_q) \doteq F^{-1}_{Q(n_q)}(\alpha / 2)$$

To calculate the warning limits, we assume that the probabilities (p_0) that a point falls on the central region for these two charts are equal when the process is in-control. That is,

$$p_0 = P(w_l(n_q) < BL < w_u(n_q) \,|\, LCL(n_q) < BL < UCL(n_q), BL \dot{\sim} Q(n_q)), \text{ q} = 1,2 \tag{16}$$

From equation (16), we get

$$p_0 = \frac{F_{Q(n_q)}(w_u(n_q)) - F_{Q(n_q)}(w_l(n_q))}{1 - \alpha} \tag{17}$$

Let

$$P(w_l(n_q) < BL < w_u(n_q) \,|\, BL \dot{\sim} Q(n_q)) = 2P(w_l(n_q) < BL < CL(n_q) \,|\, BL \dot{\sim} Q(n_q)) \tag{18}$$

Hence, from equation (16) and equation (18),

$$2(p^*(n_q) - F_{Q(n_q)}(w_l(n_q))) = p_0(1 - \alpha),$$

and we get $w_l(n_q)$ in (19)

$$w_l(n_q) = F^{-1}_{Q(n_q)}(p^*(n_q) - \frac{1}{2}p_0(1-\alpha)) \quad (19)$$

To get $w_u(n_q)$, let

$$P(w_l(n_q) < BL < w_u(n_q) \mid BL \overset{\bullet}{\sim} Q(n_q)) = 2P(w_u(n_q) < BL < UCL(n_q) \mid BL \overset{\bullet}{\sim} Q(n_q)) \quad (20)$$

From equation (16) and (19),

$$2(F_{Q(n_q)}(w_u(n_q)) - p^*(n_q)) = p_0(1-\alpha),$$

so

$$w_u(n_q) = F^{-1}_{Q(n_q)}(p^*(n_q) + \frac{1}{2}p_0(1-\alpha)) \quad (21)$$

3.4 The Out-of-control Approximate Distribution of BL

When process is out-of-control,

$$BL \overset{\bullet}{\sim} Q^*(n_q) = A' r_1'(n_q)\chi^2_{v_1'(n_q)} + B' r_2'(n_q)\chi^2_{v_2'(n_q)}, \text{ q = 1, 2,} \quad (22)$$

where

$$r_1'(n_q) = 1 + \frac{\tau_{11}}{n_q + \tau_{11}}, v_1'(n_q) = n_q + \frac{\tau_{11}^2}{n_q + 2\tau_{11}} \text{ and } r_2'(n_q) = 1 + \frac{\tau_{12}}{n_q + \tau_{12}},$$

$$v_2'(n_q) = n_q + \frac{\tau_{12}^2}{n_q + 2\tau_{12}}, q = 1,2.$$

3.5 Performance Measurement of the VSSI BL Chart

In order to compare performance with the FP BL chart, we let them have same average sample size and sampling interval under in-control process.

$$p_0 n_1 + (1-p_0)n_2 = n_0 \quad (23)$$

$$p_0 h_1 + (1-p_0)h_2 = h_0 \quad (24)$$

From equation (23) and (24), we have

$$p_0 = \frac{n_2 - n_0}{n_2 - n_1} = \frac{h_2 - h_0}{h_2 - h_1} \tag{25}$$

And given (n_0, n_1, n_2, h_0, h_2), put $p_0 = \frac{n_2 - n_0}{n_2 - n_1}$ in equation (24)

$$h_1 = \frac{h_0(n_2 - n_1) - h_2(n_0 - n_1)}{n_2 - n_0} \tag{26}$$

We use ATS_1 as our performance measurement. When the process is out-of-control, we apply Markov chain method to find ATS_1. Assume that the process is out-of-control at the beginning, we observe the possible process states at the end of the sampling interval.

State 1: If statistic BL falls within central region, next time, we choose (h_1, n_1).

State2: If statistic BL falls warning region, next time, we choose (h_2, n_2).

State3: If statistic BL falls in the action region, then the true alarm occurred and the process is adjusted.

Define transient matrix P^*

$$P^* = \begin{bmatrix} p_{11}^* & p_{12}^* & p_{13}^* \\ p_{21}^* & p_{22}^* & p_{23}^* \\ p_{31}^* & p_{32}^* & p_{33}^* \end{bmatrix}$$

where

$$p_{11}^*(h_1, n_1) = P(w_l(n_1) < BL < w_u(n_1) \,|\, BL \overset{\bullet}{\sim} Q^*(n_1)) \quad = F_{Q^*(n_1)}(w_u(n_1)) - F_{Q^*(n_1)}(w_l(n_1))$$

$$p_{12}^*(h_1, n_1) = P(w_u(n_1) < BL < UCL(n_1) \ or \ LCL(n_1) < BL < w_l(n_1) \,|\, BL \overset{\bullet}{\sim} Q^*(n_1))$$

$$= (F_{Q^*(n_1)}(UCL(n_1)) - F_{Q^*(n_1)}(w_u(n_1))) + (F_{Q^*(n_1)}(w_l(n_1)) - F_{Q^*(n_1)}(LCL(n_1))$$

$$p_{13}^*(h_1,n_1) = P(BL > UCL(n_1)\, or\, BL < LCL(n_1) \mid BL \overset{\bullet}{\sim} Q^*(n_1)) = 1 - p_{11}^*(h_1,n_1) - p_{12}^*(h_1,n_1)$$

$$p_{21}^*(h_2,n_2) = p_{11}^*(h_2,n_2), p_{22}^*(h_2,n_2) = p_{12}^*(h_2,n_2), p_{23}^*(h_2,n_2) = p_{13}^*(h_2,n_2), p_{31}^* = p_{32}^* = 0, p_{33}^* = 1$$

Let $Q^* = \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix}$,

then

$$p_1 = P(w_l(n_1) < BL < w_u(n_1) \mid LCL(n_1) < BL < UCL(n_1), BL \overset{\bullet}{\sim} Q^*(n_1)) .$$

Denote $\pi^{*}{}' = (\pi_1, \pi_2)$ with initial probabilities for state 1 and 2, where $\pi^{*}{}'$ could be obtained by solving the equation

$$\pi^{*}{}' Q^* = \pi^{*}{}'$$

$$s.t \sum_{j=1}^{2} .\pi^{*}{}_{1j} = 1,$$

and the solution is

$$\pi^{*}{}' = \left(\frac{p_{21}^*}{1 + p_{21}^* - p_{11}^*}, \frac{1 - p_{11}^*}{1 + p_{21}^* - p_{11}^*} \right)$$

Hence, the ATS_1 is obtained by (27)

$$ATS_1 = \pi^{*}{}'(I - Q^*)^{-1} h, \tag{27}$$

where $h = (h_1, h_2)$.

3.6 Example for the VSSI Bivariate Loss Chart

Continue with the example of the BL chart, with out-of-control data (19, 21, 22, 23, 24, 25 subgroups) (See Table 2.), we could get δ_1= -1.28, δ_2= -1.47, δ_3= 0.85, δ_4= 1.02, δ_5= 0.57, δ_6= -0.79, ρ_1= -0.3. Then, we find the optimal (n_1, n_2, h_1, h_2, p (n_1), p (n_2)) under specified δ_1 , δ_2 , δ_3 , δ_4 , δ_5 , δ_6 , ρ_1 and calculate control limits for the VSSI BL chart. The construction procedure is as follows:

Step 1: Give $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \end{pmatrix}$, $K_{11} = 0.5, K_{12} = K_{22} = 1$, n = 4, $\alpha = 0.0027$,

$n_0 = 4, h_0 = 1$ and $c_4 = 0.9213$.

Step 2: Use $\overline{\overline{Y}} = \begin{pmatrix} 19.45 \\ 18.38 \end{pmatrix}$ to estimate μ_0. Use $S = \begin{pmatrix} 0.62 & 0.04 \\ 0.04 & 0.62 \end{pmatrix}$ to estimate Σ_0.

And $\hat{\rho}_0 = 0.06$.

Step 3: Give $\delta_1 = -1.28$, $\delta_2 = -1.47$, $\delta_3 = 0.85$, $\delta_4 = 1.02$, $\delta_6 = 0.57$, $\delta_7 = -0.79$ and $\rho_1 = -0.3$.

Find optimal (n_1, n_2, h_1, h_2, $p(n_1)$, $p(n_2)$) by

$$
\begin{aligned}
&Minimize\ ATS_1 \\
&s.t. \\
&2 \le n_1 < n_0 < n_2 \le 25 \\
&0 < h_2 < h_0 < h_1 \le 4 \\
&LCL(n_q) < w_l(n_q) < CL(n_q), q = 1,2 \\
&CL(n_q) < w_u(n_q) < UCL(n_q), q = 1,2
\end{aligned}
$$

Step 4: The optimal solutions of (n_1, n_2, h_1, h_2, $p(n_1)$, $p(n_2)$) are:

$p(n_1) = 0.445$, $p(n_2) = 0.445$, $n_1 = 2$, $n_2 = 20$, $h_1 = 1.1125$, $h_2 = 0.1$.

And we have $ATS_1 = 0.1$.

Step 5: The optimal control limits are determined:

$$
\begin{array}{ll}
UCL(n_1) = 5.251 & UCL(n_2) = 2.135 \\
w_u(n_1) = 2.167 & w_u(n_2) = 1.503 \\
CL(n_1) = 0.907 & CL(n_2) = 1.127 \\
w_l(n_1) = 0.069 & w_l(n_2) = 0.606 \\
LCL(n_1) = 0.038 & LCL(n_2) = 0.543
\end{array}
$$

The sampling action in shown in Figure 17 below:

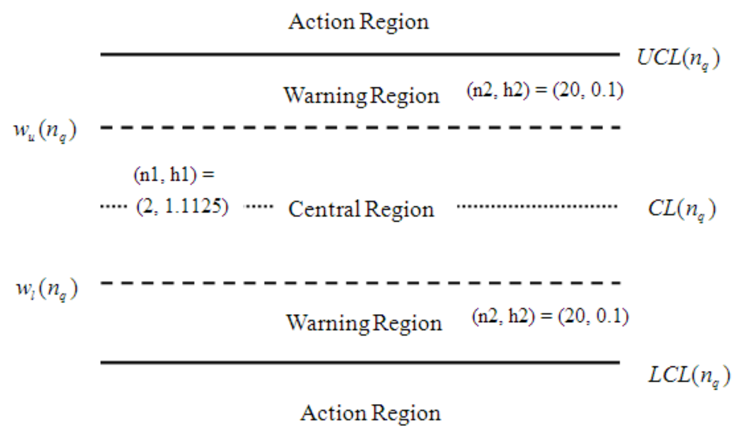

Figure 17. The Optimal VSSI BL Chart

Step 6: We have 76 data after having deleted all out-of-control data from 100 original data. Suppose we first take sample of size two. We calculate BL and plot it on the control chart based on sample size (n_1) two.

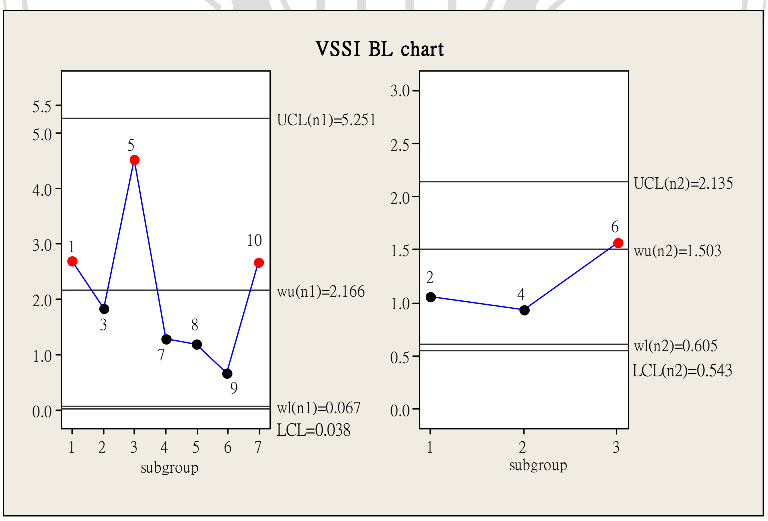

Figure 18. The Optimal VSSI BL Chart of Phase I

The first point on the left chart in Figure 18 falls between $w_u(n_1)$ and UCL (n_1), thus, next time we should take sample of size 20 and the sampling interval is 0.1. Plot the second point on the right chart. Following the same step, we found all points fall between control limits. Thus, we could now use these charts to track the following process.

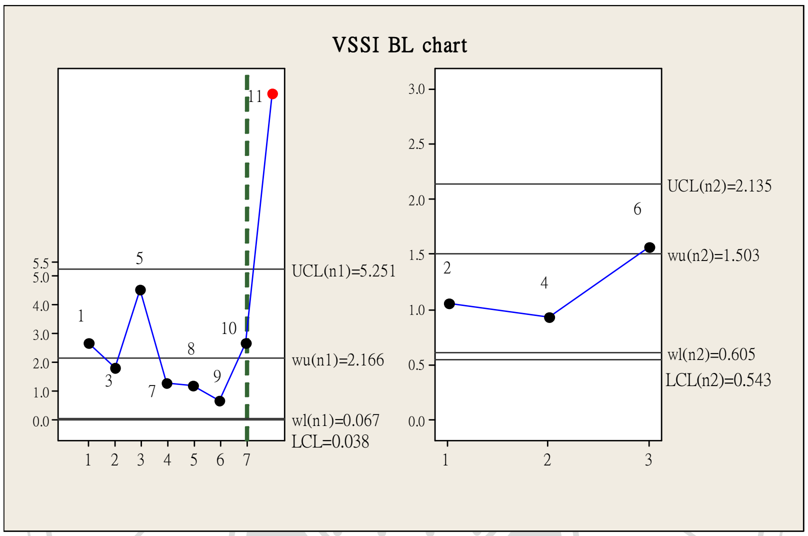

Figure 19. The optimal VSSI BL chart of Phase II

Figure 19 shows that the process is out-of-control (See sample 11.). We have to stop the process and investigate the unusual cause.

From this example, we could calculate the ATS_1 of the optimal VSSI BL chart is 0.1 and the ATS_1 of the specified BL chart is 1.22. Note that we set sampling interval for the specified BL chart is 1. The optimal VSSI BL chart could save 92.8% time to detect signals.

CHAPTER 4. ATS_1 Analysis of the VSSI Bivariate Loss Chart and ATS_1 Comparison between the BL Chart and the VSSI BL Chart

To compare the performance with the BL chart, we set $n_0 = 5, h_0 = 1, \alpha = 0.0027$, $K_{12} = K_{22} = 1$ and $\rho_0 = 0.5$ and let $\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}$ and $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\delta_6 \\ -\delta_7 \end{pmatrix}$.

4.1 The Specified $(n_q, h_q, p(n_q))$ VSSI Bivariate Loss Chart

To investigate the performance of the specified VSSI BL chart in detecting smaller shifts and the impact of K_{11}/K_{22} on ATS_1, we calculate ATS_1 under the combinations of $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1)$ with specified $(n_q, h_q, p(n_q))$ and K_{11}/K_{22}, where K_{11}/K_{22} = 0.5, 1, 2, 4. Also, we compare the performance with the BL chart by 'ARL1 saved%' with the specified VSSI BL chart to detect signal.

Table 20. Variuos levels of $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1)$ and Specified *(h_2, n_1, n_2)*

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	h_2	(n_1, n_2)
0.01	0.01	1.2	1.1	0.01	0.01	0.1	0.1	(2,25)
0.1	0.1	1.5	1.5	0.1	0.1	0.3	0.5	(3,15)
0.5	0.5	2	2	1	1	0.5	0.9	(4,10)

Note that we set $p(n_q) = 0.5$, q = 1, 2.

Table 21. 27 Combinations of $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1, (h_1, h_2)$ and (n_1, n_2)

No.	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	h_1	h_2	(n_1, n_2)
1	0.01	0.01	1.2	1.1	0.01	0.01	0.1	1.135	0.1	(2,25)
2	0.01	0.01	1.5	1.5	0.1	0.1	0.3	1.18	0.1	(3,15)
3	0.01	0.01	2	2	1	1	0.5	1.18	0.1	(4,10)
4	0.01	0.1	1.2	1.5	0.1	1	0.5	1.1	0.5	(4,10)
5	0.01	0.1	1.5	2	1	0.01	0.1	1.075	0.5	(2,25)
6	0.01	0.1	2	1.1	0.01	0.1	0.3	1.1	0.5	(3,15)
7	0.01	0.5	1.2	2	1	0.1	0.3	1.02	0.9	(3,15)
8	0.01	0.5	1.5	1.1	0.1	1	0.5	1.02	0.9	(4,10)
9	0.01	0.5	2	1.5	1	0.01	0.1	1.015	0.9	(2,25)
10	0.1	0.1	1.5	1.5	1	1	0.3	1.18	0.1	(4,10)
11	0.1	0.1	2	2	0.01	0.01	0.5	1.135	0.1	(2,25)
12	0.1	0.1	1.2	1.1	0.1	0.1	0.1	1.18	0.1	(3,15)
13	0.1	0.5	1.5	2	0.01	0.1	0.1	1.1	0.5	(3,15)
14	0.1	0.5	2	1.1	0.1	1	0.3	1.1	0.5	(4,10)
15	0.1	0.5	1.2	1.5	1	0.01	0.5	1.075	0.5	(2,25)
16	0.1	0.01	1.5	1.1	0.1	0.01	0.5	1.015	0.9	(2,25)
17	0.1	0.01	2	1.5	1	0.1	0.1	1.02	0.9	(3,15)
18	0.1	0.01	1.2	2	0.01	1	0.3	1.02	0.9	(4,10)
19	0.5	0.5	2	2	0.1	0.1	0.5	1.18	0.1	(3,15)
20	0.5	0.5	1.2	1.1	1	1	0.1	1.18	0.1	(4,10)
21	0.5	0.5	1.5	1.5	0.01	0.01	0.3	1.135	0.1	(2,25)
22	0.5	0.01	2	1.1	1	0.01	0.3	1.075	0.5	(2,25)
23	0.5	0.01	1.2	1.5	0.01	0.1	0.5	1.1	0.5	(3,15)
24	0.5	0.01	1.5	2	0.1	1	0.1	1.1	0.5	(4,10)
25	0.5	0.1	2	1.5	0.01	1	0.1	1.02	0.9	(4,10)
26	0.5	0.1	1.2	2	0.1	0.01	0.3	1.015	0.9	(2,25)
27	0.5	0.1	1.5	1.1	1	0.1	0.1	1.02	0.9	(3,15)

Table 22. ATS_1 for Specified VSSI BL Chart with $K_{11}/K_{22} = 0.5$,

$n_0 = 5, h_0 = 1, \rho_0 = 0.5, \alpha = 0.0027$, $p(n_q) = 0.5, q = 1,2$ (Based on Table 21.)

No.	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	$VSSI.ATS_1$	$Spe.ATS_1$	Saved %
1	0.046	0.370	4.676	10.521	0.942	1.219	1.287	1.289	7.07	115.00	93.85%
2	0.125	0.622	3.952	8.374	0.740	1.289	1.967	1.974	1.96	4.37	55.06%
3	0.725	2.022	7.621	10.031	1.581	2.816	4.384	4.401	0.90	1.11	18.77%
4	0.429	1.318	5.386	10.023	1.014	1.884	4.307	4.427	2.28	5.54	58.87%
5	0.060	0.460	5.917	11.752	1.111	1.200	1.256	1.258	1.77	1.37	**-29.32%**
6	0.124	0.618	3.922	8.347	0.736	1.281	1.974	1.982	2.41	5.61	57.00%
7	0.166	0.796	5.117	8.084	0.949	1.650	1.874	1.878	1.41	1.21	**-16.56%**
8	0.429	1.318	5.386	10.023	1.014	1.884	4.307	4.427	2.69	6.29	57.31%
9	0.060	0.460	5.917	11.752	1.111	1.200	1.256	1.258	1.72	1.22	**-40.77%**
10	0.725	2.022	7.621	10.031	1.581	2.816	4.384	4.401	1.30	1.57	17.27%
11	0.046	0.370	4.676	10.521	0.942	1.219	1.287	1.289	1.05	1.37	23.66%
12	0.125	0.622	3.952	8.374	0.740	1.289	1.967	1.974	9.83	82.37	88.06%
13	0.124	0.618	3.922	8.347	0.736	1.281	1.974	1.982	1.31	1.67	21.79%
14	0.429	1.318	5.386	10.023	1.014	1.884	4.307	4.427	1.60	3.67	56.30%
15	0.060	0.460	5.917	11.752	1.111	1.200	1.256	1.258	2.06	1.64	**-26.06%**
16	0.046	0.371	4.691	10.559	0.945	1.218	1.286	1.287	4.00	9.28	56.95%
17	0.166	0.796	5.117	8.084	0.949	1.650	1.874	1.878	1.61	1.39	**-15.97%**
18	0.414	1.276	5.222	9.831	0.981	1.825	4.292	4.474	1.50	2.38	36.71%
19	0.125	0.622	3.952	8.374	0.740	1.289	1.967	1.974	0.81	1.14	28.58%
20	0.725	2.022	7.621	10.031	1.581	2.816	4.384	4.401	1.31	1.69	22.72%
21	0.046	0.370	4.676	10.521	0.942	1.219	1.287	1.289	1.57	2.31	32.10%
22	0.060	0.460	5.917	11.752	1.111	1.200	1.256	1.258	1.56	1.25	**-24.45%**
23	0.124	0.618	3.922	8.347	0.736	1.281	1.974	1.982	2.13	4.22	49.62%
24	0.429	1.318	5.386	10.023	1.014	1.884	4.307	4.427	1.04	1.62	35.51%
25	0.414	1.276	5.222	9.831	0.981	1.825	4.292	4.474	1.44	2.15	33.08%
26	0.046	0.371	4.691	10.559	0.945	1.218	1.286	1.287	1.90	1.92	0.74%
27	0.166	0.796	5.117	8.084	0.949	1.650	1.874	1.878	1.86	1.65	**-12.96%**

Note that ATS_1 Saved% = [(Spe.ATS_1 - VSSI.ATS_1) / Spe.ATS_1]•100%.

Table 23. ATS_1 for Specified VSSI BL Chart with $K_{11}/K_{22} = 1$ and

$n_0 = 5, h_0 = 1, \rho_0 = 0.5, \alpha = 0.0027, p(n_q) = 0.5, q = 1,2$ (Based on Table 21.)

NO.	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	ATS_1	$FPATS_1$	Saved %
1	0.061	0.485	5.717	12.544	1.209	1.750	2.162	2.166	7.92	259.07	96.94%
2	0.164	0.809	4.848	10.200	0.956	1.639	3.313	3.382	1.91	7.45	74.31%
3	0.921	2.535	9.190	16.317	1.988	3.496	7.297	7.456	0.77	1.35	43.30%
4	0.511	1.574	6.155	11.276	1.211	2.231	5.198	7.877	5.19	10.35	49.82%
5	0.099	0.701	8.045	17.566	1.698	1.928	2.022	2.024	1.76	1.71	**-2.73%**
6	0.163	0.804	4.808	10.105	0.950	1.628	3.327	3.405	0.18	4.69	96.06%
7	0.259	1.174	6.910	12.941	1.380	2.345	3.043	3.051	1.67	1.64	**-1.75%**
8	0.511	1.574	6.155	11.276	1.211	2.231	5.198	7.877	5.37	7.60	29.35%
9	0.099	0.701	8.045	17.566	1.698	1.928	2.022	2.024	1.63	1.29	**-26.58%**
10	0.921	2.535	9.190	16.317	1.988	3.496	7.297	7.456	1.22	3.07	60.41%
11	0.061	0.485	5.717	12.544	1.209	1.750	2.162	2.166	1.15	1.59	27.57%
12	0.164	0.809	4.848	10.200	0.956	1.639	3.313	3.382	19.87	195.36	89.83%
13	0.163	0.804	4.808	10.105	0.950	1.628	3.327	3.405	1.32	2.60	49.28%
14	0.511	1.574	6.155	11.276	1.211	2.231	5.198	7.883	1.22	3.07	60.40%
15	0.099	0.701	8.045	17.566	1.698	1.928	2.022	2.024	2.25	2.49	9.92%
16	0.061	0.486	5.744	12.613	1.213	1.757	2.155	2.158	4.83	13.07	63.04%
17	0.259	1.174	6.910	12.941	1.380	2.345	3.043	3.051	1.50	1.39	**-7.68%**
18	0.496	1.531	5.981	10.928	1.177	2.171	5.051	7.648	2.00	3.60	44.47%
19	0.164	0.809	4.848	10.200	0.956	1.639	3.313	3.382	0.74	1.24	39.97%
20	0.921	2.535	9.190	16.317	1.988	3.496	7.297	7.456	1.16	4.33	73.19%
21	0.061	0.485	5.717	12.544	1.209	1.750	2.162	2.166	1.70	3.33	48.82%
22	0.099	0.701	8.045	17.566	1.698	1.928	2.022	2.024	1.27	1.19	**-6.82%**
23	0.163	0.804	4.808	10.105	0.950	1.628	3.327	3.405	2.27	8.30	72.64%
24	0.511	1.574	6.155	11.276	1.211	2.231	5.198	7.877	0.73	1.97	62.85%
25	0.496	1.531	5.981	10.928	1.177	2.171	5.051	7.648	1.22	1.96	37.98%
26	0.061	0.486	5.744	12.613	1.213	1.757	2.155	2.158	2.11	3.15	33.01%
27	0.259	1.174	6.910	12.941	1.380	2.345	3.043	3.051	1.68	1.66	**-1.41%**

Note that ATS_1 Saved %= [(Spe.ATS_1 - VSSI.ATS_1) / Spe.ATS_1]•100%.

Table 24. ATS_1 for Specified VSSI BL Chart with $K_{11}/K_{22} = 2$ and

$n_0 = 5, h_0 = 1, \rho_0 = 0.5, \alpha = 0.0027, p(n_q) = 0.5, q = 1,2$ (Based on Table 21.)

NO.	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	ATS_1	$FPATS_1$	Saved %
1	0.080	0.647	8.183	18.411	1.649	2.133	2.252	2.255	5.39	38.14	85.88%
2	0.218	1.087	6.908	14.648	1.295	2.254	3.444	3.457	1.83	3.86	52.54%
3	1.178	3.333	12.764	17.526	2.600	4.665	7.613	7.647	0.89	1.14	21.74%
4	0.629	1.967	7.981	14.962	1.509	2.813	6.720	8.501	5.25	16.26	67.73%
5	0.158	1.076	12.765	20.286	2.091	2.229	2.328	2.330	1.98	1.24	**-59.74%**
6	0.217	1.080	6.846	14.587	1.286	2.237	3.459	3.472	1.82	3.11	41.57%
7	0.399	1.784	10.835	13.982	2.108	3.179	3.365	3.371	1.76	1.34	**-31.79%**
8	0.629	1.967	7.981	14.962	1.509	2.813	6.720	8.501	22.20	7.21	**-207.86%**
9	0.158	1.076	12.765	20.286	2.091	2.229	2.328	2.330	1.58	1.08	**-45.99%**
10	1.178	3.333	12.764	17.526	2.600	4.665	7.613	7.647	1.30	1.67	22.33%
11	0.080	0.647	8.183	18.411	1.649	2.133	2.252	2.255	1.05	1.37	23.59%
12	0.218	1.087	6.908	14.648	1.295	2.254	3.444	3.457	5.97	30.62	80.51%
13	0.217	1.080	6.846	14.587	1.286	2.237	3.459	3.472	1.51	2.23	32.51%
14	0.629	1.967	7.981	14.962	1.509	2.813	6.720	10.349	1.28	2.63	51.24%
15	0.158	1.076	12.765	20.286	2.091	2.229	2.328	2.330	2.83	1.58	**-79.40%**
16	0.081	0.650	8.237	18.545	1.658	2.129	2.247	2.250	3.25	7.20	54.87%
17	0.399	1.784	10.835	13.982	2.108	3.179	3.365	3.371	1.25	1.08	**-15.54%**
18	0.616	1.926	7.777	14.531	1.477	2.751	6.551	8.713	2.60	6.10	57.36%
19	0.218	1.087	6.908	14.648	1.295	2.254	3.444	3.457	0.82	1.15	28.32%
20	1.178	3.333	12.764	17.526	2.600	4.665	7.613	7.647	1.25	1.73	27.66%
21	0.080	0.647	8.183	18.411	1.649	2.133	2.252	2.255	1.55	2.26	31.44%
22	0.158	1.076	12.765	20.286	2.091	2.229	2.328	2.330	1.16	1.02	**-14.03%**
23	0.217	1.080	6.846	14.587	1.286	2.237	3.459	3.472	2.30	5.11	55.05%
24	0.629	1.967	7.981	14.962	1.509	2.813	6.720	8.501	0.89	2.24	60.43%
25	0.616	1.926	7.777	14.531	1.477	2.751	6.551	8.713	1.14	1.77	35.39%
26	0.081	0.650	8.237	18.545	1.658	2.129	2.247	2.250	2.18	2.54	13.92%
27	0.399	1.784	10.835	13.982	2.108	3.179	3.365	3.371	1.35	1.12	**-20.53%**

Note that ATS_1 Saved% = [(Spe.ATS_1 - VSSI.ATS_1) / Spe.ATS_1]•100%.

Table 25. ATS_1 for Specified VSSI BL Chart with $K_{11}/K_{22} = 4$ and

$n_0 = 5, h_0 = 1, \rho_0 = 0.5, \alpha = 0.0027, p(n_q) = 0.5, q = 1,2$ (Based on Table 21.)

NO.	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	ATS_1	$FPATS_1$	Saved %
1	0.110	0.906	13.425	20.620	1.984	2.119	2.215	2.218	5.51	3.46	**-59.15%**
2	0.300	1.553	11.243	14.076	1.887	3.056	3.247	3.253	1.85	1.52	**-21.45%**
3	1.589	4.751	15.994	16.197	3.663	6.600	7.151	7.165	1.17	1.02	**-14.85%**
4	0.821	2.644	11.878	17.050	2.015	3.876	7.145	7.184	3.25	4.53	28.11%
5	0.248	1.701	19.404	19.877	2.318	2.461	2.564	2.567	2.14	1.04	**-105.92%**
6	0.299	1.543	11.137	14.108	1.874	3.057	3.252	3.258	1.49	1.24	**-19.87%**
7	0.616	2.860	13.552	13.706	3.079	3.349	3.501	3.507	1.60	1.09	**-47.23%**
8	0.821	2.644	11.878	17.050	2.015	3.876	7.145	7.184	2.08	2.40	13.32%
9	0.248	1.701	19.404	19.877	2.318	2.461	2.564	2.567	1.49	1.01	**-47.41%**
10	1.589	4.751	15.994	16.197	3.663	6.600	7.151	7.165	1.39	1.12	**-24.07%**
11	0.110	0.906	13.425	20.620	1.984	2.119	2.215	2.218	1.30	1.09	**-19.27%**
12	0.300	1.553	11.243	14.076	1.887	3.056	3.247	3.253	3.66	3.20	**-14.12%**
13	0.299	1.543	11.137	14.108	1.874	3.057	3.252	3.258	1.56	1.29	**-20.61%**
14	0.821	2.644	11.878	17.050	2.015	3.876	7.145	7.184	1.21	1.52	20.48%
15	0.248	1.701	19.404	19.877	2.318	2.461	2.564	2.567	3.01	1.13	**-167.40%**
16	0.111	0.910	13.534	20.582	1.983	2.118	2.214	2.216	3.07	1.66	**-84.70%**
17	0.616	2.860	13.552	13.706	3.079	3.349	3.501	3.507	1.15	1.01	**-13.50%**
18	0.809	2.603	11.623	17.208	1.985	3.810	7.206	7.250	2.42	3.18	23.94%
19	0.300	1.553	11.243	14.076	1.887	3.056	3.247	3.253	0.90	1.03	12.67%
20	1.589	4.751	15.994	16.197	3.663	6.600	7.151	7.165	1.38	1.08	**-27.68%**
21	0.110	0.906	13.425	20.620	1.984	2.119	2.215	2.218	1.88	1.26	**-49.06%**
22	0.248	1.701	19.404	19.877	2.318	2.461	2.564	2.567	1.25	1.00	**-25.26%**
23	0.299	1.543	11.137	14.108	1.874	3.057	3.252	3.258	2.28	1.75	**-30.28%**
24	0.821	2.644	11.878	17.050	2.015	3.876	7.145	7.184	1.23	1.42	13.30%
25	0.809	2.603	11.623	17.208	1.985	3.810	7.206	7.250	1.23	1.24	0.89%
26	0.111	0.910	13.534	20.582	1.983	2.118	2.214	2.216	2.58	1.36	**-90.11%**
27	0.616	2.860	13.552	13.706	3.079	3.349	3.501	3.507	1.17	1.01	**-15.60%**

Note that ATS_1 Saved %= [(Spe.ATS_1 - VSSI.ATS_1) / Spe.ATS_1]•100%.

From Table 22 to Table 25, we found that the VSSI BL chart performs poorer than the BL chart under some combinations from the value of 'ARL1 Saved%'. We note that the VSSI BL chart performs poorly most of the time when h_2 is larger ($h_2 =$ 0.5 or 0.9) and the difference of n_1 and n_2 is larger ((2, 25), (3, 15)). This shows that our decision on sample sizes and sampling intervals are important. On the other hand, some specified $p(n_q)$ may cause the poorer performance. We should consider the combination of $(n_q, h_q, p(n_q))$ carefully. Comparing different K_{11}/K_{22}, we found that when $K_{11}/K_{22} = 4$, the VSSI BL chart has the worst performance. This is quite different from the BL chart.

4.2 The VSSI Bivariate Loss Chart With Optimal ($p(n_q)$)

In terms of the results of the specified $(n_q, h_q, p(n_q))$ VSSI BL chart, next, we try to find optimal $p(n_q)$ to minimize ATS_1 with specified (n_q, h_q). We set $n_0 = 5$, $h_0 = 1$, $\rho_0 = 0.5$, $\alpha = 0.0027$. Based on the following four cases, we could find the optimal solution for $p^*(n_q)$ and ATS_1^*. We use PORT routine in R-program to find the optimal solution.

Case 1: Let $n_0 = 5, h_0 = 1, (n_1, n_2) = (2,25), (3,15), (4,10), h_2 = 0.1, 0.5, 0.9$

and $\alpha = 0.0027, K_{11} / K_{22} = 0.5$

$$\begin{aligned}
&Minimize\ ATS_1 \\
&s.t \\
&0 < p(n_q) < 1 \\
&CL(n_q) < w_u(n_q) < UCL(n_q)
\end{aligned}$$

$$LCL(n_q) < w_l(n_q) < CL(n_q)$$

$$P(CL(n_q) < BL < w_u(n_q)) = P(w_l(n_q) < BL < CL(n_q)), q = 1,2$$

Case 2: Let $n_0 = 5, h_0 = 1, (n_1, n_2) = (2,25), (3,15), (4,10), h_2 = 0.1, 0.5, 0.9$

and $\alpha = 0.0027$, $K_{11} / K_{22} = 1$

$$Minimize\ ATS_1$$
$$s.t$$
$$0 < p(n_q) < 1$$
$$CL(n_q) < w_u(n_q) < UCL(n_q)$$
$$LCL(n_q) < w_l(n_q) < CL(n_q)$$
$$P(CL(n_q) < BL < w_u(n_q)) = P(w_l(n_q) < BL < CL(n_q)), q = 1,2$$

Case 3: Let $n_0 = 5, h_0 = 1, (n_1, n_2) = (2,25), (3,15), (4,10), h_2 = 0.1, 0.5, 0.9$

and $\alpha = 0.0027$, $K_{11} / K_{22} = 2$

$$Minimize\ ATS_1$$
$$s.t$$
$$0 < p(n_q) < 1$$
$$CL(n_q) < w_u(n_q) < UCL(n_q)$$
$$LCL(n_q) < w_l(n_q) < CL(n_q)$$
$$P(CL(n_q) < BL < w_u(n_q)) = P(w_l(n_q) < BL < CL(n_q)), q = 1,2$$

Case 4: Let $n_0 = 5, h_0 = 1, (n_1, n_2) = (2,25), (3,15), (4,10), h_2 = 0.1, 0.5, 0.9$

and $\alpha = 0.0027$, $K_{11} / K_{22} = 4$

$$\begin{aligned}
&Minimize\ ATS_1 \\
&s.t \\
&0 < p(n_q) < 1 \\
&CL(n_q) < w_u(n_q) < UCL(n_q) \\
&LCL(n_q) < w_l(n_q) < CL(n_q) \\
&P(CL(n_q) < BL < w_u(n_q)) = P(w_l(n_q) < BL < CL(n_q)), q = 1,2
\end{aligned}$$

Table 26. ATS_1 for VSSI BL Chart with Optimal $p(n_q)$, $K_{11}/K_{22} = 0.5$ and

$n_0 = 5$, $h_0 = 1$, $\rho_0 = 0.5$, $\alpha = 0.0027$ (Based on Table 21.)

$p(n_1)$	$p(n_2)$	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt. $p(n_q)$ VSSI ATS_1	Spe.ATS_1	Saved% for spe.AT_1	VSSI.ATS_1	Saved% for Spe.VSSI
0.565	0.565	0.046	0.563	10.476	10.521	0.942	1.238	1.289	1.289	6.57	115.00	94.29%	7.07	7.08%
0.417	0.583	0.125	0.128	3.145	8.374	0.740	1.468	1.974	1.974	1.41	4.37	67.64%	1.96	28.00%
0.417	0.583	0.725	0.737	6.417	10.031	1.581	3.223	4.401	4.401	0.77	1.11	30.31%	0.90	14.22%
0.417	0.417	0.429	0.437	4.491	10.023	1.014	1.023	4.013	4.427	1.82	5.54	67.13%	2.28	20.08%
0.435	0.565	0.060	0.061	4.605	11.752	1.111	1.214	1.258	1.258	1.34	1.37	1.87%	1.77	24.12%
0.417	0.417	0.321	0.329	5.238	12.494	1.485	1.496	4.331	6.745	1.38	5.61	75.36%	2.41	42.70%
0.417	0.583	0.157	0.161	3.914	8.022	0.909	1.736	1.855	1.855	**1.28**	**1.21**	**-6.16%**	1.41	8.92%
0.417	0.417	0.429	0.437	4.491	10.023	1.014	1.023	4.013	4.427	2.09	6.29	66.79%	2.69	22.22%
0.435	0.565	0.060	0.061	4.605	11.752	1.111	1.214	1.258	1.258	**1.47**	**1.22**	**-20.80%**	1.72	14.19%
0.417	0.583	0.725	0.737	6.417	10.031	1.581	3.223	4.401	4.401	0.99	1.57	36.77%	1.30	23.57%
0.435	0.565	0.046	0.047	3.649	10.521	0.942	1.238	1.289	1.289	0.79	1.37	42.16%	1.05	24.24%
0.417	0.583	0.125	0.128	3.145	8.374	0.740	1.468	1.974	1.974	**10.26**	82.37	87.55%	**9.83**	**-4.33%**
0.417	0.417	0.124	0.128	3.122	8.347	0.736	0.743	1.965	1.982	1.15	1.67	31.27%	1.31	12.11%
0.417	0.417	0.429	0.437	4.491	10.023	1.014	1.023	4.013	4.427	1.18	3.67	67.86%	1.60	26.46%
0.435	0.565	0.060	0.061	4.605	11.752	1.111	1.214	1.258	1.258	1.60	1.64	2.20%	2.06	22.42%
0.435	0.565	0.046	0.047	3.661	10.559	0.945	1.237	1.287	1.287	3.15	9.28	66.09%	4.00	21.23%
0.417	0.583	0.166	0.170	4.066	8.084	0.949	1.771	1.878	1.878	1.33	1.39	4.51%	1.61	17.66%
0.417	0.417	0.414	0.421	4.354	9.831	0.981	0.990	3.896	4.474	1.30	2.38	45.44%	1.50	13.80%
0.417	0.417	0.125	0.128	3.145	8.374	0.740	0.747	1.958	1.974	0.73	1.14	36.40%	0.81	10.94%
0.417	0.417	0.725	0.737	6.417	10.031	1.581	1.595	4.364	4.401	1.02	1.69	39.68%	1.31	21.95%
0.435	0.565	0.046	0.047	3.649	10.521	0.942	1.238	1.289	1.289	1.07	2.31	53.55%	1.57	31.59%
0.435	0.565	0.060	0.061	4.605	11.752	1.111	1.214	1.258	1.258	**1.28**	**1.25**	**-2.54%**	1.56	17.61%
0.417	0.583	0.124	0.128	3.122	8.347	0.736	1.459	1.982	1.982	1.76	4.22	58.32%	2.13	17.27%
0.417	0.417	0.429	0.437	4.491	10.023	1.014	1.023	4.013	4.427	0.85	1.62	47.34%	1.04	18.35%
0.417	0.417	0.414	0.421	4.354	9.831	0.981	0.990	3.896	4.474	1.23	2.15	42.80%	1.44	14.53%
0.435	0.565	0.046	0.047	3.661	10.559	0.945	1.237	1.287	1.287	1.62	1.92	15.78%	1.90	15.15%
0.417	0.583	0.166	0.170	4.066	8.084	0.949	1.771	1.878	1.878	1.45	1.65	11.99%	1.86	22.09%

Table 27. ATS_1 for VSSI BL Chart with Optimal $p(n_q)$, $K_{11}/K_{22} = 1$ and

$n_0 = 5$, $h_0 = 1$, $\rho_0 = 0.5$, $\alpha = 0.0027$ (Based on Table 21.)

$p(n_1)$	$p(n_2)$	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt. $p(n_q)$ VSSI ATS_1	Spe.ATS_1	Saved% for spe.AT_1	VSSI.ATS_1	Saved% for Spe.VSSI
0.435	0.565	0.061	0.062	4.507	12.544	1.209	1.912	2.166	2.166	6.07	259.07	97.66%	7.92	23.34%
0.417	0.417	0.164	0.169	3.893	10.200	0.956	0.964	3.167	3.382	1.26	7.45	83.03%	1.91	33.92%
0.417	0.417	0.921	0.936	7.780	16.317	1.988	2.006	6.948	7.456	0.59	1.35	56.57%	0.77	23.41%
0.417	0.417	0.511	0.521	5.178	11.276	1.211	1.223	4.637	7.877	3.95	10.35	61.84%	5.19	23.95%
0.435	0.565	0.099	0.100	6.331	17.566	1.698	1.953	2.024	2.024	1.34	1.71	21.78%	1.76	23.86%
0.417	0.417	0.321	0.329	5.238	12.494	1.485	1.496	4.331	6.745	**1.38**	4.69	70.53%	**0.18**	**-648.60%**
0.417	0.583	0.247	0.254	5.402	12.791	1.335	2.577	3.008	3.008	1.55	1.64	5.21%	1.67	6.84%
0.417	0.417	0.511	0.521	5.178	11.276	1.211	1.223	4.637	7.877	4.77	7.60	37.22%	5.37	11.15%
0.435	0.565	0.099	0.100	6.331	17.566	1.698	1.953	2.024	2.024	**1.43**	**1.29**	**-11.04%**	1.63	12.28%
0.417	0.417	0.921	0.936	7.780	16.317	1.988	2.006	6.948	7.456	0.86	3.07	71.93%	1.22	29.12%
0.435	0.435	0.061	0.062	4.507	12.544	1.209	1.212	2.158	2.166	0.96	1.59	39.79%	1.15	16.87%
0.417	0.583	0.164	0.169	3.893	10.200	0.956	1.859	3.381	3.382	15.67	195.36	91.98%	19.87	21.13%
0.417	0.417	0.163	0.168	3.863	10.105	0.950	0.958	3.153	3.405	1.01	2.60	61.22%	1.32	23.54%
0.417	0.417	0.511	0.521	5.178	11.276	1.211	1.223	4.637	7.883	0.99	3.07	67.63%	1.22	18.26%
0.435	0.565	0.099	0.100	6.331	17.566	1.698	1.953	2.024	2.024	1.76	2.49	29.60%	2.25	21.84%
0.435	0.435	0.061	0.062	4.528	12.613	1.213	1.216	2.150	2.158	3.83	13.07	70.71%	4.83	20.76%
0.417	0.583	0.259	0.266	5.549	12.941	1.380	2.654	3.051	3.051	1.36	1.39	2.30%	1.50	9.26%
0.417	0.417	0.496	0.505	5.034	10.928	1.177	1.189	4.508	7.648	1.80	3.60	49.85%	2.00	9.68%
0.417	0.417	0.164	0.169	3.893	10.200	0.956	0.964	3.167	3.382	0.53	1.24	57.46%	0.74	29.14%
0.417	0.417	0.921	0.936	7.780	16.317	1.988	2.006	6.948	7.456	0.90	4.33	79.25%	1.16	22.61%
0.435	0.435	0.061	0.062	4.507	12.544	1.209	1.212	2.158	2.166	1.30	3.33	61.06%	1.70	23.92%
0.435	0.565	0.099	0.100	6.331	17.566	1.698	1.953	2.024	2.024	1.13	1.19	4.99%	1.27	11.06%
0.417	0.417	0.163	0.168	3.863	10.105	0.950	0.958	3.153	3.405	1.86	8.30	77.52%	2.27	17.85%
0.417	0.417	0.511	0.521	5.178	11.276	1.211	1.223	4.637	7.877	0.65	1.97	66.91%	0.73	10.92%
0.417	0.417	0.496	0.505	5.034	10.928	1.177	1.189	4.508	7.648	1.15	1.96	41.39%	1.22	5.49%
0.435	0.565	0.061	0.062	4.528	12.613	1.213	1.918	2.158	2.158	1.86	3.15	41.07%	2.11	12.03%
0.417	0.583	0.259	0.266	5.549	12.941	1.380	2.654	3.051	3.051	1.47	1.66	11.16%	1.68	12.39%

Table 28. ATS_1 for VSSI BL Chart with Optimal $p(n_q)$, $K_{11}/K_{22} = 2$ and

$n_0 = 5$, $h_0 = 1$, $\rho_0 = 0.5$, $\alpha = 0.0027$ (Based on Table 21.)

$p(n_1)$	$p(n_2)$	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt. $p(n_q)$ VSSI ATS_1	Spe.ATS_1	Saved% for spe.AT_1	VSSI.ATS_1	Saved% for Spe.VSSI
0.435	0.565	0.080	0.082	6.386	18.411	1.649	2.167	2.255	2.255	5.28	38.14	86.16%	5.39	1.95%
0.417	0.417	0.218	0.224	5.497	14.648	1.295	1.306	3.428	3.457	1.35	3.86	65.09%	1.83	26.45%
0.417	0.417	1.178	1.198	10.726	17.526	2.600	2.623	7.572	7.647	0.77	1.14	32.88%	0.89	14.24%
0.417	0.417	0.629	0.641	6.671	14.962	1.509	1.524	5.971	8.501	5.63	16.26	65.39%	5.25	-7.26%
0.435	0.565	0.158	0.160	10.015	20.286	2.091	2.254	2.330	2.330	**1.45**	**1.24**	**-16.69%**	1.98	26.95%
0.417	0.583	0.398	0.407	6.768	16.670	1.871	3.412	8.800	8.844	1.70	3.11	45.31%	1.82	6.40%
0.417	0.583	0.386	0.396	8.529	13.813	2.061	3.192	3.321	3.321	**1.39**	**1.34**	**-4.01%**	1.76	21.08%
0.583	0.583	0.629	2.512	14.838	14.962	1.509	3.246	8.499	8.501	**9.51**	**7.21**	**-31.91%**	22.20	57.15%
0.435	0.565	0.158	0.160	10.015	20.286	2.091	2.254	2.330	2.330	**1.38**	**1.08**	**-27.84%**	1.58	12.43%
0.417	0.417	1.178	1.198	10.726	17.526	2.600	2.623	7.572	7.647	1.00	1.67	39.94%	1.30	22.67%
0.435	0.565	0.080	0.082	6.386	18.411	1.649	2.167	2.255	2.255	0.79	1.37	42.13%	1.05	24.26%
0.417	0.583	0.218	0.224	5.497	14.648	1.295	2.567	3.457	3.457	4.25	30.62	86.13%	5.97	28.85%
0.417	0.417	0.217	0.223	5.450	14.587	1.286	1.297	3.442	3.472	1.27	2.23	43.02%	1.51	15.58%
0.417	0.417	0.629	0.641	6.671	14.962	1.509	1.524	5.971	10.349	0.85	2.63	67.78%	1.28	33.92%
0.435	0.565	0.158	0.160	10.015	20.286	2.091	2.254	2.330	2.330	**1.90**	**1.58**	**-20.38%**	2.83	32.90%
0.435	0.435	0.081	0.082	6.426	18.545	1.658	1.662	2.244	2.250	2.63	7.20	63.49%	3.25	19.11%
0.417	0.583	0.399	0.409	8.673	13.982	2.108	3.240	3.371	3.371	**1.12**	**1.08**	**-3.77%**	1.25	10.19%
0.417	0.417	0.616	0.628	6.507	14.531	1.477	1.492	5.824	8.713	2.14	6.10	64.84%	2.60	17.53%
0.417	0.417	0.218	0.224	5.497	14.648	1.295	1.306	3.428	3.457	0.73	1.15	36.37%	0.82	11.23%
0.417	0.417	1.178	1.198	10.726	17.526	2.600	2.623	7.572	7.647	1.06	1.73	38.50%	1.25	14.98%
0.435	0.565	0.080	0.082	6.386	18.411	1.649	2.167	2.255	2.255	1.07	2.26	52.87%	1.55	31.25%
0.435	0.553	0.158	0.160	10.015	20.286	2.091	2.250	2.330	2.330	**1.04**	**1.02**	**-1.76%**	1.16	10.76%
0.417	0.417	0.217	0.223	5.450	14.587	1.286	1.297	3.442	3.472	2.00	5.11	60.88%	2.30	12.97%
0.417	0.417	0.629	0.641	6.671	14.962	1.509	1.524	5.971	8.501	0.70	2.24	68.80%	0.89	21.16%
0.417	0.417	0.616	0.628	6.507	14.531	1.477	1.492	5.824	8.713	1.05	1.77	40.62%	1.14	8.10%
0.435	0.565	0.081	0.082	6.426	18.545	1.658	2.162	2.250	2.250	1.83	2.54	27.86%	2.18	16.19%
0.417	0.583	0.399	0.409	8.673	13.982	2.108	3.240	3.371	3.371	**1.18**	**1.12**	**-5.19%**	1.35	12.72%

Table 29. ATS_1 for VSSI BL Chart with Optimal $p(n_q)$, $K_{11}/K_{22} = 4$ and

$n_0 = 5$, $h_0 = 1$, $\rho_0 = 0.5$, $\alpha = 0.0027$ (Based on Table 21.)

$p(n_1)$	$p(n_2)$	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt. $p(n_q)$ VSSI ATS_1	$SpeATS_1$	Saved% for spe.AT_1	VSSI.ATS_1	Saved% for Spe.VSSI
0.435	0.565	0.110	0.112	10.310	20.620	1.984	2.144	2.218	2.218	**4.11**	**3.46**	**-18.85%**	5.51	25.32%
0.417	0.583	0.300	0.309	8.813	14.076	1.887	3.122	3.253	3.253	1.04	1.52	31.89%	1.85	43.92%
0.417	0.583	1.589	1.617	15.621	16.197	3.663	6.831	7.165	7.165	1.02	1.02	0.39%	1.17	13.27%
0.417	0.583	0.821	0.837	9.772	17.050	2.015	4.517	7.184	7.184	3.13	4.53	30.82%	3.25	3.77%
0.435	0.565	0.248	0.252	17.549	19.877	2.318	2.487	2.567	2.567	**1.82**	**1.04**	**-75.24%**	2.14	14.90%
0.417	0.417	0.519	0.532	9.948	22.639	2.540	2.561	5.034	5.057	**1.48**	**1.24**	**-19.25%**	1.49	0.52%
0.417	0.583	0.604	0.619	13.138	13.548	3.038	3.348	3.461	3.461	**1.59**	**1.09**	**-46.02%**	1.60	0.83%
0.417	0.583	0.821	0.837	9.772	17.050	2.015	4.517	7.184	7.184	1.77	2.40	26.33%	2.08	15.01%
0.435	0.565	0.248	0.252	17.549	19.877	2.318	2.487	2.567	2.567	**1.38**	**1.01**	**-37.05%**	1.49	7.03%
0.417	0.583	1.589	1.617	15.621	16.197	3.663	6.831	7.165	7.165	**1.23**	**1.12**	**-9.76%**	1.39	11.54%
0.435	0.565	0.110	0.112	10.310	20.620	1.984	2.144	2.218	2.218	0.93	1.09	14.70%	1.30	28.48%
0.417	0.583	0.300	0.309	8.813	14.076	1.887	3.122	3.253	3.253	3.02	3.20	5.75%	3.66	17.41%
0.417	0.583	0.299	0.307	8.732	14.108	1.874	3.125	3.258	3.258	1.08	1.29	16.61%	1.56	30.85%
0.417	0.417	0.821	0.837	9.772	17.050	2.015	2.036	7.096	7.184	1.02	1.52	32.73%	1.21	15.41%
0.435	0.565	0.248	0.252	17.549	19.877	2.318	2.487	2.567	2.567	**2.56**	**1.13**	**-127.22%**	3.01	15.03%
0.435	0.438	0.111	0.112	10.391	20.582	1.983	2.020	2.211	2.216	**2.49**	**1.66**	**-49.96%**	3.07	18.81%
0.417	0.583	0.616	0.632	13.288	13.706	3.079	3.392	3.507	3.507	**1.13**	**1.01**	**-12.17%**	1.15	1.17%
0.417	0.417	0.809	0.825	9.571	17.208	1.985	2.005	7.149	7.250	2.31	3.18	27.41%	2.42	4.57%
0.417	0.583	0.300	0.309	8.813	14.076	1.887	3.122	3.253	3.253	0.60	1.03	41.37%	0.90	32.86%
0.417	0.583	1.589	1.617	15.621	16.197	3.663	6.831	7.165	7.165	**1.19**	**1.08**	**-10.10%**	1.38	13.76%
0.435	0.565	0.110	0.112	10.310	20.620	1.984	2.144	2.218	2.218	1.26	1.26	0.36%	1.88	33.16%
0.435	0.435	0.248	0.252	17.549	19.877	2.318	2.319	2.561	2.567	**1.13**	**1.00**	**-13.45%**	1.25	9.43%
0.417	0.583	0.299	0.307	8.732	14.108	1.874	3.125	3.258	3.258	1.65	1.75	6.02%	2.28	27.86%
0.417	0.417	0.821	0.837	9.772	17.050	2.015	2.036	7.096	7.184	1.08	1.42	24.26%	1.23	12.65%
0.417	0.417	0.809	0.825	9.571	17.208	1.985	2.005	7.149	7.250	1.17	1.24	5.38%	1.23	4.53%
0.435	0.565	0.111	0.112	10.391	20.582	1.983	2.143	2.216	2.216	**1.91**	**1.36**	**-40.48%**	2.58	26.11%
0.417	0.583	0.616	0.632	13.288	13.706	3.079	3.392	3.507	3.507	**1.15**	**1.01**	**-13.73%**	1.17	1.62%

From the results of Table 26 to Table 29, we found that through finding optimal $p(n_q)$, the performance of the VSSI BL chart is better than BL chart with specified VSSIs ($n_q, h_q, p(n_q)$). But, we still could find that the VSSI BL chart with optimal $p(n_q)$ has worse performance in some combinations than the BL chart, especially when $K_{11}/K_{22} = 4$. The combinations of K_{11}/K_{22} and ($n_q, h_q, p(n_q)$) affect ATS_1 a lot.

4.3 The OptimalVSSI Bivariate Loss Chart

To investigate the performance of the optimal VSSI BL chart, we give combinations of $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \rho_1$ (See Table 21) and find optimal $n_q, h_q, p(n_q)$ to minimize ATS_1 with specified K_{11}/K_{22} and ρ_0. Based on the following four Case with specified $K_{11}/K_{22} = 0.5$, 1, 2, 4 and $\rho_0 = 0.5$, we may find optimal solutions using PORT routine in R – program.

Case 1: Let $n_0 = 5, h_0 = 1, \alpha = 0.0027$.

$$
\begin{aligned}
& \textit{Minimize}\ ATS_1 \\
& s.t \\
& 2 \le n_1 < n_0 < n_2 \le 25, \\
& 0.1 \le h_2 < h_0 < h_1 < 4, \\
& 0 < p(n_q) < 1, q = 1,2 \\
& CL(n_q) < w_u(n_q) < UCL(n_q) \\
& LCL(n_q) < w_l(n_q) < CL(n_q) \\
& P(CL(n_q) < BL < w_u(n_q)) = P(w_l(n_q) < BL < CL(n_q)), q = 1,2
\end{aligned}
$$

Table 30. ATS_1 for Optimal VSSI BL Chart with $K_{11}/K_{22} = 0.5$ and

$$n_0 = 5,\ h_0 = 1,\ \rho_0 = 0.5,\ \alpha = 0.0027$$

NO	$p(n_1)$	$p(n_2)$	h_1	h_2	n_1	n_2	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt.ATS_1*	Spe.ATS_1	Saved%
1	0.419	0.581	1.18	0.1	2.00	19.90	0.046	0.079	3.297	10.521	0.850	1.452	1.542	1.542	5.74	115.00	95.01%
2	0.467	0.543	1.10	0.1	2.89	24.60	0.115	0.302	4.011	8.603	0.947	1.254	1.316	1.316	1.61	4.37	63.21%
3	0.435	0.565	1.14	0.1	3.30	16.09	0.568	0.674	7.058	11.992	2.028	2.855	2.982	2.982	0.61	1.11	44.82%
4	0.417	0.417	1.00	0.1	4.99	25.00	0.560	0.560	5.141	8.393	1.598	1.598	2.024	2.027	1.31	5.54	76.29%
5	0.442	0.558	1.13	0.1	3.18	17.62	0.175	0.294	4.338	7.598	0.993	1.547	1.628	1.628	0.84	1.37	38.39%
6	0.404	0.404	1.00	0.1	4.99	24.24	0.281	0.281	3.372	5.391	0.936	0.936	1.332	1.335	1.36	5.61	75.76%
7	0.439	0.561	1.13	0.1	3.20	17.46	0.187	0.248	4.463	7.594	1.032	1.583	1.662	1.662	0.75	1.21	38.28%
8	0.473	0.554	1.14	0.1	2.63	20.46	0.222	0.753	5.729	12.495	1.478	2.230	2.368	2.369	1.72	6.29	72.60%
9	0.444	0.556	1.12	0.1	3.27	18.02	0.184	0.257	4.391	7.385	1.005	1.521	1.599	1.599	0.74	1.22	39.11%
10	0.428	0.572	1.16	0.1	3.19	15.10	0.543	0.767	6.857	12.358	1.969	2.993	3.132	3.133	0.82	1.57	47.60%
11	0.475	0.528	1.05	0.1	3.81	24.60	0.189	0.278	4.104	6.818	0.936	1.227	1.305	1.305	0.77	1.37	44.15%
12	0.420	0.580	1.18	0.1	2.00	19.95	0.046	0.088	3.349	10.656	0.861	1.463	1.552	1.553	4.99	82.37	93.94%
13	0.466	0.466	3.32	0.1	2.11	6.12	0.054	1.110	1.988	10.259	0.358	1.509	2.102	4.460	0.36	1.67	78.15%
14	0.462	0.462	1.11	0.1	2.63	24.73	0.222	0.537	5.761	12.503	1.591	1.873	2.040	2.044	1.43	3.67	60.91%
15	0.433	0.567	1.14	0.1	3.07	17.10	0.164	0.212	4.200	7.840	0.978	1.585	1.667	1.667	0.99	1.64	39.70%
16	0.463	0.523	1.11	0.1	2.51	24.94	0.084	0.265	3.955	9.291	0.944	1.225	1.289	1.290	2.92	9.28	68.53%
17	0.475	0.569	1.02	0.1	4.71	18.65	0.340	0.340	2.321	5.288	1.068	1.558	1.579	1.579	0.27	1.39	80.88%
18	0.480	0.551	1.11	0.1	2.58	25.00	0.205	0.674	5.955	12.248	1.548	1.951	2.035	2.035	1.06	2.38	55.54%
19	0.475	0.528	1.07	0.1	3.53	25.00	0.168	0.326	4.146	7.357	0.954	1.229	1.300	1.300	0.66	1.14	41.71%
20	0.429	0.571	1.16	0.1	3.32	14.70	0.573	0.758	6.858	11.929	1.945	3.045	3.199	3.199	1.00	1.69	41.02%
21	0.450	0.450	2.66	0.1	2.03	6.61	0.048	0.948	2.051	10.424	0.386	1.419	2.140	4.103	0.60	2.31	73.97%
22	0.465	0.571	1.17	0.1	2.72	16.79	0.129	0.537	4.537	8.796	0.968	1.611	1.691	1.692	0.82	1.25	34.45%
23	0.447	0.447	1.11	0.1	3.85	14.32	0.194	0.245	3.448	6.822	0.716	0.786	2.051	2.063	1.58	4.22	62.54%
24	0.482	0.545	1.10	0.1	2.71	25.00	0.235	0.727	6.170	12.305	1.598	1.943	2.027	2.027	0.85	1.62	47.35%
25	0.481	0.539	1.08	0.1	3.29	24.80	0.312	0.717	5.923	10.829	1.543	1.949	2.047	2.047	0.94	2.15	56.09%
26	0.458	0.458	2.94	0.1	2.01	6.39	0.047	1.012	2.017	10.518	0.373	1.463	2.124	4.228	0.49	1.92	74.28%
27	0.461	0.583	1.19	0.1	2.75	15.74	0.139	0.577	4.573	8.773	0.976	1.718	1.805	1.805	1.04	1.65	36.89%

Note that ATS_1 Saved % = [(Spe.ATS_1 - Opt.ATS_1^*) / Spe.ATS_1]•100%.

Table 31. ATS_1 for Optimal VSSI BL Chart with $K_{11}/K_{22} = 1$ and

$$n_0 = 5,\ h_0 = 1,\ \rho_0 = 0.5,\ \alpha = 0.0027$$

NO	$p(n_1)$	$p(n_2)$	h_1	h_2	n_1	n_2	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt.ATS₁*	Spe.ATS₁	Saved%
1	0.435	0.565	1.14	0.1	2.00	25.00	0.061	0.062	4.507	12.544	1.209	1.912	2.166	2.166	6.07	259.07	97.66%
2	0.470	0.470	1.09	0.1	3.78	16.67	0.248	0.555	4.679	9.086	1.013	1.359	3.051	3.081	1.61	7.45	78.45%
3	0.480	0.520	1.04	0.1	4.09	24.65	0.944	1.155	10.409	16.097	2.986	3.386	3.580	3.580	0.72	1.35	46.67%
4	0.474	0.474	1.05	0.1	3.89	25.00	0.493	0.504	6.818	11.413	1.897	1.908	3.784	3.795	2.14	10.35	79.36%
5	0.460	0.540	1.09	0.1	3.52	20.61	0.327	0.467	6.486	11.053	1.566	2.203	2.331	2.331	0.94	1.71	45.24%
6	0.495	0.495	1.01	0.1	4.78	25.00	0.348	0.376	6.169	8.057	1.215	1.246	2.197	2.198	4.15	4.69	11.60%
7	0.460	0.540	1.08	0.1	3.56	20.42	0.347	0.478	6.650	11.081	1.611	2.254	2.385	2.385	0.89	1.64	45.63%
8	0.474	0.474	1.68	0.1	2.00	9.00	0.144	1.539	4.876	15.860	1.127	2.586	4.336	8.170	1.70	7.6	77.60%
9	0.464	0.536	1.08	0.1	3.62	21.27	0.342	0.480	6.615	10.780	1.589	2.153	2.276	2.276	0.76	1.29	41.03%
10	0.471	0.529	1.06	0.1	3.76	22.38	0.855	1.186	9.887	16.889	2.890	3.627	3.835	3.835	1.08	3.07	64.69%
11	0.487	0.487	1.03	0.1	4.43	24.43	0.312	0.372	5.472	8.321	1.198	1.266	2.206	2.208	0.84	1.59	47.23%
12	0.435	0.565	1.14	0.1	2.00	25.00	0.061	0.062	4.561	12.699	1.223	1.934	2.184	2.184	5.30	195.36	97.29%
13	0.484	0.484	1.03	0.1	4.30	24.88	0.301	0.350	5.359	8.464	1.213	1.270	2.204	2.206	1.06	2.6	59.40%
14	0.470	0.470	2.80	0.1	3.00	6.00	0.332	2.399	3.946	12.897	0.807	2.782	3.930	9.500	0.73	3.07	76.14%
15	0.454	0.546	1.09	0.1	3.43	20.25	0.313	0.351	6.341	11.339	1.554	2.235	2.363	2.363	1.16	2.49	53.43%
16	0.470	0.470	1.07	0.1	3.53	25.00	0.220	0.343	4.940	9.333	1.213	1.365	2.155	2.158	3.40	13.07	73.97%
17	0.470	0.470	1.00	0.1	4.99	22.26	0.555	0.555	3.307	8.106	1.674	2.208	2.237	2.237	0.28	1.39	79.77%
18	0.476	0.476	1.07	0.1	3.41	24.94	0.394	0.779	6.637	11.761	1.843	2.210	3.921	3.941	1.27	3.6	64.64%
19	0.488	0.488	1.02	0.1	4.50	24.96	0.323	0.334	5.623	8.358	1.222	1.235	2.186	2.187	0.72	1.24	41.71%
20	0.471	0.529	1.06	0.1	3.85	20.94	0.880	1.232	9.821	16.681	2.821	3.768	4.029	4.030	1.16	4.33	73.18%
21	0.482	0.482	1.03	0.1	4.24	24.97	0.293	0.313	5.269	8.495	1.208	1.232	2.166	2.169	1.23	3.33	63.18%
22	0.473	0.542	1.09	0.1	2.96	24.86	0.241	0.635	6.990	12.960	1.695	1.946	2.032	2.032	0.80	1.19	32.42%
23	0.465	0.465	1.46	0.1	2.00	10.83	0.061	0.754	3.717	12.579	0.773	1.650	3.136	4.556	1.23	8.3	85.15%
24	0.485	0.485	1.03	0.1	4.37	24.56	0.573	0.617	7.258	10.841	1.884	1.929	3.847	3.853	0.91	1.97	53.71%
25	0.457	0.457	1.35	0.1	2.00	12.74	0.138	1.030	5.350	15.355	1.363	2.366	4.329	6.971	0.20	1.96	89.95%
26	0.481	0.481	1.71	0.1	2.00	8.79	0.061	0.981	3.430	12.613	0.658	1.740	3.072	5.391	0.42	3.15	86.62%
27	0.470	0.547	1.10	0.1	3.42	19.77	0.325	0.753	6.807	11.495	1.587	2.312	2.445	2.445	0.96	1.66	42.14%

Note that ATS_1 Saved % = [(Spe.ATS_1 - Opt.ATS_1^*) / Spe.ATS_1]•100%.

Table 32. ATS_1 for Optimal VSSI BL Chart with K_{11}/K_{22} =2 and

$$n_0 = 5,\ h_0 = 1,\ \rho_0 = 0.5,\ \alpha = 0.0027$$

NO	$p(n_1)$	$p(n_2)$	h_1	h_2	n_1	n_2	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt.ATS_1*	Spe.ATS_1	Saved%
1	0.420	0.580	1.18	0.100	2.00	20.11	0.080	0.139	5.801	18.412	1.496	2.524	2.675	2.675	4.24	38.14	88.87%
2	0.448	0.448	2.10	0.100	2.00	7.45	0.081	1.428	3.994	18.624	0.769	2.396	3.984	6.483	1.18	3.86	69.47%
3	0.442	0.558	1.12	0.100	3.41	16.82	0.960	1.133	12.114	20.342	3.413	4.757	4.980	4.980	0.63	1.14	44.64%
4	0.461	0.461	1.13	0.100	2.78	20.18	0.354	0.989	8.165	17.849	2.190	2.877	4.506	4.526	2.68	16.26	83.50%
5	0.400	0.600	1.25	0.100	2.72	13.26	0.320	0.648	7.882	15.163	1.917	3.513	3.662	3.663	0.72	1.24	41.70%
6	0.387	0.613	1.28	0.100	2.60	12.70	0.301	0.557	7.667	16.030	1.911	3.691	3.846	3.846	0.80	3.11	74.33%
7	0.471	0.471	1.07	0.100	3.52	24.85	0.525	0.830	8.728	15.892	2.379	2.698	3.793	3.799	1.73	1.34	-29.21%
8	0.471	0.471	1.07	0.100	3.52	24.85	0.525	0.830	8.728	15.892	2.379	2.698	3.793	3.799	1.73	7.21	75.99%
9	0.404	0.596	1.23	0.100	2.79	13.82	0.337	0.542	8.065	14.781	1.965	3.402	3.542	3.542	0.62	1.08	42.97%
10	0.436	0.564	1.14	0.100	3.30	16.17	0.920	1.144	11.844	20.934	3.353	4.900	5.133	5.133	0.86	1.67	48.37%
11	0.471	0.529	1.06	0.100	3.75	24.10	0.322	0.412	7.036	12.102	1.624	2.179	2.319	2.319	0.75	1.37	44.96%
12	0.422	0.578	1.18	0.100	2.00	20.16	0.081	0.171	5.903	18.624	1.514	2.544	2.695	2.695	3.88	30.62	87.32%
13	0.470	0.533	1.10	0.100	2.88	25.00	0.200	0.551	7.048	14.977	1.656	2.170	2.283	2.284	1.05	2.23	52.85%
14	0.475	0.475	1.05	0.100	3.96	24.33	0.622	0.660	8.892	15.025	2.361	2.403	3.861	3.865	1.15	2.63	56.23%
15	0.375	0.625	1.36	0.100	2.18	12.07	0.196	0.628	7.060	18.698	1.807	3.793	3.950	3.952	0.91	1.58	42.71%
16	0.460	0.460	1.11	0.100	3.55	17.06	0.296	0.639	6.396	12.652	1.382	1.807	3.050	3.059	1.93	7.20	73.14%
17	0.422	0.529	1.22	0.100	3.19	12.36	0.444	0.444	4.146	13.198	1.878	3.883	3.934	3.934	0.24	1.08	77.54%
18	0.474	0.474	1.06	0.100	3.73	24.81	0.559	0.809	8.609	15.013	2.327	2.584	3.885	3.891	1.78	6.10	70.80%
19	0.473	0.527	1.05	0.100	3.86	24.66	0.342	0.364	7.215	11.896	1.658	2.162	2.299	2.300	0.67	1.15	42.01%
20	0.433	0.567	1.14	0.100	3.33	15.40	0.930	1.181	11.671	20.793	3.280	5.068	5.331	5.332	1.05	1.73	39.41%
21	0.446	0.446	2.65	0.100	2.07	6.59	0.893	1.655	3.571	18.032	0.674	2.465	3.732	7.194	0.59	2.26	73.80%
22	0.451	0.606	1.25	0.100	2.60	13.51	0.292	1.349	8.862	15.808	1.939	3.469	3.606	3.606	0.60	1.02	40.77%
23	0.458	0.542	1.11	0.100	3.02	21.68	0.220	0.491	6.574	14.507	1.556	2.381	2.551	2.552	1.90	5.11	62.79%
24	0.483	0.483	1.04	0.100	4.22	24.64	0.675	0.844	9.302	14.603	2.372	2.544	3.823	3.826	0.98	2.24	56.31%
25	0.482	0.482	1.04	0.100	4.11	24.84	0.638	0.871	8.981	14.354	2.328	2.559	3.884	3.888	0.91	1.77	48.52%
26	0.469	0.531	1.06	0.100	3.69	24.57	0.316	0.336	7.040	12.245	1.646	2.151	2.279	2.279	1.25	2.54	50.61%
27	0.403	0.597	1.23	0.100	2.83	13.54	0.358	0.577	8.154	14.767	1.987	3.507	3.653	3.654	0.66	1.12	41.22%

Note that ATS_1 Saved% = [(Spe.ATS_1 - Opt.ATS_1^*) / Spe.ATS_1]•100%.

Table 33. ATS_1 for Optimal VSSI BL Chart with $K_{11}/K_{22} = 4$ and

$n_0 = 5,\ h_0 = 1,\ \rho_0 = 0.5,\ \alpha = 0.0027$

NO	$p(n_1)$	$p(n_2)$	h_1	h_2	n_1	n_2	$LCL(n_1)$	$wl(n_1)$	$wu(n_1)$	$UCL(n_1)$	$LCL(n_2)$	$wl(n_2)$	$wu(n_2)$	$UCL(n_2)$	Opt.ATS_1*	Spe.ATS_1	Saved%
1	0.349	0.651	1.43	0.1	2.00	11.25	0.110	0.344	6.331	20.620	1.536	3.925	4.100	4.101	1.16	3.46	66.52%
2	0.376	0.624	1.33	0.1	2.25	12.48	0.155	0.423	7.270	18.575	1.672	3.639	3.790	3.791	0.80	1.52	47.59%
3	0.307	0.693	1.61	0.1	2.28	9.03	0.660	1.322	11.597	27.555	3.416	7.607	7.802	7.804	0.43	1.02	57.57%
4	0.404	0.596	1.24	0.1	2.12	15.96	0.259	0.588	9.790	30.952	2.667	4.561	4.759	4.760	1.33	4.53	70.56%
5	0.13	0.87	3.66	0.1	2.05	6.00	0.264	0.424	4.150	19.440	1.610	7.123	7.199	7.199	0.29	1.04	72.01%
6	0.39	0.61	1.28	0.1	2.96	11.46	0.291	0.635	7.577	14.275	1.563	3.826	4.081	4.083	0.91	1.24	26.47%
7	0.133	0.867	3.70	0.1	2.00	6.00	0.254	0.594	4.171	20.105	1.637	7.209	7.285	7.287	0.30	1.09	72.70%
8	0.433	0.577	1.17	0.1	3.10	15.04	0.556	1.165	10.573	21.791	2.585	4.677	5.006	5.006	1.22	2.40	49.26%
9	0.128	0.872	3.67	0.1	2.03	6.00	0.258	0.337	4.096	19.610	1.610	7.124	7.199	7.199	0.28	1.01	71.82%
10	0.285	0.715	1.79	0.1	2.14	8.25	0.583	1.631	10.458	29.229	3.201	8.215	8.422	8.426	0.46	1.12	59.01%
11	0.4	0.6	1.24	0.1	2.66	13.89	0.229	0.400	8.076	15.670	1.779	3.295	3.430	3.431	0.61	1.09	43.69%
12	0.351	0.649	1.43	0.1	2.00	11.31	0.111	0.364	6.443	20.817	1.558	3.948	4.122	4.123	1.12	3.20	65.13%
13	0.395	0.605	1.26	0.1	2.56	13.37	0.211	0.468	7.862	16.441	1.741	3.436	3.582	3.582	0.73	1.29	43.11%
14	0.474	0.474	1.05	0.1	3.90	24.36	0.794	0.884	13.282	17.452	3.050	3.080	3.430	3.433	1.11	1.52	27.06%
15	0.128	0.872	3.70	0.1	2.00	6.00	0.248	0.377	4.050	19.877	1.610	7.124	7.199	7.199	0.29	1.13	74.34%
16	0.354	0.646	1.43	0.1	2.18	10.88	0.141	0.485	6.475	18.954	1.506	4.014	4.214	4.216	0.91	1.66	44.94%
17	0.474	0.474	1.67	0.1	2.00	9.03	0.254	3.326	13.225	20.105	2.403	4.941	5.124	5.157	0.75	1.01	25.82%
18	0.418	0.418	2.74	0.1	2.00	6.55	0.221	3.009	6.225	32.329	1.413	4.530	6.683	10.742	1.38	3.18	56.73%
19	0.402	0.598	1.23	0.1	2.64	14.19	0.229	0.393	8.250	15.902	1.823	3.272	3.403	3.403	0.57	1.03	44.75%
20	0.474	0.474	1.57	0.1	2.00	9.73	0.502	4.307	16.272	31.256	3.597	6.969	7.287	7.329	0.98	1.08	9.16%
21	0.391	0.609	1.27	0.1	2.54	13.21	0.207	0.422	7.738	16.359	1.720	3.434	3.578	3.579	0.72	1.26	42.89%
22	0.252	0.748	2.35	0.1	2.00	7.00	0.248	1.492	6.703	19.877	1.887	6.179	6.284	6.294	0.36	1.00	63.53%
23	0.368	0.632	1.37	0.1	2.02	12.24	0.113	0.396	6.886	20.709	1.638	3.707	3.859	3.861	0.85	1.75	51.41%
24	0.434	0.434	3.30	0.1	2.45	6.00	0.358	3.745	6.326	27.177	1.317	4.822	6.697	11.570	0.54	1.42	62.10%
25	0.452	0.452	3.12	0.1	2.65	6.00	0.413	3.869	6.540	25.439	1.298	4.798	6.763	11.680	0.51	1.24	59.18%
26	0.385	0.615	1.29	0.1	2.49	12.82	0.198	0.413	7.592	16.658	1.696	3.515	3.662	3.662	0.75	1.36	44.61%
27	0.172	0.828	3.40	0.1	2.03	6.11	0.265	1.268	4.843	19.796	1.670	7.077	7.162	7.171	0.32	1.01	68.30%

Note that ATS_1 Saved% = [(Spe.ATS_1 - Opt.ATS_1^*) / Spe.ATS_1]•100%.

From Table 30 to Table 33, we could see that the performance of the optimal VSSI BL chart is better than the BL chart in detecting small process shifts. Also, we found the same result as previous method, the optimal VSSI BL chart has better performance with $K_{11}/K_{22} = 4$.

Chapter 5. Performance Comparison With Some Existing Methods

5.1 ARL_1 Comparison of the BL Chart and Khoo's Max Bivariate Control Chart

To compare ARL_1 of the BL chart with Khoo's Max Bivariate chart. The following is the approach of Max Bivariate chart.

For in-control process,

$$Y \sim BN(\mu_0, \Sigma_0), \mu = T$$

Define

$$T_i^2 = (\overline{Y} - \mu)'\Sigma^{-1}(\overline{Y} - \mu) \sim \chi_2^2,$$

$$2(n-1)|S_i|^{1/2} / |\Sigma_0|^{1/2} \sim \chi_{2n-4}^2$$

Let

$U_i = \Phi^{-1}[H_2(T_i^2)], i = 1,2,\dots$

Here $\Phi(\cdot)$ denotes the standard normal distribution function while $H_2(\cdot)$ represents the chi-square distribution function with two degree of freedom.

$$V_i = \Phi^{-1}[H_{2n-4}\{2(n-1)|S_i|^{1/2} / |\Sigma_0|^{1/2}\}], i = 1,2,\dots$$

$$U_i \sim N(0,1), V_i \sim N(0,1)$$

Statistics: $C_i = \max\{|U_i|, |V_i|\}$

The control limits are as below:

$$UCL = \Phi^{-1}\left(\frac{1+\sqrt{1-\alpha}}{2}\right)$$

$$LCL = 0$$

Case 1: Refer to Khoo (2005)

Let $n = 10, \alpha = 0.005, K_{11} = 1, K_{12} = 0, K_{22} = 1$ and $\mu = T$.

For in-control process,

$$Y \sim BN(\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

For out-of-control process,

$$Y \sim BN(\mu_1 = \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} b^2 & 0 \\ 0 & 1 \end{pmatrix})$$

where δ is the mean shift and b is the covariance matrix shift.

Table 34. Compare ARL_1 for the BL Chart and Max Bivariate Chart
(See Khoo (2005) Case 1)

	$\delta=0$		$\delta=0.4$		$\delta=0.8$		$\delta=0.9$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	200.00	200.55	131.57	279.02	25.73	207.87	16.45	123.23
1.05	155.49	171.04	80.84	213.99	15.62	137.47	10.34	85.18
1.1	103.61	135.28	48.81	152.97	10.16	91.18	6.98	59.93
1.15	65.23	101.95	30.35	107.68	7.04	63.78	5.02	43.78
1.2	41.25	74.12	19.74	75.54	5.15	45.30	3.80	32.44
1.25	26.92	54.47	13.48	53.77	3.96	32.85	3.02	25.07
1.3	18.28	40.35	9.63	39.72	3.17	25.30	2.49	19.61
1.5	5.70	15.24	3.65	14.62	1.76	10.43	1.52	8.91
2	1.61	3.61	1.38	3.46	1.10	2.98	1.06	2.82
2.5	1.16	1.89	1.10	1.84	1.02	1.72	1.01	1.67
3	1.08	1.39	1.06	1.37	1.01	1.32	1.00	1.30

Table 34. Continues

	$\delta=1$		$\delta=1.1$		$\delta=1.75$		$\delta=2$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	10.74	65.69	7.22	34.20	1.38	1.92	1.10	1.28
1.05	7.03	44.80	4.93	26.82	1.23	1.92	1.06	1.29
1.1	4.94	37.02	3.60	21.84	1.14	1.92	1.03	1.31
1.15	3.68	28.38	2.79	18.05	1.09	1.91	1.02	1.32
1.2	2.89	22.03	2.27	15.11	1.05	1.90	1.01	1.34
1.25	2.37	17.89	1.92	12.54	1.03	1.89	1.00	1.34
1.3	2.01	14.90	1.68	10.67	1.02	1.88	1.00	1.35
1.5	1.35	7.34	1.22	6.14	1.00	1.80	1.00	1.37
2	1.03	2.66	1.02	2.48	1.00	1.50	1.00	1.31
2.5	1.00	1.63	1.00	1.58	1.00	1.29	1.00	1.20
3	1.00	1.29	1.00	1.27	1.00	1.16	1.00	1.13

Case 2: Refer to Khoo (2005)

Let $n = 10, \alpha = 0.005$, $K_{11} = 1, K_{12} = 0, K_{22} = 1$ and $\mu = T$.

For in-control process,

$$Y \sim BN(\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

For out-of-control process,

$$Y \sim BN(\mu_1 = \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix})$$

where δ is the mean shift and b is the covariance matrix shift.

Table 35. Compare ARL_1 for the BL Chart and Max Bivariate Chart (See Khoo (2005) Case 2)

	$\delta = 0$		$\delta = 0.4$		$\delta = 0.8$		$\delta = 0.9$	
b	*BL*	*Max−Bi*	*BL*	*Max−Bi*	*BL*	*Max−Bi*	*BL*	*Max−Bi*
1	200.00	200.55	131.57	279.02	25.73	207.87	16.45	123.23
1.05	109.27	132.53	56.10	152.93	12.40	101.34	8.50	68.66
1.1	48.10	69.65	25.47	71.15	6.96	49.19	5.08	37.22
1.15	23.00	35.91	13.23	35.78	4.42	26.63	3.40	21.72
1.2	12.45	19.98	7.77	19.50	3.10	15.17	2.50	13.13
1.25	7.54	11.91	5.06	11.53	2.35	9.43	1.98	8.35
1.3	5.01	7.72	3.59	7.45	1.90	6.34	1.65	5.75
1.5	1.87	2.51	1.60	2.47	1.21	2.26	1.15	2.18
2	1.05	1.12	1.03	1.11	1.01	1.10	1.00	1.09
2.5	1.00	1.01	1.00	1.01	1.00	1.01	1.00	1.01
3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 35. Continues

	$\delta = 1$		$\delta = 1.1$		$\delta = 1.75$		$\delta = 2$	
b	*BL*	*Max−Bi*	*BL*	*Max−Bi*	*BL*	*Max−Bi*	*BL*	*Max−Bi*
1	10.74	65.69	7.22	34.20	1.38	1.92	1.10	1.28
1.05	5.96	42.04	4.30	24.27	1.20	1.90	1.05	1.29
1.1	3.79	26.00	2.90	17.30	1.11	1.87	1.02	1.30
1.15	2.68	16.52	2.16	12.17	1.06	1.83	1.01	1.30
1.2	2.06	10.71	1.73	8.39	1.03	1.77	1.00	1.29
1.25	1.69	7.25	1.47	6.08	1.02	1.69	1.00	1.27
1.3	1.46	5.15	1.31	4.48	1.01	1.60	1.00	1.25
1.5	1.10	2.11	1.06	2.01	1.00	1.31	1.00	1.16
2	1.00	1.09	1.00	1.08	1.00	1.04	1.00	1.03
2.5	1.00	1.01	1.00	1.01	1.00	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Case 3: Refer to Case 3 of Khoo (2005)

Let $n=10, \alpha=0.005$, $K_{11}=1, K_{12}=0, K_{22}=1$ and $\mu=T$.

For in-control process,

$$Y \sim BN(\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

For out-of-control process,

$$Y \sim BN(\mu_1 = \begin{pmatrix} \delta \\ \delta \end{pmatrix}, \Sigma_1 = \begin{pmatrix} b^2 & 0 \\ 0 & 1 \end{pmatrix})$$

where δ is the mean shift and b is the covariance matrix shift.

Table 36. Compare ARL_1 for the BL Chart and Max Bivariate Chart (See Khoo (2005) Case 3)

	$\delta=0$		$\delta=0.4$		$\delta=0.6$		$\delta=0.7$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	200.00	200.55	73.82	326.60	20.74	164.40	11.20	71.14
1.05	155.49	171.04	46.67	229.01	14.21	119.09	8.10	55.85
1.1	103.61	135.28	30.00	152.75	10.06	86.06	6.05	44.59
1.15	65.23	101.95	19.94	104.97	7.38	61.70	4.67	34.51
1.2	41.25	74.12	13.77	71.54	5.61	44.56	3.73	27.18
1.25	26.92	54.47	9.90	50.63	4.41	33.14	3.06	21.65
1.3	18.28	40.35	7.39	36.92	3.58	25.39	2.58	17.42
1.5	5.70	15.24	3.15	13.62	1.98	10.32	1.62	8.19
2	1.61	3.61	1.32	3.28	1.15	2.89	1.09	2.66
2.5	1.16	1.89	1.07	1.77	1.03	1.65	1.02	1.59
3	1.08	1.39	1.02	1.34	1.01	1.29	1.00	1.26

Table 36. Continues

	$\delta=0.8$		$\delta=0.9$		$\delta=1$		$\delta=1.1$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	6.40	27.91	3.94	12.13	2.62	6.05	1.89	3.47
1.05	4.88	21.14	3.16	11.07	2.20	5.73	1.66	3.36
1.1	3.84	20.65	2.61	10.08	1.90	5.41	1.50	3.26
1.15	3.12	17.72	2.21	9.26	1.69	5.13	1.37	3.17
1.2	2.60	15.22	1.93	8.37	1.52	4.84	1.28	3.09
1.25	2.23	12.91	1.71	7.50	1.40	4.59	1.21	2.98
1.3	1.95	10.97	1.55	6.79	1.31	4.30	1.16	2.89
1.5	1.38	6.26	1.21	4.61	1.11	3.32	1.05	2.49
2	1.05	2.42	1.02	2.14	1.01	1.88	1.00	1.65
2.5	1.01	1.51	1.00	1.43	1.00	1.37	1.00	1.29
3	1.00	1.22	1.00	1.19	1.00	1.16	1.00	1.13

Case 4: Refer to Case 4 of Khoo (2005)

Let $n=10, \alpha=0.005$, $K_{11}=1, K_{12}=0, K_{22}=1$ and $\mu=T$.

For in-control process,

$$Y \sim BN(\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

For out-of-control process,

$$Y \sim BN(\mu_1 = \begin{pmatrix} \delta \\ \delta \end{pmatrix}, \Sigma_1 = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix})$$

where δ is the mean shift and b is the covariance matrix shift.

Table 37. Compare ARL_1 for the BL Chart and Max Bivariate Chart
(See Khoo (2005) Case 4)

	$\delta = 0$		$\delta = 0.4$		$\delta = 0.6$		$\delta = 0.7$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	200	200.55	73.82	326.6	20.74	164.4	11.2	71.14
1.05	109.27	132.53	32.79	152.15	11.11	85.76	6.67	44.91
1.1	48.1	69.65	16.5	67.13	6.71	44.45	4.41	27.54
1.15	23	35.91	9.4	33.36	4.47	24.67	3.17	17.03
1.2	12.45	19.98	5.95	18.08	3.23	14.3	2.44	10.9
1.25	7.54	11.91	4.11	10.96	2.49	9.03	1.98	7.37
1.3	5.01	7.72	3.05	7.21	2.03	6.09	1.69	5.25
1.5	1.87	2.51	1.51	2.41	1.27	2.23	1.18	2.1
2	1.05	1.12	1.03	1.11	1.01	1.1	1.01	1.09
2.5	1	1.01	1	1.01	1	1.01	1	1.01
3	1	1	1	1	1	1	1	1

Table 37. Continues

	$\delta = 0.8$		$\delta = 0.9$		$\delta = 1$		$\delta = 1.1$	
b	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$	BL	$Max-Bi$
1	6.4	27.91	3.94	12.13	2.62	6.05	1.89	3.47
1.05	4.22	20.68	2.84	10.03	2.05	5.46	1.59	3.27
1.1	3.03	15.2	2.2	8.33	1.7	4.82	1.39	3.09
1.15	2.34	10.78	1.81	6.67	1.47	4.24	1.27	2.88
1.2	1.91	7.8	1.55	5.35	1.33	3.68	1.18	2.64
1.25	1.63	5.77	1.39	4.32	1.23	3.16	1.12	3.16
1.3	1.44	4.32	1.27	3.46	1.16	2.7	1.08	2.16
1.5	1.11	1.96	1.07	1.8	1.04	1.63	1.02	1.49
2	1	1.08	1	1.07	1	1.06	1	1.05
2.5	1	1.01	1	1.01	1	1	1	1
3	1	1	1	1	1	1	1	1

From the results of Table 34 to Table 37, we found that the BL chart has smaller ARL_1 than Max Bivariate chart. Also, we found BL chart and Max Bivariate chart have similar performance when process had large shifts ($b \ge 1.25$ or $\delta \ge 1.75$). We set $K_{11}/K_{22} = 1$ for the BL chart for 4 cases based on our data analysis which indicates that when two quality characteristics are independent, the BL chart performs better with $K_{11}/K_{22} = 1$.

5.2 ARL_1 Comparison for the BL Chart, MSE Chart, Max-CUSUM Chart and Max- MEWMA Chart

To compare ARL_1 of the BL chart with other existing charts: MSE chart (Spiring and Cheng (1998).), Max-CUSUM chart (Cheng and Thaga (2005).) and Max-MEWMA chart (Xie et. al (1999).), (what are they and refer to which papers!! You are lazy!!), we have the following:

for in-control process,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim BN(\mu = \begin{pmatrix} \mu_{y_1} \\ \mu_{y_2} \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}), \mu = T$$

where $-1 \le \rho \le 1$;

for out-of-control process,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim BN(\mu = \begin{pmatrix} \mu_{y_1} + \delta \\ \mu_{y_2} + \delta \end{pmatrix}, b^2 \Sigma = \begin{pmatrix} b^2\sigma_1^2 & b^2\rho\sigma_1\sigma_2 \\ b^2\rho\sigma_1\sigma_2 & b^2\sigma_2^2 \end{pmatrix})$$

where δ is the mean shift and b is the covariance matrix shift.

Table 38. ARL comparison for both BL Chart and Max-CUSUM Chart with

$n = 2, K_{11} / K_{22} = 4, \rho = 0.1, ARL_0 = 250$

	$\delta = 0$		$\delta = 0.25$		$\delta = 0.50$		$\delta = 1.00$		$\delta = 1.5$		$\delta = 2.00$		$\delta = 2.50$	
b	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*
1.00	250.13	250.00	**16.72**	**79.56**	**6.89**	**39.50**	**2.77**	**9.30**	**1.68**	**2.99**	1.34	1.46	1.14	1.08
1.25	**9.43**	**19.23**	**13.01**	**15.60**	**5.71**	**9.76**	**2.6**	**3.56**	1.65	1.69	1.29	1.14	1.1	1.02
1.50	7.48	7.27	11.53	6.30	5.39	4.52	2.66	2.18	1.61	1.31	1.24	1.05	1.1	1.00
2.00	5.46	2.74	9	2.55	4.69	2.12	2.51	1.40	1.6	1.09	1.2	1.01	1.39	1.00
2.50	4.47	1.76	7.38	1.69	4.21	1.52	2.48	1.18	1.58	1.03	1.17	1.00	1.09	1.00
3.00	3.9	1.40	6.26	1.37	3.85	1.28	2.44	1.09	1.55	1.01	1.15	1.00	1.05	1.00
4.00	4.88	1.15	4.88	1.14	3.34	1.10	2.35	1.03	1.5	1.00	1.15	1.00	1.05	1.00

Table 39. ARL comparison for BL Chart and Max-CUSUM Chart with

$n = 2, K_{11} / K_{22} = 4, \rho = 0.6, ARL_0 = 200$

	$\delta = 0$		$\delta = 0.50$		$\delta = 1.00$		$\delta = 1.5$		$\delta = 2.00$		$\delta = 2.50$		$\delta = 3.00$	
b	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*	*MaxC*	*BL*
1.00	200.2	200.00	**6.2**	**26.10**	**3.6**	**7.49**	**2.6**	**2.73**	2	1.43	1.7	1.08	1.4	1.01
1.50	11.3	5.61	3.7	3.82	3.4	2.04	2.6	1.30	2.1	1.06	1.8	1.01	1.4	1.00
2.00	7.5	2.40	4.1	1.95	3.2	1.37	2.6	1.09	2.2	1.01	1.8	1.00	1.6	1.00
2.50	5.8	1.63	3.7	1.45	3.1	1.17	2.6	1.03	2	1.00	1.9	1.00	1.3	1.00
3.00	4.9	1.34	3.4	1.24	2.9	1.09	2.5	1.01	2	1.00	1.5	1.00	1.2	1.00

From Table 38, the BL chart with $\rho = 0.1$ has smaller ARL_1 when $b \geq 1.5$ and $\delta \geq 1.5$. From previous analysis, using the BL chart needs more samples (n = 5) to detect process. Thus, in this case, the BL chart is insensitive to small shift of the mean vector and the covariance matrix when we set n = 2. From Table 39, the BL chart with $\rho = 0.6$ has better performance than the Max-CUSUM chart, except the case that only the mean vector shifted.

Table 40. ARL for BL Chart, Max-CUSUM Chart, the Max-MEWMA Chart and Multivariate MSE chart under $n=2, K_{11}/K_{22}=4, \rho=0.6, ARL_0=200$

	$\delta=0$				$\delta=0.50$				$\delta=1.00$			
b	*MaxC*	*MaxE*	*MSE*	*BL*	*MaxC*	*MaxE*	*MSE*	*BL*	*MaxC*	*MaxE*	*MSE*	*BL*
1.00	200.2	200.1	200.0	200.00	**6.2**	**18.2**	**22.3**	**26.10**	**3.6**	**5.3**	**7.6**	**7.49**
1.50	11.3	7.5	13.3	5.61	3.7	6.1	7.3	3.82	3.4	4.1	4.7	2.04
2.00	7.5	3.4	3.1	2.40	4.1	3.2	2.8	1.95	3.2	2.9	2.7	1.37
2.50	5.8	2.4	2.0	1.63	3.7	2.4	1.6	1.45	3.1	2.2	1.4	1.17
3.00	4.9	2.0	1.5	1.34	3.4	2.0	1.4	1.24	2.9	1.9	1.2	1.09

Table 40. Continues

	$\delta=1.5$				$\delta=2.00$			
b	*MaxC*	*MaxE*	*MSE*	*BL*	*MaxC*	*MaxE*	*MSE*	*BL*
1.00	**2.6**	**3.2**	**4.1**	**2.73**	2	2.3	2.1	1.43
1.50	2.6	3.0	3.3	1.30	2.1	2.3	1.9	1.06
2.00	2.6	2.4	2.6	1.09	2.2	2.1	1.7	1.01
2.50	2.6	2.1	1.4	1.03	2	1.9	1.3	1.00
3.00	2.5	1.8	1.2	1.01	2	1.7	1.0	1.00

Table 40. Continues

	$\delta=2.50$				$\delta=3.00$			
b	*MaxC*	*MaxE*	*MSE*	*BL*	*MaxC*	*MaxE*	*MSE*	*BL*
1.00	1.7	2.0	1.5	1.08	1.4	1.7	1.2	1.01
1.50	1.8	1.7	1.4	1.01	1.4	1.2	1.2	1.00
2.00	1.8	1.8	1.3	1.00	1.6	1.6	1.1	1.00
2.50	1.9	1.7	1.2	1.00	1.3	1.5	1.1	1.00
3.00	1.5	1.1	1.0	1.00	1.2	1.0	1.0	1.0

From Table 40, we found that the performance of the BL chart in detecting small shift is poorer than other charts. But, when process had larger shifts the performance of the BL chart is better, especially in detecting both the mean vector and covariance matrix shifts. Comparing with MSE chart, Max-CUSUM chart and Max-MEWMA chart, if there is only mean shift, the BL chart has smaller ARL_1 when $\delta>1.5$.

5.3 ARL_1 Comparison for the Bivariate Loss Chart, EWMA V- Chart, EWMA M-Chart, MEWMA Chart and $|S|$ - Chart

For the following comparisons, we only discuss the covariance matrix shifts. Also, the correlations of two quality characteristics are set at ρ = -0.2, 0, 0.5, 0.8. For each case the sample size n = 4, 8. Note that for below tables 41 to 48, we refer to Yeh *et al*. (2003), except the BL chart.

(1). Independent Case

Table 41. Comparison of ARL for Changes in Variability with n = 4 and $\rho = 0$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 2)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	BL chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0)	128.42	114.06	12.61	123.60	102.28	1.210
(1.10,1.10,0)	93.40	**64.44**	88.93	94.05	**87.54**	1.464
(1.25,1.00,0)	72.96	51.73	68.18	80.96	35.04	1.563
(1.50,1.00,0)	31.80	20.36	32.09	35.09	9.52	2.250
(1.25,1.25,0)	31.81	**17.38**	39.22	29.24	**22.85**	2.441
(1.50,1.50,0)	12.60	7.03	15.04	8.32	5.84	5.063

Table 42. Comparison of ARL for Changes in Variability with n = 8 and $\rho = 0$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 2)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	BL chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0)	125.54	**77.29**	113.66	114.14	**85.13**	1.210
(1.10,1.10,0)	93.19	**28.13**	86.49	64.20	**64.83**	1.464
(1.25,1.00,0)	76.30	**21.50**	70.53	50.68	**22.12**	1.563
(1.50,1.00,0)	32.01	8.24	32.07	15.76	5.16	2.250
(1.25,1.25,0)	31.25	**7.29**	39.83	12.85	**12.23**	2.441
(1.50,1.50,0)	12.41	3.74	14.84	2.99	2.81	5.063

(2). Medium correlation

Table 43. Comparison of ARL for Changes in Variability with n = 4 and $\rho = 0.5$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 0.5)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	BL chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0.5)	**125.67**	**141.25**	**83.94**	146.53	**143.64**	0.908
(1.10,1.10,0.5)	107.29	133.87	**65.24**	133.60	**89.91**	1.098
(1.25,1.00,0.5)	87.96	126.77	**53.88**	129.65	**77.07**	1.172
(1.50,1.00,0.5)	39.33	41.73	28.92	68.85	28.90	1.688
(1.25,1.25,0.5)	40.16	33.01	33.39	57.74	24.18	1.831

Table 44. Comparison of ARL for Changes in Variability with n = 8 and $\rho = 0.5$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 0.5)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	BL chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0.5)	**125.57**	**131.28**	**83.24**	142.61	**133.19**	0.908
(1.10,1.10,0.5)	106.07	123.70	**65.78**	132.41	**67.38**	1.098
(1.25,1.00,0.5)	89.37	**91.11**	**52.59**	122.66	**58.75**	1.172
(1.50,1.00,0.5)	39.82	**16.54**	**28.80**	40.21	**17.78**	1.688
(1.25,1.25,0.5)	40.99	**12.90**	32.83	29.52	**13.12**	1.831
(1.50,1.50,0.5)	14.75	4.41	14.71	4.68	3.01	3.797

(3). Negative Correlation

Table 45. Comparison of ARL for Changes in Variability with n = 4 and $\rho = -0.2$ (w = 0.2, p = 2, ARL_0=200, K_{11} / K_{22} = 2)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\lvert S \rvert$-chart	*BL* chart	$\lvert\Sigma\rvert / \lvert\Sigma_0\rvert$
(1.10,1.00,-0.2)	129.36	126.33	107.15	128.95	65.91	1.162
(1.10,1.10, -0.2)	96.72	73.58	82.99	98.89	54.78	1.406
(1.25,1.00, -0.2)	77.57	58.06	66.22	88.19	23.09	1.500
(1.50,1.00, -0.2)	53.29	22.03	30.89	38.60	7.29	2.160
(1.25,1.25, -0.2)	33.18	19.23	37.30	32.17	15.60	2.344
(1.50,1.50, -0.2)	12.70	7.22	15.06	8.74	4.44	4.860

Table 46. Comparison of ARL for Changes in Variability with n = 8 and $\rho = -0.2$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 2)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\lvert S \rvert$-chart	*BL* chart	$\lvert\Sigma\rvert / \lvert\Sigma_0\rvert$
(1.10,1.00,-0.2)	125.55	98.78	107.81	124.76	48.33	1.162
(1.10,1.10, -0.2)	95.69	32.90	82.65	71.20	36.06	1.406
(1.25,1.00, -0.2)	78.03	25.27	66.49	57.94	13.66	1.500
(1.50,1.00, -0.2)	32.83	9.01	31.19	17.57	4.20	2.160
(1.25,1.25, -0.2)	32.99	7.73	38.74	14.25	8.08	2.344
(1.50,1.50, -0.2)	12.81	3.81	14.87	3.15	2.28	4.860

(4). **Higher Correlation**

Table 47. Comparison of ARL for Changes in Variability with n = 4 and $\rho = 0.8$ (w=0.2, p=2, ARL_0=200, K_{11}/K_{22}=0.5)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\lvert S \rvert$-chart	BL chart	$\lvert\Sigma\rvert / \lvert\Sigma_0\rvert$
(1.10,1.00,0.8)	**92.32**	**24.95**	**60.77**	**139.17**	**143.52**	0.436
(1.10,1.10, 0.8)	96.37	**39.38**	**47.79**	141.92	**91.55**	0.527
(1.25,1.00, 0.8)	90.48	**46.63**	**41.11**	145.85	**77.51**	0.563
(1.50,1.00, 0.8)	52.14	125.12	**24.29**	148.43	**29.69**	0.810
(1.25,1.25, 0.8)	56.55	138.84	27.16	148.54	25.16	0.879
(1.50,1.50, 0.8)	21.04	32.57	13.43	57.91	6.39	1.823

Table 48. Comparison of ARL for Changes in Variability with n = 8 and $\rho = 0.8$ (w = 0.2, p = 2, ARL_0 = 200, K_{11} / K_{22} = 0.5)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\lvert S \rvert$-chart	BL chart	$\lvert\Sigma\rvert / \lvert\Sigma_0\rvert$
(1.10,1.00,0.8)	**93.11**	**8.57**	**61.09**	**58.71**	**84.89**	0.436
(1.10,1.10, 0.8)	**92.61**	**13.21**	**48.42**	88.59	**25.57**	0.527
(1.25,1.00, 0.8)	**87.80**	**15.89**	**41.06**	97.51	**42.15**	0.563
(1.50,1.00, 0.8)	51.17	79.57	24.06	140.89	NA	0.810
(1.25,1.25, 0.8)	55.99	118.50	26.89	145.57	6.85	0.879
(1.50,1.50, 0.8)	20.73	13.06	13.28	30.13	2.20	1.823

From the above Tables (41 to 48), we found that for each case, if we take larger size samples, ARL_1 will decrease for all charts. This is reasonable. Comparing results of different ρ, the BL chart outperforms others when correlation is negative ($\rho = -0.2$). Basically, the BL chart has better performance than M-chart and $|S|$ -chart. Comparing with V – chart and MEWMA chart, the BL chart has better performance when the process is in the following case.

1. Only one σ shifted, σ should be larger than 1.10.
2. Two σ shifted, both σ should be larger than 1.10 for independent case and larger than 1.25 for medium and higher correlation case.

When only the covariance matrix shifted, the performance of the BL chart is poorer than V-chart and MEWMA chart in detecting small shifts, especially only one σ shifted. We also found that K_{11}/K_{22} affects the performance of the BL chart.

Chapter 6. Conclusion and Future Research

Our proposed control charts are constructed based on the approximate distribution of the average bivariate loss, the BL statistics we used could monitor both the mean vector and the covariance matrix simultaneously. These are single charts which could detect the average loss of products when process is out-of-control, at the same time, we could get the process shifted to more loss or less loss. Unlike most existed multivariate schemes, our proposed control charts are more flexible. We allow more possible situations, for example, the mean vector might shift from target vector when process is in-control, or each variable of mean vector (δ_1, δ_2) and covariance matrix (δ_3, δ_4) has different shift scale. We also discussed that two quality characteristics might have different level of correlation when process is out-of-control. From the results of data analyses, our proposed control charts could detect process change quickly when both the mean vector and the covariance matrix had shifted, especially the small shifts. We found that the ratio of K_{11}/K_{22} affects the performance of our proposed control charts. When $K_{11}/K_{22} = 2$ or 4, the performance is better. Thus, we suggest that when deciding the quality characteristics, we put more important quality characteristic in Y_1 and the other in Y_2. In comparison studies, the results showed that our proposed method performs better than Max Bivariate chart, Max CUSUM chart and MEWMA chart when both the mean vector and the covariance matrix had shifted. If only the mean vector had shifted, Max CUSUM chart and MEWMA chart perform better in detecting small shifts given n = 2. But our proposed chart need to take more samples like n = 4 or 5 to detect. If only the covariance matrix had shifted, our proposed chart has better performance than $|S|$ chart, EWMA M-chart. Comparing with EWMA V-chart and MEWMA chart, though our proposed chart does not detect quickly in some situations, it still has some advantages. The EWMA V-chart mainly monitors the covariance matrix, to monitor both the mean vector and the covariance matrix, we need to combine MEWMA chart or EWMA M-chart. On the other hand, our proposed BL chart is easy to use and is a single chart to do two charts' work.

In the future, we could extend two quality characteristics to three or more quality characteristics. To improve the power in detecting small shifts, we could construct the EWMA BL chart and the CUSUM BL chart. Also, we could discuss more about the effect of the ratio K_{11}/K_{22} on performance. We could define some specific rules about the choice of K_{11}/K_{22} with different correlations of quality characteristics.

Appendices

Appendix A

The exact distribution of BL:

$$BL=\frac{1}{n}K_{11}\sum_{j=1}^{n}(Y_{1j}-T_1+\frac{K_{12}}{2K_{11}}(Y_{2j}-T_2))^2+\frac{1}{n}(K_{22}-\frac{K_{12}^2}{4K_{11}})\sum_{j=1}^{n}(Y_{2j}-T_2)^2$$

Let $BL=\Delta_1+\Delta_2$,

where $\Delta_1=\frac{1}{n}K_{11}\sum_{j=1}^{n}(Y_{1j}-T_1+\frac{K_{12}}{2K_{11}}(Y_{2j}-T_2))^2, \Delta_2=\frac{1}{n}(K_{22}-\frac{K_{12}^2}{4K_{11}})\sum_{j=1}^{n}(Y_{2j}-T_2)^2$

From Δ_1, we can separate it into two parts:

$$\Delta_1=\frac{1}{n}K_{11}\sum_{j=1}^{n}[Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}-(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)+(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(T_1+\frac{K_{12}}{2K_{11}}T_2)]^2$$

$$=\frac{1}{n}K_{11}[\sum_{j=1}^{n}(Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}-(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2))^2+\sum_{j=1}^{n}((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2]$$

Let $\Delta_1=\frac{1}{n}K_{11}[\Delta_{11}+\Delta_{12}]$, where $\Delta_{11}=\sum_{j=1}^{n}(Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}-(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2))^2$

$$\Delta_{12}=\sum_{j=1}^{n}((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2.$$

Since we know that

$$Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}\sim N(\mu_{y1}+\frac{K_{12}}{2K_{11}}\mu_{y2},\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12})\quad j=1,\ldots,n.$$

We define the sample variance of $Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}$ as

$$S^2=\frac{\sum_{j=1}^{n}(Y_{1j}+\frac{K_{12}}{2K_{11}}Y_{2j}-(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2))^2}{(n-1)}$$, thus, we can get $\Delta_{11}=(n-1)S^2$.

And then, we know that $\dfrac{(n-1)S^2}{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}}\sim\chi_{n-1}^2$,

So $\Delta_{11}\sim(\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12})\chi_{n-1}^2$. (28)

For Δ_{12}, we separate it into two parts.

$$\Delta_{12}=\sum_{j=1}^{n}((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_{21}})+(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_{21}})-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2$$

$$=\sum_{j=1}^{n}((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_2}))^2+\sum_{j=1}^{n}((\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_{21}})-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2$$

Let $\Delta_{12}=\Delta_{121}+\Delta_{122}$

where $\Delta_{121}=\sum_{j=1}^{n}((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_2}))^2$,

$\Delta_{122}=\sum_{j=1}^{n}((\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_{21}})-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2$.

For Δ_{121}, we know that

$$\frac{n((\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_2}))^2}{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}}=(\frac{(\overline{Y}_1+\frac{K_{12}}{2K_{11}}\overline{Y}_2)-(\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_2})}{\sqrt{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}/n}})^2\sim\chi_1^2$$

$$\Delta_{121}\sim(\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12})\chi_1^2 \tag{29}$$

For Δ_{122},

$$\frac{\Delta_{122}}{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}}=\frac{n((\mu_{y_1}+\frac{K_{12}}{2K_{11}}\mu_{y_2})-(T_1+\frac{K_{12}}{2K_{11}}T_2))^2}{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}}$$

$$=\frac{n(\delta_5\sigma_1+\frac{K_{12}}{2K_{11}}\delta_6\sigma_2)^2}{\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12}} \tag{30}$$

From equations (29) and (30) , we can get

$$\Delta_{12}\sim(\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12})\chi_{1,\tau_{01}}^2 \tag{31}$$

where $\chi_{1,\tau_{01}}^2$ is a non-central parameter with 1 degree of freedom and non-centrality parameter τ_{01}. Here $\tau_{01}=\dfrac{n(\delta_5\sigma_1+\frac{K_{12}}{2K_{11}}\delta_6\sigma_2)^2}{(\sigma_1^2+(\frac{K_{12}}{2K_{11}})^2\sigma_2^2+\frac{K_{12}}{K_{11}}\sigma_{12})}$.

Since $\Delta_1=\frac{1}{n}K_{11}[\Delta_{11}+\Delta_{12}]$, from equations (28) and (31),

$$\Delta_1 \sim \frac{1}{n} K_{11}(\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2 \sigma_2^2 + \frac{K_{12}}{K_{11}} \sigma_{12}) \chi^2_{n,\tau_{01}} \quad (32)$$

For Δ_2,

$$\Delta_2 = \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}}) \sum_{j=1}^{n} (Y_{2j} - T_2)^2$$

$$= \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}})[\sum_{j=1}^{n} (Y_{2j} - \overline{Y}_2)^2 + \sum_{j=1}^{n} (\overline{Y}_2 - T_2)^2]$$

Let $\Delta_2 = \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}})[\Delta_{21} + \Delta_{22}]$,

Where $\Delta_{21} = \sum_{j=1}^{n} (Y_{2j} - \overline{Y}_2)^2$, $\Delta_{22} = \sum_{j=1}^{n} (\overline{Y}_2 - T_2)^2$.

We know that $\Delta_{21} \sim \sigma_2^2 \chi^2_{n-1}$. (33)

Since $\Delta_{22} = \sum_{j=1}^{n} (\overline{Y}_2 - \mu_{y_2})^2 + \sum_{j=1}^{n} (\mu_{y_2} - T_2)^2$,

Let $\Delta_{22} = \Delta_{221} + \Delta_{222}$, where $\Delta_{221} = \sum_{j=1}^{n} (\overline{Y}_2 - \mu_{y_2})^2$, $\Delta_{222} = \sum_{j=1}^{n} (\mu_{y_2} - T_2)^2$.

We know that $\Delta_{221} \sim \sigma_2^2 \chi_1^2$. (34)

For Δ_{222},

We can get $\frac{\Delta_{222}}{\sigma_2^2} = \frac{n(\mu_{y_2} - T_2)}{\sigma_2^2} = \frac{n\delta_6^2 \sigma_2^2}{\sigma_2^2}$ $(\because T_2 = \mu_{y_2} - \delta_6 \sigma_2)$ (35)

From equations (34) and (35),

$$\Delta_{22} \sim \sigma_2^2 \chi^2_{1,\tau_{02}} \quad (34)$$

where $\chi^2_{1,\tau_{02}}$ is a non-central parameter with 1 degree of freedom and non-centrality parameter τ_{02}. Here $\tau_{02} = \frac{n\delta_7^2 \sigma_2^2}{\sigma_2^2} = n\delta_6^2$.

Since $\Delta_2 = \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}})[\Delta_{21} + \Delta_{22}]$, from equations (33) and (36),

We know that $\Delta_2 \sim \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}}) \sigma_2^2 \chi^2_{n,\tau_{02}}$. (37)

Since $BL = \Delta_1 + \Delta_2$, from equations (31) and (37), we can get

$$BL \sim \frac{1}{n} K_{11}(\sigma_1^2 + (\frac{K_{12}}{2K_{11}})^2 \sigma_2^2 + \frac{K_{12}}{K_{11}} \sigma_{12}) \chi^2_{n,\tau_{01}} + \frac{1}{n}(K_{22} - \frac{K_{12}^2}{4K_{11}}) \sigma_2^2 \chi^2_{n,\tau_{02}}.$$

Appendix B

Table 49. The ARL_1 for BL Chart with Small Shifts Given n = 5

$\alpha = 0.0027$ and $\rho_0 = 0.1$.

No.								$K_{11}/K_{22} = 0.5$	$K_{11}/K_{22} = 1$	$K_{11}/K_{22} = 2$	$K_{11}/K_{22} = 4$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	**Spe.ARL$_1$**	**Spe.ARL$_1$**	**Spe.ARL$_1$**	**Spe.ARL$_1$**
1	0.1	0.1	1.1	1.1	0.1	0.1	0.1	96.43	94.63	105.01	8.73
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	22.42	26.72	39.41	4.90
3	0.1	0.1	2	2	1	1	0.8	16.70	17.54	19.54	2.49
4	0.1	0.5	1.1	1.5	0.5	1	0.8	1.78	1.96	2.65	1.61
5	0.1	0.5	1.5	2	1	0.1	0.1	2.16	2.44	3.20	1.52
6	0.1	0.5	2	1.1	0.1	0.5	0.5	1.54	1.74	1.98	1.16
7	0.1	1	1.1	2	1	0.5	0.5	1.08	1.12	1.25	1.11
8	0.1	1	1.5	1.1	0.5	1	0.8	1.03	1.06	1.15	1.05
9	0.1	1	2	1.5	1	0.1	0.1	1.01	1.03	1.07	1.01
10	0.5	0.5	1.5	1.5	1	1	0.5	1.32	1.38	1.63	1.41
11	0.5	0.5	2	2	0.1	0.1	0.8	1.21	1.26	1.41	1.18
12	0.5	0.5	1.1	1.1	0.5	0.5	0.1	1.44	1.45	1.12	1.01
13	0.5	1	1.5	2	0.1	0.5	0.1	1.88	1.58	1.52	1.23
14	0.5	1	2	1.1	0.5	1	0.5	1.41	1.28	1.26	1.08
15	0.5	1	1.1	1.5	1	0.1	0.8	1.20	1.15	1.03	1.00
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	1.20	1.39	2.11	2.03
17	0.5	0.1	2	1.5	1	0.5	0.1	1.13	1.34	2.03	1.55
18	0.5	0.1	1.1	2	0.1	1	0.5	1.27	1.62	1.33	1.05
19	1	1	2	2	0.5	0.5	0.8	1.15	1.10	1.09	1.06
20	1	1	1.1	1.1	1	1	0.1	1.18	1.07	1.04	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.06	1.02	1.00	1.00
22	1	0.1	2	1.1	1	0.1	0.5	1.06	1.11	1.29	1.47
23	1	0.1	1.1	1.5	0.1	0.5	0.8	1.15	1.20	1.32	1.05
24	1	0.1	1.5	2	0.5	1	0.1	1.11	1.16	1.11	1.01
25	1	0.5	2	1.5	0.1	1	0.1	1.12	1.12	1.18	1.18
26	1	0.5	1.1	2	0.5	0.1	0.5	1.15	1.12	1.12	1.01
27	1	0.5	1.5	1.1	1	0.5	0.8	1.07	1.05	1.02	1.00

Table 50. The ARL_1 for BL Chart with Small Shifts Given n = 5

$\alpha = 0.0027$ and $\rho_0 = 0.8$.

No.								$K_{11}/K_{22} = 0.5$	$K_{11}/K_{22} = 1$	$K_{11}/K_{22} = 2$	$K_{11}/K_{22} = 4$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1
1	0.1	0.1	1.1	1.1	0.1	0.1	0.1	81.27	430.67	51.21	4.38
2	0.1	0.1	1.5	1.5	0.5	0.5	0.5	11.87	71.89	15.11	2.87
3	0.1	0.1	2	2	1	1	0.8	3.65	14.64	5.09	1.70
4	0.1	0.5	1.1	1.5	0.5	1	0.8	1.75	3.28	2.17	1.33
5	0.1	0.5	1.5	2	1	0.1	0.1	1.72	3.70	2.15	1.27
6	0.1	0.5	2	1.1	0.1	0.5	0.5	1.13	1.64	1.32	1.07
7	0.1	1	1.1	2	1	0.5	0.5	1.08	1.29	1.18	1.05
8	0.1	1	1.5	1.1	0.5	1	0.8	1.02	1.11	1.08	1.02
9	0.1	1	2	1.5	1	0.1	0.1	1.00	1.02	1.02	1.00
10	0.5	0.5	1.5	1.5	1	1	0.5	1.38	2.08	1.78	1.20
11	0.5	0.5	2	2	0.1	0.1	0.8	1.12	1.54	1.34	1.08
12	0.5	0.5	1.1	1.1	0.5	0.5	0.1	1.14	1.19	1.04	1.00
13	0.5	1	1.5	2	0.1	0.5	0.1	2.05	2.59	1.63	1.12
14	0.5	1	2	1.1	0.5	1	0.5	1.23	1.79	1.30	1.04
15	0.5	1	1.1	1.5	1	0.1	0.8	1.07	1.06	1.01	1.00
16	0.5	0.1	1.5	1.1	0.5	0.1	0.8	1.24	2.30	2.43	1.46
17	0.5	0.1	2	1.5	1	0.5	0.1	1.06	1.97	1.82	1.21
18	0.5	0.1	1.1	2	0.1	1	0.5	1.07	1.24	1.11	1.02
19	1	1	2	2	0.5	0.5	0.8	1.16	1.24	1.17	1.03
20	1	1	1.1	1.1	1	1	0.1	1.11	1.09	1.01	1.00
21	1	1	1.5	1.5	0.1	0.1	0.5	1.01	1.01	1.00	1.00
22	1	0.1	2	1.1	1	0.1	0.5	1.06	1.34	1.61	1.18
23	1	0.1	1.1	1.5	0.1	0.5	0.8	1.09	1.26	1.12	1.02
24	1	0.1	1.5	2	0.5	1	0.1	1.02	1.08	1.02	1.00
25	1	0.5	2	1.5	0.1	1	0.1	1.14	1.42	1.40	1.07
26	1	0.5	1.1	2	0.5	0.1	0.5	1.08	1.17	1.04	1.00
27	1	0.5	1.5	1.1	1	0.5	0.8	1.01	1.03	1.00	1.00

Table 51. The ARL_1 for BL Chart with Smaller Shifts Given n = 5

$\alpha = 0.0027$ and $\rho_0 = 0.5$.

No.								$K_{11}/K_{22}=0.5$	$K_{11}/K_{22}=1$	$K_{11}/K_{22}=2$	$K_{11}/K_{22}=4$
	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1	Spe.ARL_1
1	0.01	0.01	1.2	1.1	0.01	0.01	0.1	115.00	259.07	38.14	3.46
2	0.01	0.01	1.5	1.5	0.1	0.1	0.3	4.37	7.45	3.86	1.52
3	0.01	0.01	2	2	1	1	0.5	1.11	1.35	1.14	1.02
4	0.01	0.1	1.2	1.5	0.1	1	0.5	5.54	10.35	16.26	4.53
5	0.01	0.1	1.5	2	1	0.01	0.1	1.37	1.71	1.24	1.04
6	0.01	0.1	2	1.1	0.01	0.1	0.3	5.61	4.69	3.11	1.24
7	0.01	0.5	1.2	2	1	0.1	0.3	1.21	1.64	1.34	1.09
8	0.01	0.5	1.5	1.1	0.1	1	0.5	6.29	7.60	7.21	2.40
9	0.01	0.5	2	1.5	1	0.01	0.1	1.22	1.29	1.08	1.01
10	0.1	0.1	1.5	1.5	1	1	0.3	1.57	3.07	1.67	1.12
11	0.1	0.1	2	2	0.01	0.01	0.5	1.37	1.59	1.37	1.09
12	0.1	0.1	1.2	1.1	0.1	0.1	0.1	82.37	195.36	30.62	3.20
13	0.1	0.5	1.5	2	0.01	0.1	0.1	1.67	2.60	2.23	1.29
14	0.1	0.5	2	1.1	0.1	1	0.3	3.67	3.07	2.63	1.52
15	0.1	0.5	1.2	1.5	1	0.01	0.5	1.64	2.49	1.58	1.13
16	0.1	0.01	1.5	1.1	0.1	0.01	0.5	9.28	13.07	7.20	1.66
17	0.1	0.01	2	1.5	1	0.1	0.1	1.39	1.39	1.08	1.01
18	0.1	0.01	1.2	2	0.01	1	0.3	2.38	3.60	6.10	3.18
19	0.5	0.5	2	2	0.1	0.1	0.5	1.14	1.24	1.15	1.03
20	0.5	0.5	1.2	1.1	1	1	0.1	1.69	4.33	1.73	1.08
21	0.5	0.5	1.5	1.5	0.01	0.01	0.3	2.31	3.33	2.26	1.26
22	0.5	0.01	2	1.1	1	0.01	0.3	1.25	1.19	1.02	1.00
23	0.5	0.01	1.2	1.5	0.01	0.1	0.5	4.22	8.30	5.11	1.75
24	0.5	0.01	1.5	2	0.1	1	0.1	1.62	1.97	2.24	1.42
25	0.5	0.1	2	1.5	0.01	1	0.1	2.15	1.96	1.77	1.24
26	0.5	0.1	1.2	2	0.1	0.01	0.3	1.92	3.15	2.54	1.36
27	0.5	0.1	1.5	1.1	1	0.1	0.1	1.65	1.66	1.12	1.01

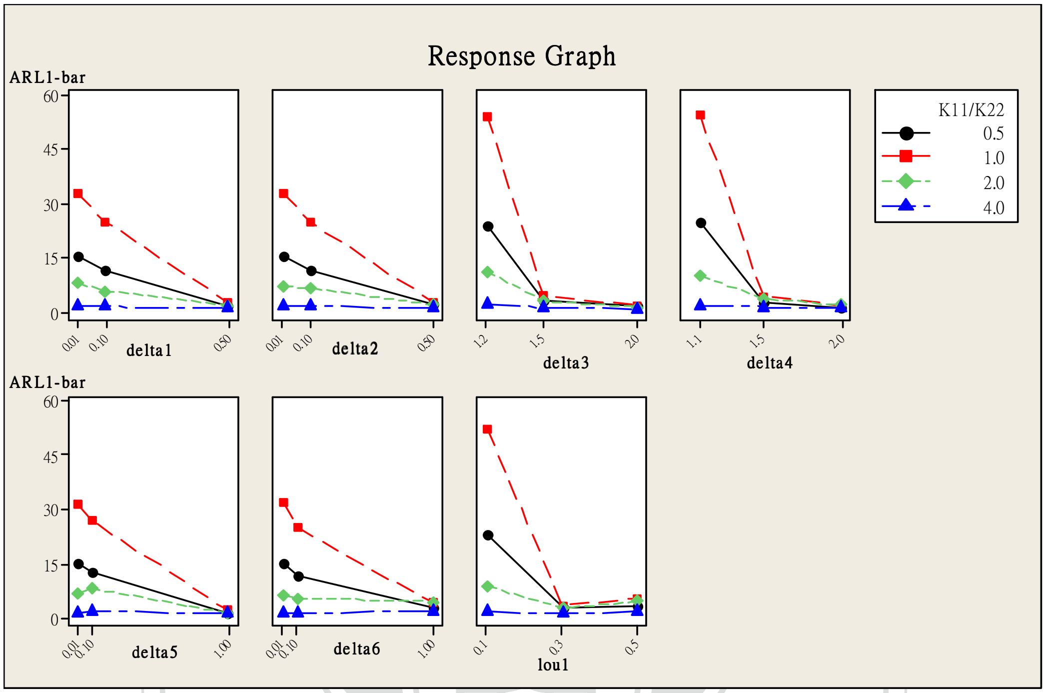

Figure 20. Response Graph of ARL$_1$-bar with $\rho_0 = 0.5$ Based on Table 51

Table 52. The Optimal BL Chart with Smaller Shifts Given $\rho_0 = 0.5$ and $\alpha = 0.0027$

							$K_{11}/K_{22}=0.5$			$K_{11}/K_{22}=1$			$K_{11}/K_{22}=2$			$K_{11}/K_{22}=4$		
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	ρ_1	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$	n^*	h^*	$Opt.ATS_1^*$
0.01	0.01	1.2	1.1	0.01	0.01	0.1	25.00	0.1	0.10	25.00	0.1	0.11	25.00	0.1	0.10	25.00	0.1	0.10
0.01	0.01	1.5	1.5	0.1	0.1	0.3	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.01	0.01	2	2	1	1	0.5	18.63	0.1	0.10	21.89	0.1	0.10	19.61	0.1	0.10	16.15	0.1	0.10
0.01	0.1	1.2	1.5	0.1	1	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.01	0.1	1.5	2	1	0.01	0.1	23.21	0.1	0.10	24.93	0.1	0.10	21.71	0.1	0.10	17.61	0.1	0.10
0.01	0.1	2	1.1	0.01	0.1	0.3	25.00	0.1	0.10	14.48	0.1	0.16	23.79	0.1	0.10	25.00	0.1	0.10
0.01	0.5	1.2	2	1	0.1	0.3	21.46	0.1	0.10	24.80	0.1	0.10	23.57	0.1	0.10	20.28	0.1	0.10
0.01	0.5	1.5	1.1	0.1	1	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.01	0.5	2	1.5	1	0.01	0.1	20.81	0.1	0.10	21.91	0.1	0.10	18.74	0.1	0.10	15.44	0.1	0.10
0.1	0.1	1.5	1.5	1	1	0.3	23.06	0.1	0.10	25.00	0.1	0.10	24.04	0.1	0.10	19.67	0.1	0.10
0.1	0.1	2	2	0.01	0.01	0.5	24.19	0.1	0.10	25.00	0.1	0.10	24.17	0.1	0.10	19.99	0.1	0.10
0.1	0.1	1.2	1.1	0.1	0.1	0.1	24.38	0.1	0.10	25.00	0.1	0.10	20.85	0.1	0.10	17.09	0.1	0.10
0.1	0.5	1.5	2	0.01	0.1	0.1	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	23.09	0.1	0.10
0.1	0.5	2	1.1	0.1	1	0.3	25.00	0.1	0.10	7.14	0.1	0.22	10.04	0.1	0.14	25.00	0.1	0.10
0.1	0.5	1.2	1.5	1	0.01	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	22.36	0.1	0.10
0.1	0.01	1.5	1.1	0.1	0.01	0.5	25.00	0.1	0.10	8.96	0.1	0.18	25.00	0.1	0.10	25.00	0.1	0.10
0.1	0.01	2	1.5	1	0.1	0.1	24.30	0.1	0.10	23.69	0.1	0.10	18.97	0.1	0.10	15.27	0.1	0.10
0.1	0.01	1.2	2	0.01	1	0.3	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.5	0.5	2	2	0.1	0.1	0.5	20.15	0.1	0.10	22.95	0.1	0.10	20.46	0.1	0.10	17.29	0.1	0.10
0.5	0.5	1.2	1.1	1	1	0.1	19.83	0.1	0.10	24.35	0.1	0.10	21.42	0.1	0.10	17.20	0.1	0.10
0.5	0.5	1.5	1.5	0.01	0.01	0.3	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	23.60	0.1	0.10
0.5	0.01	2	1.1	1	0.01	0.3	22.20	0.1	0.10	20.09	0.1	0.10	15.58	0.1	0.10	8.00	0.1	0.10
0.5	0.01	1.2	1.5	0.01	0.1	0.5	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10
0.5	0.01	1.5	2	0.1	1	0.1	23.08	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	23.16	0.1	0.10
0.5	0.1	2	1.5	0.01	1	0.1	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	22.78	0.1	0.10
0.5	0.1	1.2	2	0.1	0.01	0.3	25.00	0.1	0.10	25.00	0.1	0.10	25.00	0.1	0.10	22.65	0.1	0.10
0.5	0.1	1.5	1.1	1	0.1	0.5	25.00	0.1	0.10	24.85	0.1	0.10	19.50	0.1	0.10	14.84	0.1	0.10

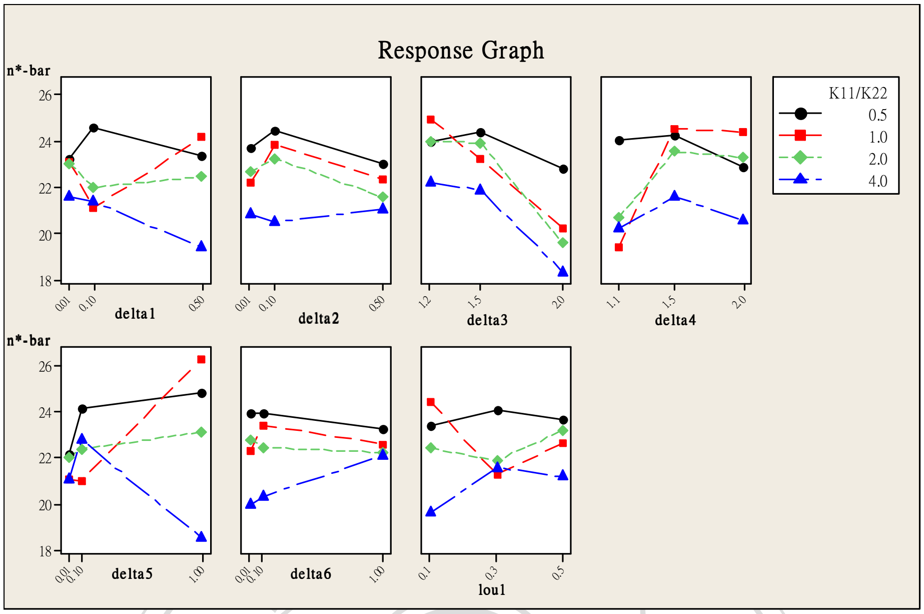

Figure 21. Response Graph for n*-bar under $\rho_0 = 0.5$ Based on Table 52

Table 53. ATS_1 Saved % of Optimal BL Chart and BL Chart for Smaller Shifts with

$$\rho_0 = 0.5$$

No.	$K_{11}/K_{22} = 0.5$			$K_{11}/K_{22} = 1$			$K_{11}/K_{22} = 2$			$K_{11}/K_{22} = 4$		
	$Spe.ATS_1$	$Opt.ATS_1$	Saved%	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %	$Spe.ATS_1$	$Opt.ATS_1$	Saved %
1	115.00	0.10	99.91%	259.07	0.11	99.96%	38.14	0.10	99.73%	3.46	0.10	97.00%
2	4.37	0.10	97.71%	7.45	0.10	98.66%	3.86	0.10	97.41%	1.52	0.10	93.00%
3	1.11	0.10	90.99%	1.35	0.10	92.59%	1.14	0.10	91.23%	1.02	0.10	90.00%
4	5.54	0.10	98.19%	10.35	0.10	99.02%	16.26	0.10	99.38%	4.53	0.10	98.00%
5	1.37	0.10	92.70%	1.71	0.10	94.15%	1.24	0.10	91.94%	1.04	0.10	90.00%
6	5.61	0.10	98.22%	4.69	0.16	96.59%	3.11	0.10	96.68%	1.24	0.10	92.00%
7	1.21	0.10	91.71%	1.64	0.10	93.90%	1.34	0.10	92.51%	1.09	0.10	91.00%
8	6.29	0.10	98.41%	7.60	0.10	98.68%	7.21	0.10	98.61%	2.40	0.10	96.00%
9	1.22	0.10	91.80%	1.29	0.10	92.25%	1.08	0.10	90.74%	1.01	0.10	90.00%
10	1.57	0.10	93.63%	3.07	0.10	96.74%	1.67	0.10	94.01%	1.12	0.10	91.00%
11	1.37	0.10	92.70%	1.59	0.10	93.71%	1.37	0.10	92.70%	1.09	0.10	91.00%
12	82.37	0.10	99.88%	195.36	0.10	99.95%	30.62	0.10	99.67%	3.20	0.10	97.00%
13	1.67	0.10	94.01%	2.60	0.10	96.15%	2.23	0.10	95.52%	1.29	0.10	92.00%
14	3.67	0.10	97.28%	3.07	0.22	92.71%	2.63	0.14	94.61%	1.52	0.10	93.00%
15	1.64	0.10	93.89%	2.49	0.10	95.99%	1.58	0.10	93.66%	1.13	0.10	91.00%
16	9.28	0.10	98.92%	13.07	0.18	98.64%	7.20	0.10	98.61%	1.66	0.10	94.00%
17	1.39	0.10	92.81%	1.39	0.10	92.81%	1.08	0.10	90.74%	1.01	0.10	90.00%
18	2.38	0.10	95.79%	3.60	0.10	97.22%	6.10	0.10	98.36%	3.18	0.10	97.00%
19	1.14	0.10	91.23%	1.24	0.10	91.94%	1.15	0.10	91.30%	1.03	0.10	90.00%
20	1.69	0.10	94.08%	4.33	0.10	97.69%	1.73	0.10	94.22%	1.08	0.10	91.00%
21	2.31	0.10	95.67%	3.33	0.10	97.00%	2.26	0.10	95.58%	1.26	0.10	92.00%
22	1.25	0.10	92.00%	1.19	0.10	91.60%	1.02	0.10	90.20%	1.00	0.10	90.00%
23	4.22	0.10	97.63%	8.30	0.10	98.79%	5.11	0.10	98.04%	1.75	0.10	94.00%
24	1.62	0.10	93.83%	1.97	0.10	94.92%	2.24	0.10	95.54%	1.42	0.10	93.00%
25	2.15	0.10	95.35%	1.96	0.10	94.90%	1.77	0.10	94.35%	1.24	0.10	92.00%
26	1.92	0.10	94.79%	3.15	0.10	96.83%	2.54	0.10	96.06%	1.36	0.10	93.00%
27	1.65	0.10	93.94%	1.66	0.10	93.98%	1.12	0.10	91.07%	1.01	0.10	90.00%

Appendix C

Comparing methods of approximating non-central chi-square distribution to central chi-square distribution (Patnaik (1949) vs. Pearson (1959))

Give n = 5, 15, 25 and $\tau = 3, 5, 10$ and we simulated 100,000 times.

From Figure 22,

1. when sample size is small, Patnaik's method is not that good, but still performs better than Pearson's method. Both methods approximate better when non-centrality parameter is small.
2. When sample size is 15, Patnaik's method approximates better when non-centrality parameter is larger. But, Pearson's method approximates better with small non-centrality parameter, even better than Patnaik's method.
3. When sample size is 25 (large), Patnaik's method approximates very well.

The result shows that Patnaik's method is better than Pearson's method. When the degree of freedom is larger, the approximation is better. Though the Pearson's method is not that good, when the degree of freedom is larger, its performance is better. Pearson's method applies to small non-centrality parameter and also needs more samples.

Blue line: exact central chi-square density

Red line: approximate central chi-square density

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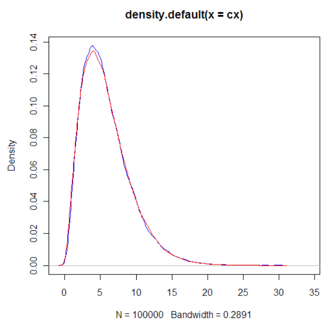
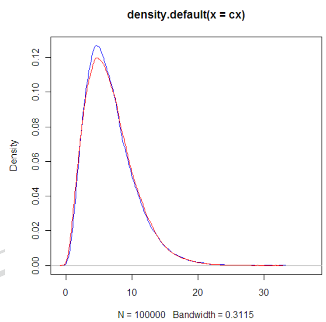
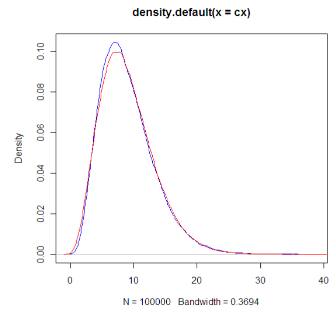
<tr><td>Patnaik's method</td><td>
</td><td>
</td><td>
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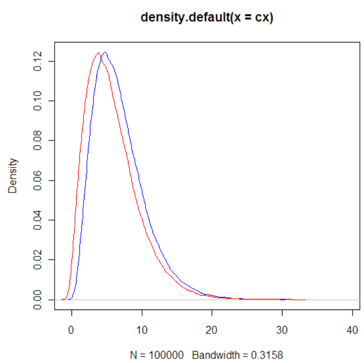
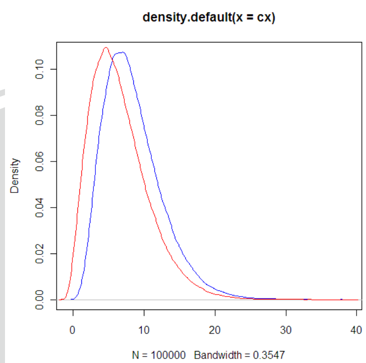
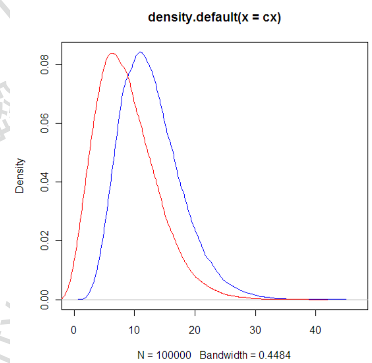
<tr><td>Pearson's method</td><td>
</td><td>
</td><td>
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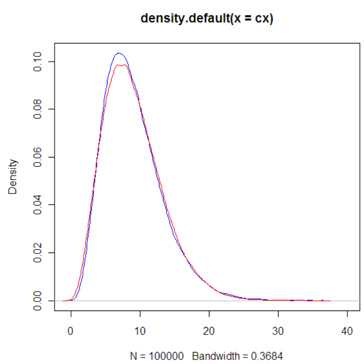
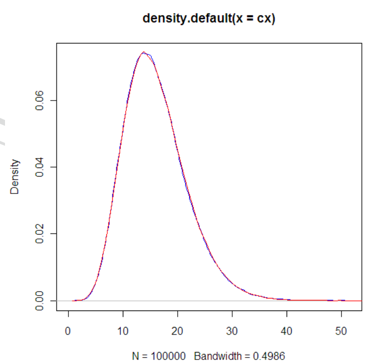
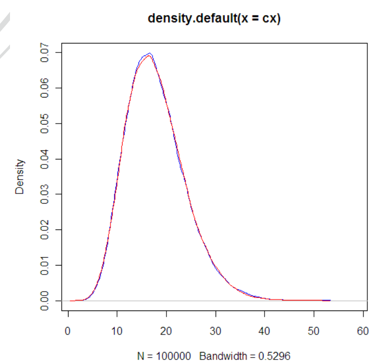
<tr><td>Patnaik's method</td><td>
</td><td>
</td><td>
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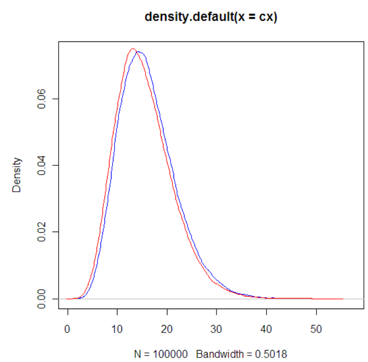
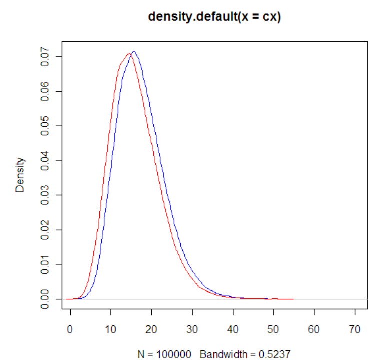
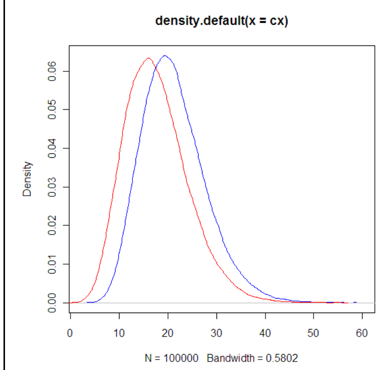
<tr><td>Pearson's method</td><td>
</td><td>
</td><td>
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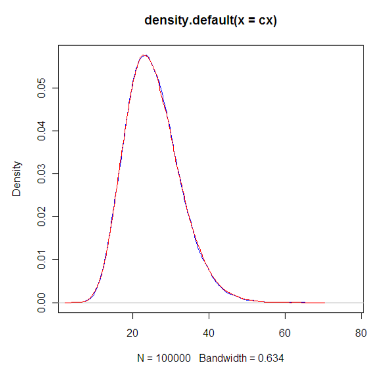
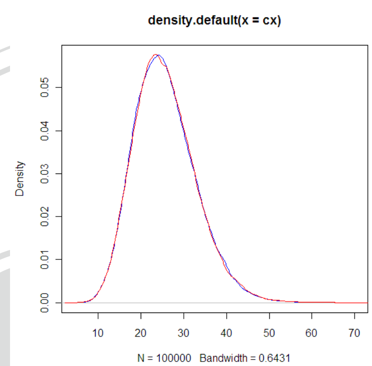
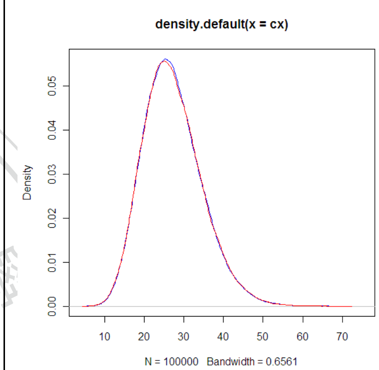
<tr><td>Patnaik's method</td><td>
</td><td>
</td><td>
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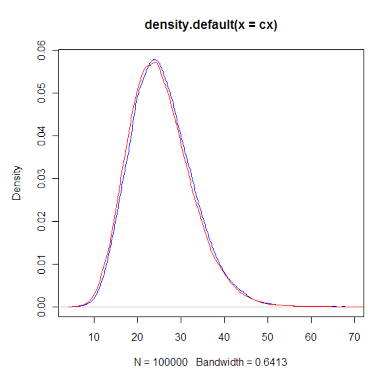
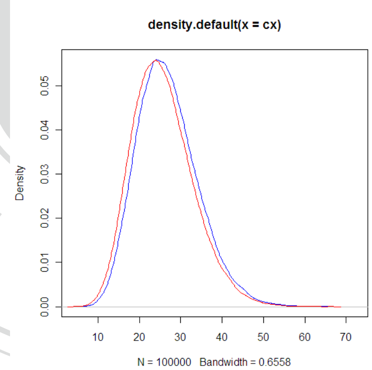
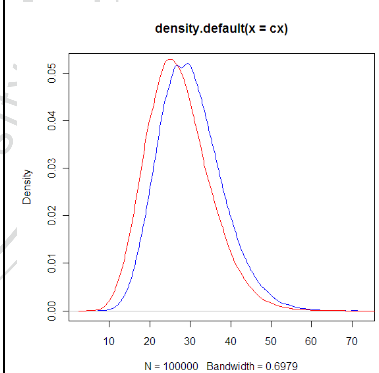
<tr><td>Pearson's method</td><td>
</td><td>
</td><td>
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Figure 22. Simulation Results for Patnaik's Method and Pearson's Method

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