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This article studies variations of mortality-linked structured products that investment banks can issue for insurance companies and annuity providers to hedge their mortality and longevity risk. We examine how different types of mortality-linked structured notes might be constructed through the purchase or sale of mortality options. We further propose bullish and bearish mortality bonds whose redemption values depend on three different kinds of underlying mortality indexes. We demonstrate how their flexible structures can enable investors with different views of future mortality trends to monetize their expectations regarding mortality rates.

The occurrence of unexpected changes in mortality rates poses a direct challenge to the calculation of fair premium rates and risk reserves for the insurance and pensions industries. Annuity providers suffer the risk that pensioners will live longer than predicted by mortality projections, while life insurers experience the risk of unexpected increases in mortality. In recent years, securitization has become an important technique for primary insurers to repackage insurance risks and sell them to investors or other insurers. Since the mid-1990s, the repackaging of catastrophe risk as CAT bonds has established an alternative asset class for investors to participate in the insurance risk market. For discussions of securitization of catastrophic property risk, see Froot [2001]; Niehaus [2002]; Cummins, Lalonde, and Phillips [2004]. In this paper, we propose several different kinds of mortality-linked securities that could be issued by investment banks for life insurers or annuity providers to manage mortality and longevity risks.

Securitization is an attractive means for the insurance industry to offload its mortality risk exposure onto capital markets. With regard to the mortality risk securitization, the Swiss Re issued the first mortality-risk-linked security in December 2003. The Swiss Re mortality bond linked its principal repayment to the experienced mortality rates in five countries, and was designed to reduce the exposure of Swiss Re to catastrophic mortality deterioration over its three-year period. That is, the bond redemption values decrease for bondholders if the realized mortality rate climbs higher. To transfer the other tail of mortality risk, i.e., longevity risk, to capital markets, the European Investment Bank (EIB) issued the 25-year longevity bond to provide a solution for U.K. pension schemes to hedge their long-term systematic longevity risks in November 2004. The EIB longevity bond was an annuity bond with floating coupon payments, and its coupon payments were linked to a cohort survivor index based on the realized mortality rates of English and Welsh males each year.[1] For a review of the life insurance and annuity

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securitizations conducted in recent years, please see Cowley and Cummins [2005]; Blake et al. [2006]; and Lane and Beckwith [2006, 2007]. Both mortality and longevity bonds are simply instruments designed to allow life insurers and pension funds to hedge their exposure to the risk of losses that may arise from payouts under unexpected increases or decreases in mortality rates. Insurance companies and annuity providers use mortality-risk-linked products to transfer or acquire reinsurance risk or manage the portfolio of underlying risks, thus prompting a greater convergence of capital markets and the insurance industry.

The coupon payments of the EIB longevity bond are proportional to the survivorship rate of the specified reference population. The bonds can be viewed as "coupon-based" longevity bonds. The Swiss Re mortality bonds pay higher coupon rates than comparable Treasury bonds as compensation for the associated risk, because the principal values of the bonds are not guaranteed to be redeemable at the maturity date. They can be classified as "principal-at-risk" bonds. From this perspective, both bonds can be viewed as mortality-linked structured notes, which are conventional, fixed-income securities combined with derivative elements.

The incorporation of derivative contracts into fixed-income debts enables coupon payments or redemption amounts at the maturity of the notes to depend on the performance of the underlying benchmark, such as interest rates, equity market indexes, exchange rates, or corporate credits. (For introductions of more complex structured products, see Das [2001].) Mortality-linked structured notes are just debt instruments whose repayments of principal or payments of interest are tied to the performance of the underlying mortality index. High-yield structured notes like Swiss Re mortality bonds target aggressive investors with strong risk appetites who are willing to put their investment at risk to obtain higher coupons. For risk-averse investors seeking to preserve capital while maintaining low-risk exposure to the underlying mortality index, principal-guaranteed structured notes emerge to meet demand.

For principal-guaranteed, mortality-linked structured notes, the coupons of notes are foregone to create a long position on options and thus maintain the upside potential of the underlying mortality index. Principal-guaranteed, mortality-linked structured notes are attractive to investors because the principal amount of investments is guaranteed, but they still have the prospect of earning extra returns if the mortality options are in-the-money at maturity. For high-yield, mortality-linked structured notes, enhanced coupons get generated by the premium income obtained from the sale of mortality options. When the sold options become in-the-money at the option maturity date, investors of high-yield, mortality-linked structured notes (i.e., the writers of mortality options) are obligated to execute the transaction, which requires risking their principal. For that reason, high-yield structured notes generally are not principal-protected.

The payoff of the mortality options is engineered into the fixed income securities, thus principal-guaranteed or high-yield, mortality-linked structured notes, depending on the positions of the mortality options. The incorporation of mortality options into vanilla bond structures provides a way for insurers or pension plans to take a position in mortality options to hedge their death benefit or annuity liability risks. It also offers an opportunity for potential investors to express their views on movements of the underlying mortality index rates. In addition, because mortality-linked securities have low correlation with returns on equity, foreign exchanges, and other financial assets, the mortality-linked product has become a particularly attractive way for investors to improve their portfolio performance. Therefore, mortality risk represents a unique and differentiated asset class that may diversify investment portfolios.

Although mortality-linked structured products can be highly customized with a wide range of mortality-rate-related underlying assets, such as specific books of liabilities of insurance companies or annuity providers, asymmetric information problems may arise as holders of mortality risk exposures, i.e., insurance companies or annuity providers, have superior knowledge of the underlying risks due to their access to better experienced mortality data. Therefore, in this paper, we do not study securitizations of specific books of insurance firms or pension plans' liabilities, but analyze structured products with their payoffs linked to publicly available mortality indexes.

Mortality-linked structured products written on publicly available indexes based on government data have the advantages of standardization and transparency. The securitization of mortality risks with a payoff linked to public demographic indexes is similar to the transfer of credit risk via synthetic collateralized debt obligations (CDOs), as synthetic CDOs are not backed by

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cash flows of underlying assets. It is also analogous to the application of the synthetic CDO technology to create tranches of credit default swap indexes (CDXs). Wills and Sherris [2010] propose a tranched longevity bond structure with payments based on a specified population mortality index. In this article, we propose three different types of mortality-linked structured products with their payoffs depending on some publicly available mortality indexes.

A wide range of mortality models have been proposed to analyze mortality dynamic processes for pricing mortality-linked securities. Milevsky and Promislow [2001] construct a stochastic hazard rate model, as opposed to a deterministic force of mortality, and use both discrete and continuous time models to price mortality options. Dahl [2004] also specifies the mortality intensity as a stochastic process. Having observed the similarities between the force of mortality and interest rates, Cairns, Blake, and Dowd [2006] show how to model mortality risks and price mortality-linked instruments using the frameworks developed for interest rate derivatives, such as short rate models, forward mortality models, and mortality market models. Additional stochastic mortality rate models are proposed by Biffis [2005], Schrager [2006], and Miltersen and Persson [2005]. Cox, Lin, and Wang [2006] argue that a good stochastic mortality model should take into account mortality jumps, and then propose a jump-diffusion model to describe mortality rate dynamics. Lin, Cox, and Pedersen [2010] combine a general mortality trend, a diffusion process, a permanent longevity jump process, and a temporary mortality jump process into their mortality rate model.

In order to price mortality-linked securities, the underlying mortality rate process needs to be specified under a risk-adjusted probability measure. However, since there are not enough traded assets that can be used to replicate the payoffs of mortality-linked securities, mortality-linked securities have to be priced in an incomplete market setting. Derivative pricing in incomplete markets suggests choosing one of the equivalent martingale measures in some economically or mathematically motivated methods, such as those based on hedging arguments, utility or equilibrium-type considerations, or distance minimization (see Schweizer [1996]; Davis [1997]; and Frittelli [2000], among others).

To price financial and insurance risks in an incomplete market, Wang [2000, 2002] postulates a framework that is based on the application of a distortion operator to the probability distribution of the risk. Both Cox et al. [2006] and Lin and Cox [2008] use the Wang transform technique to price mortality-linked securities. With regard to model calibration, although there is no liquid and deep market for trading mortality risk, several authors suggest using existing mortality-linked securities like annuity data to imply the market price of risk. For example, see Biffis [2005]; Blake et al. [2006]; and Lin and Cox [2005], among others.

Investors in capital markets can express their views on future mortality rates by taking positions in these structured notes. Blake, Cairns, and Dowd [2006] propose several variations of hypothetical mortality-linked securities, such as mortality swaps, futures, options, and swaptions, and investigate their potential applications. Because the option is a basic component usually embodied in a securitization, we specifically study the use of mortality options for constructing mortality-linked structured notes.

Given our focus on mortality-linked structured notes, especially mortality option-embedded structured notes, we first follow Cox, Lin, and Wang [2006] and model the mortality index as a jump-diffusion process, and estimate the parameters of the process using the maximum likelihood estimation method. We then apply the Wang transform to convert the physical projected mortality rates into mortality rates under the risk-neutral probability measure.[2] Finally, we describe how mortality-linked structured notes could determine their coupons or redemption values on the basis of the mortality index. For principal-guaranteed, mortality-linked structured notes, we study the relationship between the level of principal protection and units of mortality options purchased. For high-yield, mortality-linked structured notes, we analyze how to produce the extra coupon spread through the sale of different units of mortality options.

We also analyze a hypothetical debt security with both bullish and bearish mortality bonds, such that changes in the redemption value of one bond are offset by changes in the redemption value generated by another bond. The debt security with both bullish/bearish mortality bonds is attractive to issuers because it can produce known cash flow payments, regardless of the underlying mortality index realization. For investors, the bullish and bearish mortality bonds also offer a flexible way to monetize their view of future mortality trends.

The remainder of this article is organized as follows: We review the jump-diffusion mortality rate model

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employed herein, and follow with a section deriving the analytical pricing formulas for mortality call and put options under the jump-diffusion model. The next section first introduces U.S. mortality rate data and then provides the estimation results. Using the estimated model, we then analyze three different types of mortality-linked structured products: principal-guaranteed structured notes, high-yield structured notes, and bullish and bearish mortality bonds. For each structured product, we explain how its value depends on the realized underlying mortality rate and examine its fair price. The last section serves as the conclusion to our analysis.

THE MORTALITY MODEL

Mortality-linked structured notes connect coupons or principal redemptions to some designated mortality indexes. Mortality rate is the ratio of the number of deaths over a given period in an age-specific group to the entire population size of that group. The historical mortality rate time series constitutes a mortality index. To study mortality-linked securities, a realistic mortality rate model is thus required. Lin and Cox [2008] note that a sudden spike in death rates, which may result from epidemics, hurricanes, earthquakes, or man-made disasters like wars, will cause severe losses to insurance industries due to huge claim payments. They further argue that the mortality dynamics should include "normal" deviations from the trend as well as "unanticipated" mortality shocks. Since the rationale behind the design of mortality-linked structured products is for insurance companies or annuity providers to hedge their unexpected mortality risks, good mortality dynamics should incorporate mortality jumps into the model. Therefore, of the different stochastic mortality rate models suggested in the literature, we employ the jump-diffusion model that Cox, Lin, and Wang [2006] propose to specify the mortality index process. We briefly review the model in this section.

Cox et al. [2006] propose the following dynamics to model the mortality index under the physical probability measure P:

$$\frac{dq_t}{q_t} = (\alpha - \lambda k)dt + \sigma dW_t^P + (Y-1)\,dN_t^P, \tag{1}$$

where α is the instantaneous expected change percentage of the mortality rate. The mortality rate improves if α is negative. In addition, σ is the instantaneous volatility of the mortality rate if no jump events occur. The number of jumps N between time 0 and time t is governed by a Poisson distribution, $prob\,(N_t^P = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$ where λ is the jump intensity of the Poisson process N_t^P under the physical probability measure P. If the jump event occurs at time t, the mortality rate changes from q_t to Yq_t, where the jump size Y is an independent random variable. We assume $Y = \exp(m + su)$, where m and s are constants, and u is standard normally distributed; that is, $u \sim N(0, 1)$. Thus, the jump size is lognormally distributed. We define k as $k \equiv E[Y-1] = \exp\left(m + \frac{1}{2}s^2\right) - 1$ to ensure that α is the instantaneous expected change percentage of the mortality rate.

Using Itô's lemma, we can explicitly solve the stochastic differential equation for q_t under the probability measure P, q_t^P, to yield

$$q_t^P = q_0^P \exp\left\{\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma W_t^P\right\} Y^P(N_t), \tag{2}$$

where $Y^P(N_t) = \prod_{b=1}^{N_t} Y_b^P$, Y^P is the cumulative jump size under the probability measure P. The jump size Y_b^P is independently and identically distributed.

Conditional on $N_t = n$, from Equation (2), we can calculate $\widetilde{\mu}_n$, or the mean of variable $\ln(q_t^P)$, as $\widetilde{\mu}_n = \ln(q_0^P) + \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)t + nm$, and calculate $\widetilde{\sigma}_n^2$, or the variance of $\ln(q_t^P)$, as $\sigma^2 t + ns^2$.

Because $\ln\,(q_t^P)$ is lognormally distributed, the probability density function for q_t, conditional on current information set $\mathcal{F}_0$ and $N_t = n$, is

$$f(q_t^P = q \mid N_t = n) = \frac{1}{q\sqrt{\left(2\pi\widetilde{\sigma}_n^2\right)}} e^{-\frac{1}{2}\left(\frac{\ln q - \widetilde{\mu}_n}{\widetilde{\sigma}_n}\right)^2}. \tag{3}$$

Therefore, the density function of q_t is

$$f(q_t^P = q) = \sum_{n=0}^{\infty} f(q_t^P = q \mid N_t = n) \times \text{Prob}\,(N_t = n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{q\sqrt{\left(2\pi\widetilde{\sigma}_n^2\right)}} e^{-\frac{1}{2}\left(\frac{\ln q - \widetilde{\mu}_n}{\widetilde{\sigma}_n}\right)^2} \times \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{4}$$

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Henceforward, we evaluate Equation (4) by cutting off the summation of jump events at $n = 10$, because the probability of the event $N_t > 10$ is small enough to be ignored.

We can then work out the cumulative distribution function as follows:

$$\begin{aligned} F(q_t^P \le q) &= \int_0^q f(q_t^P = x)\,dx \\ &= \int_0^q \sum_{n=0}^{\infty} \frac{1}{x\sqrt{\left(2\pi\widetilde{\sigma_n^2}\right)}} e^{-\frac{1}{2}\left(\frac{\ln x - \widetilde{\mu_n}}{\widetilde{\sigma_n}}\right)^2} \times \frac{e^{-\lambda t}(\lambda t)^n}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \Phi\left(\frac{\ln q - \widetilde{\mu_n}}{\widetilde{\sigma_n}}\right), \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution.

The model specification of Equation (1) has the advantage of mathematical tractability, which allows a closed-form formula for the expected future mortality rate to be derived as in Equation (2). The model has also been widely implemented as a stock-price jump-diffusion model, for which closed-form solutions for options and other securities are available. Because of this closed-form solution, the model may provide useful mortality dynamics for mortality simulation, as well as being useful in the capital market applications we discuss in the next section.

For pricing purposes, we need the mortality rate distribution under the risk-neutral probability measure. To derive the risk-adjusted mortality rate distribution, we apply the Wang [2000, 2002] transform technique:

$$F^Q(x) = \Phi[\Phi^{-1}(F(x)) + \psi], \tag{5}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution, and $\psi > 0$ is a constant. The transform produces a risk-adjusted cumulative distribution function $F^Q(x)$. Wang [2007] further proves that normal and lognormal distribution properties are preserved under the Wang transform.

We assume that the risk-adjustment parameters of the physical mortality distribution of Brownian motions and jump sizes remain constant. That is, $\psi_W = \psi_Y = \psi$, where ψ_W is the market price of risk for Brownian motion W, and ψ_Y is the market price of risk for jump size Y. According to the Wang transform, we can change the measures of the Brownian motion and jump sizes to get the risk-neutralized distribution. Conditional on the current information set $\mathcal{F}_0$, the q_t under the risk-adjusted Q measure becomes

$$q_t^Q = q_0^Q \exp\left\{\begin{aligned}&\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)t \\ &+ \sigma\psi\sqrt{t} + \sigma W_t^Q\end{aligned}\right\}\prod_{b=1}^{N_t} Y_b^Q e^{\psi s}, \tag{6}$$

where the mean and variance for $\ln(q_t^Q)$, given the current information set $\mathcal{F}_0$ and $N_t = n$ under the Q measure, is

$$\begin{aligned} \mu_n &= \ln(q_0^Q) + \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma\psi\sqrt{t} + n(m + \psi s) \\ &= \widetilde{\mu_n} + \sigma\psi\sqrt{t} + n(\psi s), \end{aligned}$$

and $\sigma_n^2 = \widetilde{\sigma_n^2} = \sigma^2 t + ns^2$. It is thus obvious that the measure change alters the mean but not the variance.

VALUATIONS OF MORTALITY OPTIONS

To analyze mortality option-embedded structured notes, we need to know the values of the mortality options. In this section, we derive the values of mortality call and put options with one unit of notional principal. The following proposition provides the foundation for calculating the value of a European mortality call option.

Proposition 1. *Assuming that the interest rate r is constant and independent of mortality rate, the initial market value $C_0(q, K)$ of the European mortality call option with one unit of notional principal, strike rate K, and payoff as $max[(q_T - K), 0]$ at maturity $(T + 1)$ is given by*

$$\frac{C_0(q,K)}{B_0} = E_0^Q\left[\frac{(q_T - K)^+}{B_{T+1}}\right].$$

That is,

$$\begin{aligned} C_0(q,K) &= e^{-r(T+1)} E_0^Q[(q_T - K)^+] \\ &= e^{-r(T+1)} \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times e^{\mu_n + \frac{1}{2}\sigma_n^2} \\ &\quad \times \left[1 - \Phi\left(\frac{\ln K - \mu_n}{\sigma_n} - \sigma_n\right)\right] - e^{-r(T+1)} K \\ &\quad \times \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times \left[1 - \Phi\left(\frac{\ln K - \mu_n}{\sigma_n}\right)\right], \end{aligned}$$

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where $\Phi(\cdot)$ is the cumulative probability function of a standardized, normally distributed variable, and B. is the money market account accumulator, $B_t = e^{-rt}$.

Proof. The value of the European mortality call option equals the expected value of the payoff $\max[(q_T - K), 0]$ at maturity date $(T+1)$, under the risk-neutral probability measure Q. The detailed calculation of the expectation $E_0^Q[(q_T - K)^+]$ appears in Appendix A. □

Remark 2. *The European mortality put option with one unit of notional principal, strike rate K, and payoff as max$[(K - q_T), 0]$ at maturity $(T+1)$, is equal to*

$$e^{-r(T+1)}K \times \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times \left[\Phi\left(\frac{\ln K - \mu_n}{\sigma_n}\right)\right] - e^{-r(T+1)} \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times e^{\mu_n + \frac{1}{2}\sigma_n^2} \times \left[\Phi\left(\frac{\ln K - \mu_n}{\sigma_n} - \sigma_n\right)\right].$$

The derivation can be constructed by following Proposition 1, and we omit the detailed calculation here.

MORTALITY-LINKED STRUCTURED PRODUCTS

In this section, we study the use of mortality derivatives to construct mortality-linked structured products. Specifically, we demonstrate how mortality options can be combined with traditional fixed-income securities to create principal-guaranteed or high-yield, mortality-linked structured notes to attract investors with various mortality risk tolerances. Moreover, we introduce bullish and bearish mortality bonds with offsetting redemption values, which allow the issuer to remain immunized against the underlying mortality risk.

The creation of a desired mortality risk exposure within mortality-linked structured notes requires linking either the coupon or the redemption value to the nominated mortality index. The position of the incorporated mortality option reflects the nature of a structured note. A long position in mortality options implies that part of the coupon or the principal of the structured note must be sacrificed to pay for the option premium; a short position indicates a higher coupon can be generated by the premium obtained from the written options. In the former case, an option buyer is not obliged to exercise the option and does so only when it is profitable. The structure thus provides a guaranteed principal while also maintaining some exposure to the underlying mortality risk. In the latter case, an option writer receives the premium up front, but suffers potential liabilities later when that counterparty, an option buyer, chooses to exercise. An option writer's loss then gets funded through the reduction of principal. Therefore, the role played by option positions provides a means to construct different types of structured notes.

To analyze examples of mortality-linked structured notes, we download the death number and population size of the United States for each year from 1900 to 2013 from the Human Mortality Database.[3] The mortality rate is computed as the ratio of death counts to population size. Exhibit 1 plots the U.S. mortality rate for the period 1900 to 2013.

Following Lin and Cox [2008], we employ the maximum likelihood estimation method to calibrate Equation (6) to historical U.S. mortality rate data.[4] We report the estimation result that appears in Exhibit 2. To price mortality-linked structured products, which we discuss subsequently in this section, we set the parameters in the mortality process of Equation (6) to those listed in Exhibit 2.

We can determine the market price of risk ψ in the risk-neutral process of q_t from the retail market for life insurance or annuity products by Equation (5), so that the current market price of a life insurance or annuity product equals the model price under the transformed distribution. However, as there is no active and deep market for mortality risk trading, risk-adjusted stochastic mortality rate processes may not be easily calibrated to the limited market prices of mortality-linked securities. Because we lack mortality products price data to estimate the market price of risk ψ in the risk-neutral process of q_t, we will not estimate ψ via model calibration. Instead, we exogenously assume the values of the market price of risk ψ to be 0, 0.4, 0.8, and 1.2, in line with the estimates that were made in the literature, e.g., Lin and Cox [2008].

Exhibit 3 displays the Wang-transformed probability density distribution (PDF) of U.S. mortality rates taking the market price of risk ψ to be 0, 0.5, and 1. The dotted line denotes the transformed PDF of mortality rate q_t with $\psi = 0.5$, and the dashed line

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EXHIBIT 1

U.S. Historical Mortality Index

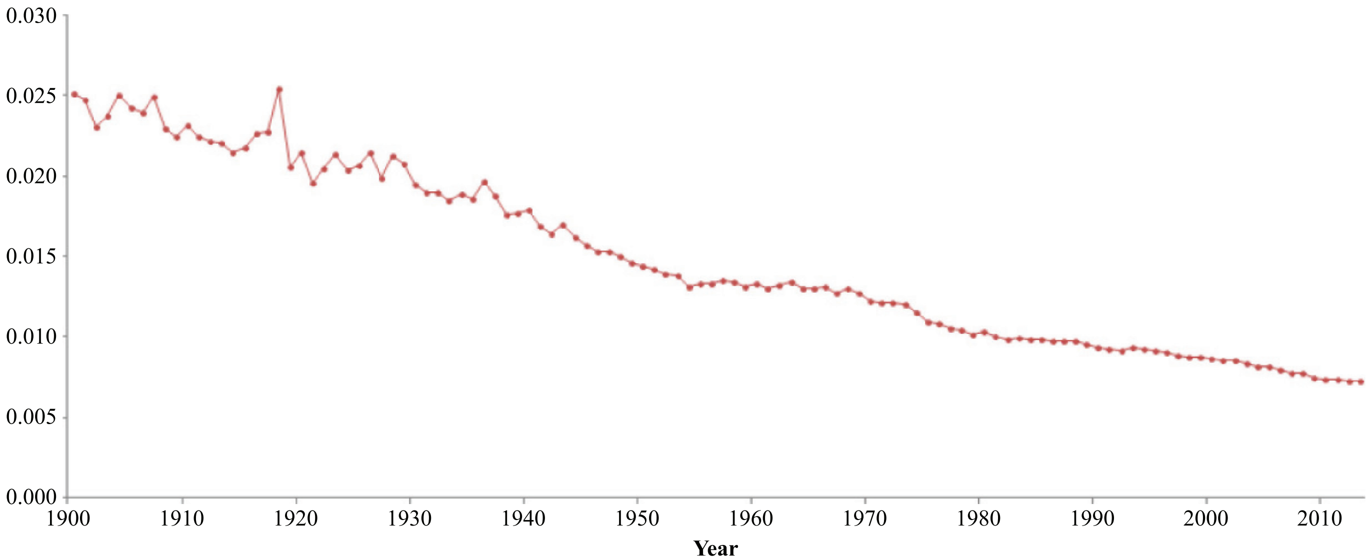

EXHIBIT 2

Parameter Estimates for the Mortality Rate Model

Parameters	α	σ	λ	m	s	k
Estimates	–0.0095	0.0280	0.0476	–0.0225	0.1035	–0.0167
S.e. of estimates	0.0029	0.0024	0.0406	0.0514	0.0458	–

Note: S.e. is the standard errors of the estimates.

denotes that with $\psi = 1$. When setting the market price of risk ψ equal to 0, the line shows the physical PDF of mortality rate q_t. Exhibit 3 shows that with a larger market price of risk, the transformed PDF of q_t shifts more to the right of the physical PDF of q_t. This means that more weights will be put on the right tail of the PDF of q_t for a larger market price of risk.

Principal-Guaranteed Mortality-Linked Structured Notes

Principal-guaranteed structured notes have engendered a lot of interest lately. The typical principal-guaranteed structure comprises a zero-coupon bond, which pays back par at maturity to deliver capital protection when the situation involves a 100% capital guarantee. The sacrificed coupon payments may be used to purchase options, which offer upside potential while limiting downside risks.

In this subsection, we introduce two kinds of principal-guaranteed, mortality-linked structured notes: one constructed through the combination of the purchase of mortality call options with a traditional fixed-income security, and another constructed through the combination of the purchase of mortality put options and a traditional fixed-income security. Investors who choose the former anticipate that the mortality index will increase; investors who select the latter expect it will decrease. Both products ensure principal protection, which typically guarantees the repayment of a pre-determined percentage of the initial investment and enables these investors to participate in the growth or decline of the underlying mortality index.

In the following example, we assume the note, with a maturity of five years, is issued at par with a price of \$5 million. For the options embedded in the note, we consider a five-year option issued in 2013 based on the U.S. mortality index with a notional amount of \$5 million and a strike price of 0.73%, set equal to the U.S. mortality rate

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Exhibit 3

Wang-Transformed Mortality Rate Probability Distribution with Different ψ

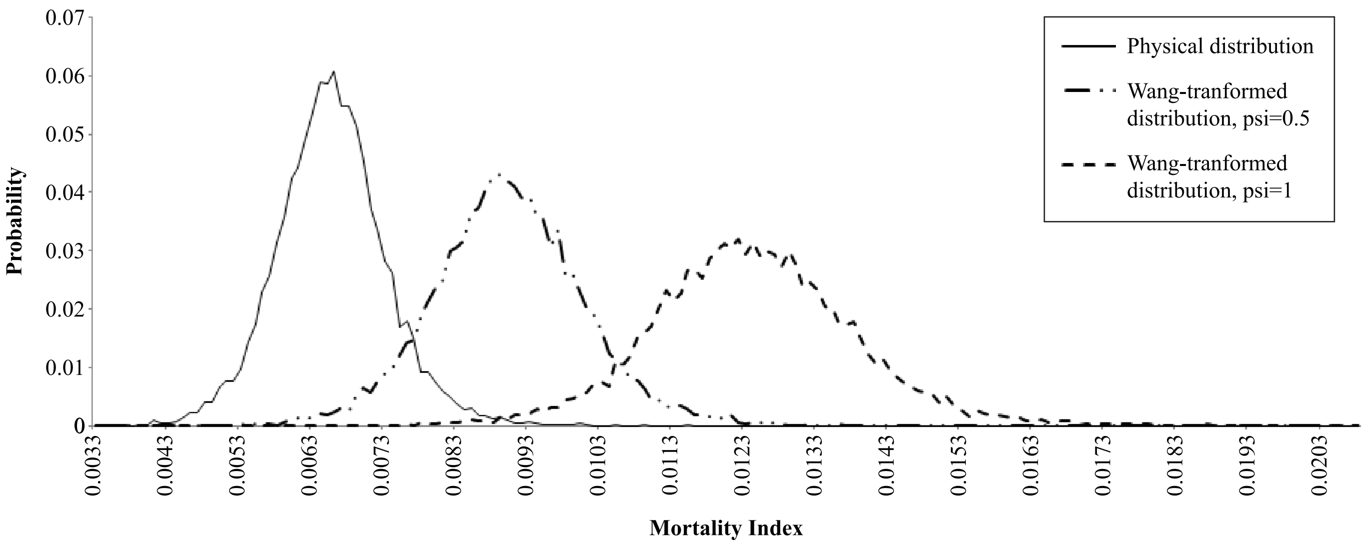

Exhibit 4

Principal-Guaranteed Structured Notes under Different Degrees of Capital Protection

			100% Principal-Guaranteed Initial Reserve ($4,680,654.32)		90% Principal-Guaranteed Initial Reserve ($4,212,588.89)	
ψ	Call Price	Put Price	Number of Calls Purchased	Number of Puts Purchased	Number of Calls Purchased	Number of Puts Purchased
0	410.03	2,006.20	778.83	159.18	1,920.37	392.49
0.4	840.53	1,266.40	379.93	252.17	936.80	621.77
0.8	1,514.60	750.16	210.84	425.70	519.88	1,049.66
1.2	2,445.30	388.14	130.60	822.76	322.01	2,028.68

Notes: ψ is the market price of risk.

of the year 2013. Therefore, all options studied in Exhibit 4 are at-the-money options. We simulate the mortality index using the estimated values reported in Exhibit 2. The initial reserve is the value that, if invested in a risk-free savings account today, could guarantee the required percentage of the note's principal repayment at the maturity date. We calculate the initial reserve as the present value of the capital-guaranteed amount at the maturity date, as listed in Exhibit 4. The difference between the bond price and the initial reserve is the amount available for purchasing options. We then calculate the number of calls (puts) to be purchased in the structured notes in the case of 100% and 90% capital protections, and report them in columns 4 and 6 (5 and 7) of Exhibit 4.

The numerical results in Exhibit 4 show that the lower the principal-guaranteed level an investor requires, the lower the initial reserve that should be deposited in the risk-free account. As a result, more cash is available for purchasing more options. Investors then receive higher payoffs from positions in which they have more purchased options if those options are valuable at maturity. The results in Exhibit 4 also show that the larger the market price of risk, the higher (lower) the call (put) price. Based on Equation (5) or Exhibit 3, a positive market price of risk ψ implies that the transformed probability density function of the mortality rate q_t lies on the right of the probability density function of q_t under the physical probability measure. It indicates that the market

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expects a greater probability of having a higher mortality rate than the actual probability suggests. This explains why the premium of a call (put) with a fixed strike rate is higher (lower) for a higher market price of risk ψ.

At maturity, the payoff of the principal-guaranteed, mortality-linked structured note that investors will receive equals the protected capital plus any profits earned from holding options. This structure highlights the intrinsic attraction of the principal-guaranteed structure for risk-averse investors seeking to create exposure to the underlying mortality risk. One variation of the basic structure involves the introduction of a participation rate into the payoff function of embedded options. Different participation levels thus allow greater customization and fit the various risk-return trade-offs demanded by individual investors.

High-Yield Mortality-Linked Structured Notes

High-yield structured notes represent a specific type of interest-paying, non-principal-protected, medium-term notes with embedded sold options. The premium on the written options gets incorporated into a bond to generate a significantly above-market coupon in return for the investor's willingness to undertake option risks. Payment of the high-yield, mortality-linked structured note at maturity is determined by the performance of the underlying mortality rates. If the value of the underlying mortality rate is equal to or smaller (greater) than the strike price at maturity for written mortality call (put) options, investors receive their initial invested principal and agreed-upon interest. However, if the underlying mortality rate is greater (smaller) than the initial strike rate at maturity for written mortality call (put) options, the initial principal outlay gets reduced by the amount of losses resulting from the written options.

In the following example, we assume the high-yield, mortality-linked structured note is initially issued into the markets at a par of $5 million. The coupon is paid semi-annually, and the maturity of the note is five years. The other option settings are the same as those described in the previous subsection. Because the note is priced at par, the expected compensation should be readily observable in the form of higher coupon rates.

The valuation of the high-yield, mortality-linked structured note involves determining the fair coupon spread s above the LIBOR rate that investors demand as compensation for risking their principal. The fair spread can be solved numerically by equating the note's expected future cash flows to its issuance price. For selling m units of mortality put options, the coupon spread s can be determined by numerically solving the following equation:

$$F = E_0^Q\left[\sum_{t=1}^{2T} P\left(0,\frac{t}{2}\right)\left(\frac{LIBOR+s}{2}\right) + P(0,T)(I_{\{q_T \geq K\}}F + I_{\{q_T<K\}}(F - m\times(K-q_T)\times F_O))\right], \tag{7}$$

where $P(0, t)$ is the discount factor for maturity t; F is the par value of the note; F_O is the notional principal of the option; T is the maturity of the structured note; K is the strike rate of the written options; m is the number of options sold; $I_{\{\cdot\}}$ denotes the indicator function; $E_0^Q[\cdot]$ denotes the expectation operator over the risk-neutral probability measure Q; $LIBOR$ is the LIBOR rate; and s is the coupon spread above the LIBOR rate. The focus of this example is not the forecast of future LIBOR rates, so for simplicity, we assume that the expected future LIBOR rates are equal to the spot LIBOR rates.[5] At maturity, if the underlying mortality index is not greater than the strike rate K, the put options sold incur losses, and the principal redemption at maturity becomes $[F - m\times F_O \times I_{\{q_T<K\}}(K-q_T)]$, where the expected value of $I_{\{q_T<K\}}(K-q_T)$ can be computed using the put option formula listed in Remark 2.

In the following numerical analysis, we assume $T = 5$, $F = \$5$ million, $F_O = \$5$ million, and $m = 100$ and 200. We set the strike rate equal to the mortality rate of 2013, that is $K = 0.73\%$. We simulate the mortality index using the estimated values reported in Exhibit 2. The volatility of the underlying index is 2.8%. The calculated fair spreads associated with the sales of 100 and 200 mortality call and put options appear in Exhibit 5. The results show that the more options an investor sells, the higher the spreads he or she can obtain. However, through the sales of options, option writers might incur potential liabilities later.

Bullish and Bearish Mortality Bonds

In this subsection, we extend the basic idea of incorporating options into a plain-vanilla bond in which

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Exhibit 5

Fair Coupon Spreads of High-Yield Structured Notes

High-Yield MLSN with Embedded Sold Options						
ψ	Call Price	$s_C^{(100)}$	$s_C^{(200)}$	Put Price	$s_P^{(100)}$	$s_P^{(200)}$
0	410.03	0.34%	0.48%	2,006.20	0.93%	1.70%
0.4	840.53	0.49%	0.81%	1,266.40	0.65%	1.15%
0.8	1,514.60	0.76%	1.33%	750.16	0.46%	0.74%
1.2	2,445.30	1.11%	2.04%	388.14	0.33%	0.47%

Notes: ψ is the market price of risk, $s_C^{(100)}$ ($s_C^{(200)}$) is the spread above LIBOR when 100 (200) call options are sold, $s_P^{(100)}$ ($s_P^{(200)}$) is the spread above LIBOR when 100 (200) put options are sold.

we divide the issue into two classes, such that one class provides the positive link and the other the negative link to the underlying mortality index. This innovative design can simultaneously meet the mortality rate expectations of different investors. Moreover, the integration of derivatives with different payoffs into bonds can create offsetting securities that produce known cash flow payments for the issuer.

For illustration purposes, we set out an indicative term sheet of a hypothetical bond with a maturity of 10 years in Exhibit 6. The issue consists of two classes: a bullish mortality bond with face value $\$P_M$ and a bearish mortality bond with face value $\$P_L$. Both classes are issued at par, with redemption values set in accordance with the formula listed in Exhibit 6. The proposed mortality bonds allow the issuers to remain immunized against the risk of mortality-linked redemption values even when both bonds have different principals, i.e., $P_M \neq P_L$. For example, if $P_M = \kappa \times P_L$, then in order to make the final redemption value of both bonds, $R_M + R_L$, equal to the principal of a plain-vanilla bond from which the two mortality bonds are generated, the participation rate of the bullish mortality bond η_M has to be set as $\eta_M = \frac{1}{\kappa}\eta_L$. For illustration purposes, we plot the redemption values at maturity for both classes of bonds when the underlying is set to case (1) $\hat{q}_T = \max_{2014 \le t \le 2023}\left(\frac{q_t}{q_{t-1}}\right)$ in Exhibit 7.

$K_M(K_L)$ is the strike rate at which, for example, when $K_D < \hat{q}_T < K_U$, and $\hat{q}_T = K_M(K_L)$, the payoff of a bullish (bearish) mortality bond apart from its principal $\$P_M(\$P_L)$ is zero. K_D is the rate at which the payoff of a bullish (bearish) mortality bond starts increasing (decreasing) when $\hat{q}_T > K_D$, but the upside (downside) payoff beyond K_U is capped and fixed. Both bullish and bearish mortality bonds limit an investor's upside potential as well as downside risk. To be more specific, for a bullish mortality bond, if the underlying index $\hat{q}_T$ is high and greater than the higher strike rate K_U, the payoff is the upper bound $P_M(1 + \eta_M(K_U - K_M))$. However, if the index $\hat{q}_T$ at the expiration date lies between the two strike rates K_D and K_U, the payoff is $P_M(1+\eta_M(\hat{q}_T - K_M))$, and if the index $\hat{q}_T$ is below the lower strike rate K_D, the payoff is the lower bound of $P_M(1 + \eta_M(K_D - K_M))$. The redemption value of a bullish mortality bond at a maturity date is similar to the payoff of a bull spread strategy. Bullish mortality bonds will have more payoffs when the realized underlying index $\hat{q}_T$ is higher. We use the subscript M to stand for the bullish mortality bonds as the bonds can be used to hedge unexpected mortality risks, i.e., a higher-than-expected underlying mortality index.

Exhibit 7 also shows that the redemption value of a bearish mortality bond at maturity date is similar to the payoff of a bear spread strategy. When the realized underlying index $\hat{q}_T$ is smaller, bearish mortality bonds will have larger payoffs. Therefore, annuity providers or pension plans can buy bearish mortality bonds to pay for the more-than-expected pension benefits due to an unexpected smaller realized underlying mortality index. Since bearish mortality bonds can be used to hedge the longevity risk, we use subscript L to stand for the bearish mortality bonds. In summary, life insurers with mortality risk exposures can use bullish mortality bonds to hedge their liabilities and pay for insurance claims, whereas annuity providers with exposures to an unexpected smaller realized underlying mortality index can buy bearish mortality bonds to hedge the longevity risk.

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EXHIBIT 6

Indicative Term Sheet of Hypothetical Bullish/Bearish Mortality Bonds

Issue Date:	2013/01/02
Maturity Date:	2023/01/02
Principal:	$\$P_M$ ($\$P_L$) for the bullish (bearish) mortality bond
Issue Price:	100% for both classes
Coupon Rate:	$c_M = (LIBOR + s_M)$ for the bullish mortality bond, and $c_L = (LIBOR + s_L)$ for the bearish mortality bond, where s_M (s_L): the spread for the coupon of the bullish (bearish) mortality bond, c_M (c_L): the coupon rate of the bullish (bearish) mortality bond, and $LIBOR$: the six-month LIBOR rate.
Underlying:	(1) $\hat{q}_T = \max_{2014 \le t \le 2023} \left(\frac{q_t}{q_{t-1}} \right)$ (2) $\hat{q}_T = \left(\prod_{t=2013}^{2023} q_t \right)^{\frac{1}{11}}$ (3) $\hat{q}_T = \sum_{t=2013}^{2023} q_t$
Redemption Values:	The redemptions of both classes are indexed to the underlying mortality rate, as follows:

$$R_M = \begin{cases} P_M \left(1 + \eta_M \left(K_U - K_M\right)\right) & \text{if } K_U < \hat{q}_T \\ P_M \left(1 + \eta_M \left(\hat{q}_T - K_M\right)\right) & \text{if } K_D < \hat{q}_T \le K_U \\ P_M \left(1 + \eta_M \left(K_D - K_M\right)\right) & \text{if } \hat{q}_T \le K_D \end{cases}$$

$$R_L = \begin{cases} P_L \left(1 - \eta_L \left(K_U - K_L\right)\right) & \text{if } K_U < \hat{q}_T \\ P_L \left(1 - \eta_L \left(\hat{q}_T - K_L\right)\right) & \text{if } K_D < \hat{q}_T \le K_U \\ P_L \left(1 - \eta_L \left(K_D - K_L\right)\right) & \text{if } \hat{q}_T \le K_D \end{cases}$$

where

R_M(R_L) is the redemption for the bullish (bearish) mortality bond,

η_M(η_L) is the participation rate for the bullish (bearish) mortality bond,

K_M (K_L) is the strike level for the bullish (bearish) mortality bond,

K_D is the rate at which a bullish (bearish) mortality bond's payoff starts increasing (decreasing), and

K_U is the rate beyond which the upside (downside) payoff of a bullish (bearish) mortality bond is capped and fixed.

The two classes of mortality bonds issued in this structure are designed to benefit from either upward or downward movement of the underlying index.

The bullish and bearish mortality bonds carry coupons of $c_M = (LIBOR + s_M)$ and $c_L = (LIBOR + s_L)$, respectively. The underlying index $\hat{q}_T$ of the two classes of bonds is set to reflect the mortality trend during the bond-holding period. For the hypothetical structured bonds, we propose three different settings for the underlying index $\hat{q}_T$ on which the redemption value of

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the bonds depends. We assume that the mortality rates are observed at equally-spaced discrete times t_j, $t_j = j\Delta$, $j = 0, 1, \ldots, h$, and $t_h = h\Delta = T$. In the first case, we set $\hat{q}_T$, the underlying index at maturity, as the maximum of the ratio of mortality rates of consecutive years, i.e., $\hat{q}_T = \max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)$. If the mortality rate steadily declines over time, then $\left\{\frac{q_{t_j}}{q_{t_{j-1}}}\right\}$ would be a decreasing sequence.

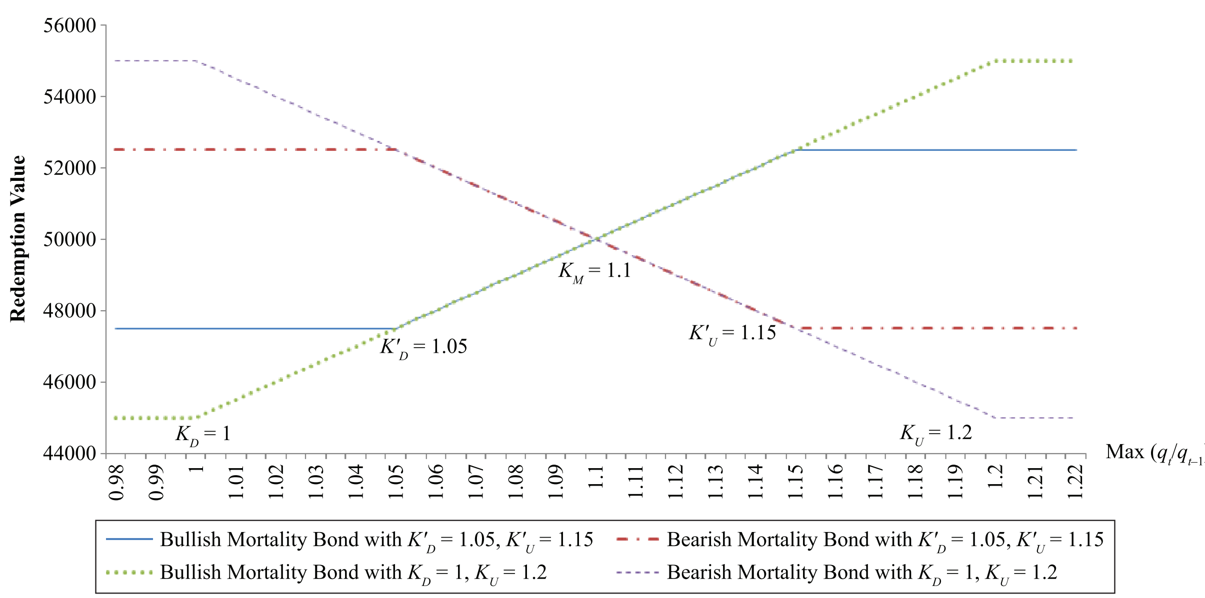
In the second case, we set $\hat{q}_T$ as the geometric average of the mortality rates observed each year, i.e., $\hat{q}_T = \left(\prod_{j=0}^{h} q_{t_j}\right)^{\frac{1}{h+1}}$. The redemption value to the holders of the structured bonds thus depends on the geometric average of the mortality rates over the life of the contract. From the viewpoint of pricing, the geometric average has a technical advantage over the arithmetic one. Structured bonds with their redemption value linked to the geometric average of the mortality rates will be less sensitive to "spikes" of the underlying mortality rates at contract maturity date. Moreover, this geometric average underlying addresses a particular hedging demand: the hedge against the exposure to the average. Insurance companies or pension plans whose objective is to control average claim payments or average annuity payments might find structured bonds with redemption value depending on the geometric average of the mortality rates over the contract period ideal for them to manage their average risk exposure efficiently. That is why we set the underlying index at maturity as the geometric average of the mortality rates over the contract period in the second case.

In the last case, we set $\hat{q}_T$ as the sum of the mortality rates over the contract period, i.e., $\hat{q}_T = \Sigma_{j=0}^{h} q_{t_j}$. Therefore, the redemption value to the structured bond holders is linked to the mortality rates in a cumulative way. In this case, if we scale $\hat{q}_T$ by the ratio of one to the number of mortality rates put in the summation, we obtain an arithmetic average. Both geometric and arithmetic averages are used in financial markets, but the use of arithmetic averages is far more common. Accordingly, the structured bonds whose redemption value depends on the cumulative mortality rates can be valued using several techniques that have been proposed in the literature for pricing Asian options on the arithmetic average.

Pricing the bullish/bearish mortality bonds involves determining the fair coupon spreads of both bonds. Similar to the calculation of spreads in the high-yield, mortality-linked structured note case in the previous subsection, we solve the fair spread $s_M(s_L)$ of the bullish/bearish mortality bonds numerically by

EXHIBIT 7

Redemption Values of Bullish and Bearish Mortality Bonds



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equating the issuance price of the bond with the bond's expected future cash flows, which consist of two parts: the coupon payments and the redemption value at maturity. For the first part of the future cash flows, to simplify the calculation, we assume that the expected future LIBOR rates equal the implied forward rates when we calculate the coupon payments.[6] For the second part, we use the following proposition to calculate the expected redemption value of the bullish mortality bonds.

Proposition 3. *The expected redemption value for the bullish mortality bond, denoted by $E_0^Q[R_M]$, is given by*

$$E_0^Q[R_M] = P_M[1+\eta_M(VT(\hat{q}_T))]$$

where

$$VT\left(\hat{q}_T\right) = K_D F^Q(\hat{q}_T \le K_D) + K_U F^Q(\hat{q}_T > K_U) - K_M + E_0^Q[\hat{q}_T I_{\{K_D<\hat{q}_T \le K_U\}}]$$

And when (i) $\hat{q}_T$ is defined as M_T, $M_T = \max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)$,

$$\begin{aligned}
&F^Q(\hat{q}_T \le K_D)\\
&= F^Q_{M_T}(M_T \le K_D)\\
&= \prod_{j=1}^{h}\sum_{n_{(t_{j-1},t_j)}=0}^{\infty}\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t_{j-1},t_j)}}}{n_{(t_{j-1},t_j)}!}\Phi\left(\frac{\ln(K_D)-\mu_{n_{(t_{j-1},t_j)}}}{\sigma_{n(t_{j-1},t_j)}}\right)
\end{aligned}$$

$$\begin{aligned}
&F^Q(\hat{q}_T > K_U)\\
&= F^Q_{M_T}(M_T > K_U)\\
&= \left[1-\prod_{j=1}^{h}\sum_{n_{(t_{j-1},t_j)}=0}^{\infty}\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t-1,t)}}}{n_{(t_{j-1},t_j)}!}\Phi\left(\frac{\ln(K_U)-\mu_{n_{(t_{j-1},t_j)}}}{\sigma_{n_{(t_{j-1},t_j)}}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&E_0^Q\left[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}\right]\\
&= E_0^Q\left[M_T I_{\{K_D<M_T\le K_U\}}\right]\\
&= \int_{K_D}^{K_U} x\left[\prod_{j=1}^{h}\sum_{n_{(t_{j-1},t_j)}=0}^{\infty}\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t_{j-1},t_j)}}}{n_{(t_{j-1},t_j)}!}\left[\Phi(SN(x+dx))-\Phi(SN(x))\right]\right]
\end{aligned}$$

where $\Phi(SN(x)) = \Phi\left(\frac{\ln(x)-\mu_{n_{(t_{j-1},t_j)}}}{\sigma_{n_{(t_{j-1},t_j)}}}\right)$.

When (ii) $\hat{q}_T$ is defined as G_T, $G_T = \left(\prod_{j=1}^{h} q_{t_j}\right)^{\frac{1}{h+1}}$,

$$\begin{aligned}
&F^Q(\hat{q}_T \le K_D)\\
&= F^Q_{X_T}(G_T \le K_D)\\
&= \frac{1}{2}+\frac{1}{2\pi}\int_0^{\infty}\frac{e^{iv\ln K_D}\phi_T^x(-v)-e^{-iv\ln K_D}\phi_T^x(-v)}{iv}dv
\end{aligned}$$

$$\begin{aligned}
&F^Q(\hat{q}_T > K_U)\\
&= F^Q_{X_T}(G_T > K_U)\\
&= \frac{1}{2}-\frac{1}{2\pi}\int_0^{\infty}\frac{e^{iv\ln K_U}\phi_T^x(-v)-e^{-iv\ln K_U}\phi_T^x(-v)}{iv}dv
\end{aligned}$$

$$\begin{aligned}
E_0^Q\left[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}\right] &= E_0^Q\left[G_T I_{\{K_D<G_T\le K_U\}}\right]\\
&= \int_{\ln K_D}^{\ln K_U} x\, f_{X_T}(x)\ dx
\end{aligned}$$

where $X_T = \ln\ G_T$, $\phi_T^x(\cdot)$ is the characteristic function of X_T, and $f_{X_T}(x)$ is the probability density function of X_T.

When (iii) $\hat{q}_T$ is defined as A_T, $A_T = \Sigma_{j=0}^{h} q_{t_j}$,

$$F^Q(\hat{q}_T \le K_D) = F^Q_{A_T}(A_T \le K_D) = \int_{-\infty}^{\ln\left(\frac{K_D}{q_0}-1\right)} f_{B_1^\Delta}(x)\,dx$$

$$F^Q(\hat{q}_T > K_U) = F^Q_{A_T}(A_T > K_U) = \int_{\ln\left(\frac{K_U}{q_0}-1\right)}^{\infty} f_{B_1^\Delta}(x)\,dx$$

$$\begin{aligned}
E_0^Q\left[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}\right] &= E_0^Q\left[A_T I_{\{K_D<A_T\le K_U\}}\right]\\
&= \int_{\ln\left(\frac{K_D}{q_0}-1\right)}^{\ln\left(\frac{K_U}{q_0}-1\right)} q_0(1+e^x)\, f_{B_1^\Delta}(x)dx
\end{aligned}$$

where $f_{B_j^\Delta}(x)$ satisfies the following recursion

$$f_{B_j^\Delta}(x) = \int_{-\infty}^{\infty} f_{Z^\Delta}(x-\ln(1+e^y))f_{B_{j+1}^\Delta}(y)dy$$
$$\text{for } j = h-1, \ldots, 1$$

with the initial condition set as $f_{B_h^\Delta} \equiv f_{Z^\Delta}$, where $Z_j^\Delta \equiv \ln(q_{t_j}) - \ln(q_{t_{j-1}})$.

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Proof. Recall the redemption value of the bullish mortality bond listed in Exhibit 6. We first define the redemption value of the bullish mortality bond as R_M^D when the underlying index $\hat{q}_T$ is not greater than K_D, as R_M^{DU} when the underlying index $\hat{q}_T$ is greater than K_D and not greater than K_U, and as R_M^U when the underlying index $\hat{q}_T$ is larger than K_U. Denote $F^Q(\cdot)$ as the cumulative probability distribution of $\hat{q}_T$ under measure Q, and $I_{\{\cdot\}}$ as the indicator function, we then have

$$\begin{aligned} E_0^Q[R_M] &= E_0^Q\Big[(R_M^D)I_{\{\hat{q}_T \le K_D\}} + (R_M^{DU})I_{\{K_D<\hat{q}_T\le K_U\}} \\ &\qquad + (R_M^U)I_{\{K_U<\hat{q}_T\}}\Big] \\ &= (R_M^D)F^Q(\hat{q}_T \le K_D) + E_0^Q\Big[(R_M^{DU})I_{\{K_D<\hat{q}_T\le K_U\}}\Big] \\ &\quad + (R_M^U)F^Q(K_U < \hat{q}_T) \end{aligned} \tag{8}$$

After substituting the redemption values of R_M^D, R_M^{DU}, and R_M^U into Equation (8), we can write $E_0^Q[R_M]$ as

$$E_0^Q[R_M] = P_M[1+\eta_M(VT(\hat{q}_T))]$$

where

$$\begin{aligned} VT(\hat{q}_T) &= K_D F^Q(\hat{q}_T \le K_D) + K_U F^Q(\hat{q}_T > K_U) - K_M \\ &\quad + E_0^Q\Big[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}\Big] \end{aligned}$$

The rest of the proof requires the computation of the (a) $F^Q(\hat{q}_T \le x)$ and (b) $E_0^Q[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}]$ terms.

We denote q_{t_j}, $j = 1, \ldots, h$, as the mortality index observed at time t_j, and assume that all observations are at equally-spaced discrete times t_j, $t_j = j\Delta$, $j = 0, 1, \ldots, h$, and $t_h = h\Delta = T$. $q_{t_h} \equiv q_T$ is the mortality index observed at time T. For the bullish mortality bonds, all three types of payoff structures depend on the mortality indexes observed at equally-spaced discrete times t_j.

Following Fusai and Meucci [2008], we consider the demeaned log-increments of mortality index q between time t_{j-1} and t_j as ξ_j^Δ, $\xi_j^\Delta \equiv \ln(q_{j\Delta}) - \ln(q_{(j-1)\Delta}) - m_j^\Delta \Delta$, where m_j^Δ is the deterministic component of the log-increments ξ_j^Δ, and

$$m_j^\Delta \Delta = \left[\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi\sqrt{\Delta}\right].$$

Next, we work out the detailed derivations of the $F^Q(\hat{q}_T \le x)$ and $E_0^Q[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}]$ terms for (i) $\hat{q}_T$ is defined as M_T,

$$M_T = \max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right),$$

in Appendix C, (ii) $\hat{q}_T$ is defined as G_T,

$$G_T = \left(\prod_{j=0}^{h} q_{t_j}\right)^{\frac{1}{h+1}},$$

in Appendix D, and (iii) $\hat{q}_T$ is defined as A_T, $A_T = \Sigma_{j=0}^h q_{t_j}$, in Appendix E, respectively. □

The derivation of the expected redemption value of the bearish mortality bond can be constructed by following Proposition 3. We omit the detailed calculation here.

On the basis of the derived analytical formula for the redemption values of both bonds, we next investigate the pricing behaviors of bullish/bearish mortality bonds subject to varying values of market price of risk ψ and contract-specific values of K_D and K_U. In the numerical analysis, we assume each bond has a principal amount of \$50,000, and assume a participation rate $\eta_M = \eta_L = \eta$ and take η to be 1 for simplicity. With regards to the benchmark case, for the first setting of $\hat{q}_T$,

$$\hat{q}_T = \max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right),$$

we take ψ equal to 0.8 and the parameters in the bond's redemption formula to $K_D = 1.05$, $K_U = 1.2$. For the second setting

$$\hat{q}_T = \left(\prod_{j=0}^{h} q_{t_j}\right)^{\frac{1}{h+1}},$$

we set ψ to be 0.4, and $K_D = 1.05$, $K_U = 1.2$. For the last setting $\hat{q}_T = \Sigma_{j=0}^h q_{t_j}$, we take ψ to be 0.4, and parameters in the bond's redemption as $K_D = 1$, $K_U = 1.2$. For all three different underlying settings, we take $K_M = K_L = 1.1$ in the benchmark case studied. We run 100,000 simulation paths to produce the simulated cumulative probability $F^Q(\hat{q}_T \le q)$ and the redemption values. Panel A of Exhibit 8 lists features of bullish and

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bearish mortality bonds at different values of market price of risk ψ and contract-specific values of K_D and K_U for the setting of

$$\hat{q}_T = \max_{1 \le j \le h} \left(\frac{q_{t_j}}{q_{t_{j-1}}} \right).$$

Panel B lists those for the setting of

$$\hat{q}_T = \left(\prod_{j=0}^{h} q_{t_j} \right)^{\frac{1}{h+1}},$$

and Panel C presents those when $\hat{q}_T$ is defined as $\hat{q}_T = \Sigma_{j=0}^{h} q_{t_j}$.

All three panels show that the expected redemption value of bullish mortality bonds, $E_0^Q[R_M]$, is an increasing function of the market price of risk ψ, which rises quite significantly with increasing value of ψ. Exhibit 8 also shows that the expected redemption value of bearish mortality bonds, $E_0^Q[R_L]$, is a decreasing function of the market price of risk ψ. We can observe that with the increasing market price of risk ψ, P_D, the probability that $\hat{q}_T$ is less than K_D, becomes smaller. Moreover, the probability that $\hat{q}_T$ is greater than K_U, P_U, increases

EXHIBIT 8

Features of Bullish and Bearish Mortality Bonds under Different Parameters

Panel A: $\hat{q}_T = \max_{2014 \le t \le 2023} \left(\frac{q_t}{q_{t-1}} \right)$

Base case: ψ = 0.8, K_D = 1.05, K_U = 1.2, η = 1

	P_D	P_U	$E_0^Q[R_M]$	$E_0^Q[R_L]$	s_M	s_L
Base case	53.01%	6.00%	48680.95	51319.05	27.83	–13.01
Sensitivity of ψ						
ψ = 0	74.60%	1.06%	47916.84	52083.16	39.66	–24.84
ψ = 0.4	64.54%	2.81%	48236.60	51763.40	34.71	–19.89
ψ = 1.2	41.14%	10.64%	49175.09	50824.91	20.18	–5.36
Sensitivity of K_D						
K_D = 1	0.13%	6.00%	48249.90	51750.10	34.50	–19.68
K_D = 1.1	83.58%	6.00%	50533.04	49466.96	–0.84	15.66
K_D = 1.15	89.58%	6.00%	52702.47	47297.53	–34.43	49.24
Sensitivity of K_U						
K_U = 1.1	53.01%	16.43%	48147.90	51852.10	36.08	–21.26
K_U = 1.15	53.01%	10.42%	48478.48	51521.52	30.96	–16.14
K_U = 1.25	53.01%	3.11%	48792.25	51207.75	26.10	–11.29

Panel B: $\hat{q}_T = \left(\prod_{t=2013}^{2023} q_t \right)^{\frac{1}{11}}$

Base case: ψ = 0.4, K_D = 1.05, K_U = 1.2, η = 1

	P_D	P_U	$E_0^Q[R_M]$	$E_0^Q[R_L]$	s_M	s_L
Base case	87.18%	0.60%	48374.97	51625.03	32.56	–17.75
Sensitivity of ψ						
ψ = 0	95.03%	0.13%	48238.78	51761.22	34.67	–19.86
ψ = 0.8	73.82%	2.09%	48667.88	51332.12	28.03	–13.21
ψ = 1.2	56.79%	5.34%	49134.43	50865.57	20.81	–5.99
Sensitivity of K_D						
K_D = 0.95	28.87%	0.60%	46132.43	53867.57	67.28	–52.46
K_D = 1	63.40%	0.60%	46974.38	53025.62	54.24	–39.43
K_D = 1.1	95.93%	0.60%	50063.42	49936.58	6.43	8.39

(*continued*)

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EXHIBIT 8 (*continued*)

Features of Bullish and Bearish Mortality Bonds under Different Parameters

	P_D	P_U	$E_0^Q[R_M]$	$E_0^Q[R_L]$	s_M	s_L
Sensitivity of K_U						
$K_U = 1.1$	87.18%	4.08%	48311.55	51688.45	33.55	–18.73
$K_U = 1.15$	87.18%	1.50%	48357.19	51642.81	32.84	–18.02
$K_U = 1.25$	87.18%	0.23%	48382.22	51617.78	32.45	–17.63

Panel C: $\hat{q}_T = \sum_{t=2013}^{2023} q_t$

Base case: $\psi = 0.4$, $K_D = 1$, $K_U = 1.2$, $\eta = 1$

	P_D	P_U	$E_0^Q[R_M]$	$E_0^Q[R_L]$	s_M	s_L
Base case	62.93%	0.66%	46691.72	53308.28	58.62	–43.80
Sensitivity of ψ						
$\psi = 0$	80.13%	0.13%	46282.25	53717.75	64.96	–50.14
$\psi = 0.8$	43.04%	2.25%	47372.26	52627.74	48.09	–33.27
$\psi = 1.2$	25.32%	5.78%	48259.72	51740.28	34.35	–19.53
Sensitivity of K_D						
$K_D = 0.95$	28.43%	0.66%	45776.38	54223.62	72.79	–57.97
$K_D = 1.05$	86.83%	0.66%	48226.23	51773.77	34.87	–20.05
$K_D = 1.1$	95.69%	0.66%	50076.57	49923.43	6.22	8.59
Sensitivity of K_U						
$K_U = 1.1$	62.93%	4.31%	46615.16	53384.84	59.81	–44.99
$K_U = 1.15$	62.93%	1.64%	46669.95	53330.05	58.96	–44.14
$K_U = 1.25$	62.93%	0.29%	46700.65	53299.35	58.48	–43.67

Notes: (1) ψ is the market price of risk; η is the participation rate; K_D is the rate below which the bullish (bearish) mortality bond has its minimum (maximum) fixed payoff; K_U is the rate above which the bullish (bearish) mortality bond has its maximum (minimum) fixed payoff. K_M (K_L) is the rate at which the payoff of a bullish (bearish) mortality bond apart from its principal is zero, $K_D < K_M$ (K_L) $< K_U$. We set $K_M = K_L = 1.1$ in the analysis.

(2) $P_D = prob(\hat{q}_T \le K_D)$, the probability that the underlying $\hat{q}_T$ is smaller than rate K_D; $P_U = prob(\hat{q}_T \ge K_U)$, the probability that the underlying $\hat{q}_T$ is larger than rate K_D.

(3) $E_0^Q[R_M]$($E_0^Q[R_L]$) is the expected redemption value of a bullish (bearish) mortality bond.

(4) s_M (s_L) is the spread for the coupon of a bullish (bearish) mortality bond, and is reported in basis points.

with the market price of risk ψ. This is due to the fact that more weights are put on the right tail of the probability density function of mortality rates for higher ψ. Exhibit 3 shows that a higher market price of risk implies that the market expects a higher probability of having a higher mortality rate than the actual probability suggests. Therefore, the larger ψ becomes, the higher the probability that $\hat{q}_T$ is greater than K_U, which results in a larger redemption value of bullish mortality bonds. Similarly, a larger ψ implies that the probability of $\hat{q}_T$ less than K_D is smaller, which gives rise to a smaller redemption value of bearish mortality bonds.

Next, we look at the sensitivity of contract-specific parameter K_D to bond features. When $\hat{q}_T$ is below K_D, a bullish mortality bond will suffer its maximum loss, which is equal to $P_M(1 + \eta(K_D - K_M))$. Exhibit 8 shows that as K_D becomes smaller, P_D, the probability of $\hat{q}_T$ being smaller than K_D, becomes lower. With a decreasing K_D, the bullish mortality bond will incur a larger maximum loss $P_M \times \eta(K_D - K_M)$ as $K_D < K_M$, which results in a smaller redemption value for a bullish mortality bond. For a bearish mortality bond, a smaller K_D implies a larger profit $P_L \times \eta(K_L - K_D)$, resulting in a greater redemption value for a bearish mortality bond.

Finally we study how the features of mortality-linked structured bonds vary with the parameter K_U. At K_U, the maximum gain of a bullish mortality bond is capped at $P_M \times \eta(K_U - K_M)$, and the maximum loss of a bearish bond is limited at $P_L \times \eta(K_L - K_U)$. For a larger K_U, the probability of $\hat{q}_T$ being more than K_D becomes smaller, which is consistent with the results shown in

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column 2, (P_U) of Sensitivity of K_U, of Exhibit 8. With an increasing K_U, the bullish mortality bond will incur a larger maximum gain $P_M \times \eta(K_L - K_M)$ as $K_U > K_M$, which results in a greater redemption value for a bullish mortality bond. An increasing K_U also implies a greater loss $P_L \times \eta(K_L - K_U)$ for a bearish mortality bond as $K_U > K_L$, giving rise to a smaller redemption value for it.

Sensitivity analysis in Exhibit 8 reveals that the value of bullish/bearish mortality bonds is highly sensitive to the market price of mortality risk ψ. This indicates that finding an exact ψ is essential for pricing mortality-linked structured products. Traders and risk managers thus need to be careful with their choice of an underlying market price of mortality risk. Since the values of mortality-linked structured bonds also vary with contract-specific parameters such as K_D and K_U, the setting of appropriate parameters in the contract may also be important. One can devise a more aggressive mortality-linked structured bond by widening the difference between K_D and K_U, which will give rise to a higher capped gain and a larger limited downside risk. A greater difference between K_D and K_U also means that the underlying mortality index $\hat{q}_T$ must move upward (downward) by a larger degree for the bullish (bearish) mortality bond holders to realize the greater maximum profit.

CONCLUSION

In recent years, securities linked to insurance risks have emerged as a specific asset class in the capital market. Securitization of mortality risks provides a way for primary insurers and annuity providers to repackage and offload their risk exposures onto those who are willing to undertake them in the financial market. Swiss Re mortality and EIB bonds represent common examples. We suggest a new perspective from which to analyze these mortality-linked securities. Specifically, we introduce the idea of structured products and study how mortality-linked structured notes may be constructed through purchases or sales of mortality options. In this regard, we extend Blake, Cairns, and Dowd's [2006] work by exploring a feasible application of mortality options in practice. The structured notes proposed herein provide a means for investors with different views of future mortality trends to monetize their mortality rate expectations by buying or selling mortality options. We further propose a security containing two bond classes, a bullish mortality bond whose payoff provides the positive link to the underlying mortality index, and a bearish mortality bond whose payoff links negatively to the underlying mortality index. In this design, issuers of bonds become immunized against the risk of mortality-linked redemption values.

We use mortality data from the United States to calibrate the jump-diffusion mortality index model. We then employ the estimated model to evaluate the proposed mortality-linked structured products. The numerical results reveal the relationship between the level of principal protections and the units of mortality options purchased in the case of principal-guaranteed structured notes. For high-yield structured notes, our numerical results indicate the calculated extra coupon spread that can be generated through the sale of different units of mortality options. We also find coupon spreads for high-yield structured notes under different participation rates. We further suggest three different settings for the underlyings of the proposed bullish/bearish mortality bonds, and provide numerical analyses to study the impact of contract specifications on bond features.

The examples provided in this article demonstrate the flexible structure of mortality-linked structured products, which investment banks could issue for life insurers or annuity providers to hedge their mortality/longevity risks. Although the market for mortality derivatives is still in its infancy, we believe that the securitization of mortality risk by issuing mortality-linked structured products will strengthen the linkage between financial and insurance markets.

APPENDIX A

In this appendix, we calculate the expected payoff of a European mortality call option. At the option maturity date, the expected payoff function is

$$
\begin{aligned}
&E_0^Q[(q_T - K)^+] \\
&\quad = \int_K^\infty (x - K) \times f^Q(q_T = x)dx \\
&\quad = \int_K^\infty (x - K) \times \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}\,x} \times e^{-\frac{1}{2}\left(\frac{\ln x - \mu_n}{\sigma_n}\right)^2} \times \frac{e^{-\lambda T}(\lambda T)^n}{n!} dx \\
&\quad = \frac{e^{-\lambda T}}{\sqrt{2\pi}} \times \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{\sigma_n n!} \\
&\qquad \times \int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x - \mu_n}{\sigma_n}\right)^2} dx - K\int_K^\infty f^Q(q_T = X)dx.
\end{aligned}
$$

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Because

$$\int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu_n}{\sigma_n}\right)^2} dx = \int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu_n}{\sigma_n}\right)^2} \times e^{\ln x} d\ln x,$$

setting $V = \ln x$ yields

$$\int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu_n}{\sigma_n}\right)^2} dx = \int_{\ln K}^\infty e^{-\frac{1}{2}\left(\frac{V-\mu_n}{\sigma_n}\right)^2} \times e^V dV.$$

We further normalize the variable V by setting $z = \frac{V-\mu_n}{\sigma_n}$ and get

$$\begin{aligned}&\int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu_n}{\sigma_n}\right)^2} dx\\ &= \int_{\frac{\ln K-\mu_n}{\sigma_n}}^\infty e^{-\frac{1}{2}z^2} \times e^{z\sigma_n+\mu_n} \times \sigma_n dz\\ &= \int_{\frac{\ln K-\mu_n}{\sigma_n}}^\infty e^{-\frac{1}{2}(z^2-2z\sigma_n+\sigma_n^2)+\frac{1}{2}\sigma_n^2+\mu_n} \times \sigma_n dz\\ &= \sigma_n e^{\mu_n+\frac{1}{2}\sigma_n^2} \times \int_{\frac{\ln K-\mu_n}{\sigma_n}}^\infty e^{-\frac{1}{2}(z-\sigma_n)^2} dz.\end{aligned}$$

Finally, letting $U = z - \sigma_n$, we obtain

$$\begin{aligned}&\int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu_n}{\sigma_n}\right)^2} dx\\ &= \sigma_n e^{\mu_n+\frac{1}{2}\sigma_n^2} \times \int_{\frac{\ln K-\mu_n}{\sigma_n}-\sigma_n}^\infty e^{-\frac{1}{2}U^2} dU\\ &= \sqrt{2\pi} \times \sigma_n \times e^{\mu_n+\frac{1}{2}\sigma_n^2} \times \int_{\frac{\ln K-\mu_n}{\sigma_n}-\sigma_n}^\infty \frac{1}{\sqrt{2\pi}} \times e^{-\frac{1}{2}U^2} dU\\ &= \sqrt{2\pi} \times \sigma_n \times e^{\mu_n+\frac{1}{2}\sigma_n^2} \times \left[1-\Phi\left(\frac{\ln K-\mu_n}{\sigma_n}-\sigma_n\right)\right].\end{aligned}$$

Therefore,

$$\int_K^\infty f^Q(q_T = x)dx = \sum_{n=0}^\infty \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times \left[1-\Phi\left(\frac{\ln K-\mu_n}{\sigma_n}\right)\right],$$

and

$$\begin{aligned}&E_0^Q[(q_T^Q - K)^+] =\\ &\sum_{n=0}^\infty \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times e^{\mu_n+\frac{1}{2}\sigma_n^2} \times \left[1-\Phi\left(\frac{\ln K-\mu_n}{\sigma_n}-\sigma_n\right)\right]\\ &-K \times \sum_{n=0}^\infty \frac{e^{-\lambda T}(\lambda T)^n}{n!} \times \left[1-\Phi\left(\frac{\ln K-\mu_n}{\sigma_n}\right)\right].\end{aligned}$$

APPENDIX B

We list the spot rates and implied forward rates used in the text in the following table.

Time	**Spot Rate**	**Forward Rate**
T	***R*(0, *T*)**	***F*(0; *T*, *T* + 0.5)**
0.5	0.4300%	0.2101%
1	0.3200%	0.5001%
1.5	0.3800%	0.6202%
2	0.4400%	1.0414%
2.5	0.5600%	1.2822%
3	0.6800%	1.7700%
3.5	0.8350%	2.0817%
4	0.9900%	2.9416%
4.5	1.2050%	2.3609%
5	1.3200%	2.8986%
5.5	1.4625%	3.1858%
6	1.6050%	3.4732%
6.5	1.7475%	3.7608%
7	1.8900%	3.3497%
7.5	1.9867%	3.5444%
8	2.0833%	3.7392%
8.5	2.1800%	3.9341%
9	2.2767%	4.1290%
9.5	2.3733%	4.3241%
10	2.4700%	4.5193%
10.5	2.5667%	

APPENDIX C

For $\hat{q}_T \equiv M_T = \max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)$, we note that according to Equation (6),

$$\left[\ln\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)\middle| N_{t_j} - N_{t_{j-1}} = (n_{(t_{j-1},t_j)})\right]$$

follows a normal distribution under the Q measure, with mean

$$\mu_{n_{(t_{j-1},t_j)}} = \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi(\sqrt{\Delta}) + n_{(t_{j-1},t_j)}(m+\psi s),$$

and variance $\sigma^2_{n_{(t_{j-1},t_j)}} = \sigma^2\delta + n_{(t_{j-1},t_j)}s^2$, where $n_{(t_{j-1},t_j)}$ is the number of jumps between time t_{j-1} and t_j. It therefore follows that

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$$f\left(\frac{q_{t_j}}{q_{t_{j-1}}}=x\middle|\mathcal{F}_0\right),$$

the probability density function for $\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)$ under the Q measure, is

$$\begin{aligned} f\left(\frac{q_{t_j}}{q_{t_{j-1}}}=x\mid\mathcal{F}_0\right) &= \sum_{n(t_{j-1},t_j)=0}^{\infty} f\left(\frac{q_{t_j}}{q_{t_{j-1}}}=x\mid\mathcal{F}_0,N_{t_j}-N_{t_{j-1}}=n(t_{j-1},t_j)\right) \\ &\quad\times \text{Prob}\,(N_{t_j}-N_{t_{j-1}}=n(t_{j-1},t_j)) \\ &= \sum_{n(t_{j-1},t_j)=0}^{\infty}\frac{1}{x\sqrt{(2\pi\sigma^2_{n_{(t_{j-1},t_j)}})}}\exp\left(-\frac{1}{2}\frac{(\ln(x)-\mu_{n_{(t_{j-1},t_j)}})^2}{\sigma^2_{n_{(t_{j-1},t_j)}}}\right) \\ &\quad\times\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t_{j-1},t_j)}}}{(n_{(t_{j-1},t_j)})!}. \end{aligned} \tag{9}$$

Thus, the cumulative distribution function for $\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)$ under the Q measure is

$$F^{Q}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\le x\mid\mathcal{F}_0\right)=\sum_{n_{(t_{j-1},t_j)}=0}^{\infty}\Phi\left(\frac{\ln(x)-\mu_{n_{(t_{j-1},t_j)}}}{\sigma_{n_{(t_{j-1},t_j)}}}\right)\times\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t_{j-1},t_j)}}}{(n_{(t_{j-1},t_j)})!}. \tag{10}$$

With the help of Equations (9) and (10), we can work out $F^{Q}(\hat{q}_T\le q)$ as

$$\begin{aligned} F^{Q}(\hat{q}_T\le q|\mathcal{F}_0) &= F^{Q}_{M_T}(M_T\le q|\mathcal{F}_0) \\ &= \text{Prob}\left(\max_{1\le j\le h}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\right)\le q\right) \\ &= \text{Prob}\left(\frac{q_{t_1}}{q_{t0}}\le q,\frac{q_{t_2}}{q_{t_1}}\le q,\ldots,\frac{q_{t_h}}{q_{t_{h-1}}}\le q\right). \end{aligned}$$

Because

$$\frac{q_{t_j}}{q_{t_{j-1}}}=\exp\left\{\begin{aligned}&\left(\alpha-\frac{1}{2}\sigma^2-\lambda k\right)\Delta+\sigma\psi\sqrt{\Delta}\\ &+\sigma\left(W^{Q}_{t_j}-W^{Q}_{t_{j-1}}\right)\end{aligned}\right\}\prod_{b=N_{t_{j-1}}}^{N_{t_j}}Y^{Q}_{b}e^{\psi_s},$$

$\left\{W^{Q}_{t_j},Y^{Q}_{b},N_{t_j}\right\}$ are mutually independent, and $W^{Q}_{t_j}$ and N_{t_j} have independent increments,

$$\left\{\frac{q_{t_1}}{q_{t_0}}\le q,\frac{q_{t_2}}{q_{t_1}}\le q,\ldots,\frac{q_{t_h}}{q_{t_{h-1}}}\le q\right\}$$

are mutually independent. We thus arrive at

$$\begin{aligned} F^{Q}_{M_T}(M_T\le q|\mathcal{F}_0) &= \prod_{j=1}^{h}\text{Prob}\left(\frac{q_{t_j}}{q_{t_{j-1}}}\le q\right) \\ &= \prod_{j=1}^{h}\sum_{n_{(t_{j-1},t_j)}=0}^{\infty}\frac{e^{-\lambda\Delta}(\lambda\Delta)^{n_{(t_{j-1},t_j)}}}{(n_{(t_{j-1},t_j)})!}\Phi\left(\frac{\ln(q)-\mu_{n_{(t_{j-1},t_j)}}}{\sigma_{n_{(t_{j-1},t_j)}}}\right). \end{aligned} \tag{11}$$

For the evaluation of the $E^{Q}_{0}[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}]$ term, we can write

$$\begin{aligned} E^{Q}_{0}\left[\hat{q}_T I_{\{K_D<\hat{q}_T\le K_U\}}\right] &= E^{Q}_{0}\left[M_T I_{\{K_D<M_T\le K_U\}}\right] \\ &= \int_{K_D}^{K_U} x\,dF^{Q}_{M_T}(M_T\le x) \\ &\simeq \int_{K_D}^{K_U} x\left[F^{Q}_{M_T}(M_T\le x+dx)-F^{Q}_{M_T}(M_T\le x)\right], \end{aligned}$$

with the approximation that $dF^{Q}_{M_T}(M_T\le x)=F^{Q}_{M_T}(M_T\le x+dx)-F^{Q}_{M_T}(M_T\le x)$ for $K_D\le M_T\le K_U$, where we can compute the two cumulative distribution functions using Equation (11).

Appendix D

When the payoff of the bullish mortality bond depends on the geometric average of the mortality index observed at each time t_j, $j=0,1,\ldots,h$, $\hat{q}_T$ is defined as G_T,

$$G_T=\left(\prod_{j=0}^{h}q_{t_j}\right)^{\frac{1}{h+1}}.$$

Following Fusai and Meucci [2008], we denote $\phi^{\xi}_{\Delta}(v)$ as the logarithm of the characteristic function of ξ^{Δ}_{j}, then

$$\begin{aligned} \phi^{\xi}_{\Delta}(v) &= \ln[E(e^{iv\xi^{\Delta}_{j}})] \\ &= -\frac{1}{2}\sigma^2 v^2\Delta+\lambda\Delta[e^{iv(m+\psi s)-\frac{1}{2}v^2 s^2}-1]. \end{aligned}$$

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We further define the logarithm of the geometric average of the mortality index as X_T, and express $X_T = \ln G_T$ in terms of the increments ξ_j^Δ as

$$
\begin{aligned}
X_T &= \ln G_T = \frac{1}{h+1}\ln\left(\prod_{j=0}^{h} q_{t_j}\right) \\
&= \frac{1}{h+1}\sum_{j=0}^{h}\ln q_{t_j} \\
&= \ln q_{t_0} + \frac{1}{h+1}\sum_{j=1}^{h}\ln\left(\frac{q_{t_j}}{q_{t_0}}\right) \\
&= \ln q_{t_0} + \frac{1}{h+1}\left[\begin{array}{l}\left(\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi\sqrt{\Delta}\right)\frac{h(h+1)}{2} \\ + \sum_{j=1}^{n}\sum_{l=1}^{j}\xi_l^\Delta \end{array}\right] \\
&= \ln q_{t_0} + \left(\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi\sqrt{\Delta}\right)\frac{h}{2} \\
&\quad + \frac{1}{h+1}\sum_{j=1}^{h}(h-j+1)\xi_j^\Delta .
\end{aligned}
$$

The characteristic function of X_T is

$$
\begin{aligned}
\phi_\Delta^X(v) &= E(e^{ivX_T}) \\
&= e^{iv\left(\ln q_{t_0} + \left(\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi\sqrt{\Delta}\right)\frac{h}{2}\right)}\prod_{j=1}^{h} E\left[e^{iv\frac{(h-j+1)}{h+1}\xi_j^\Delta}\right] \\
&= \exp\left[\begin{array}{l} iv\left(\ln q_{t_0} + \left(\left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)\Delta + \sigma\psi\sqrt{\Delta}\right)\frac{h}{2}\right) \\ + \sum_{j=1}^{h}\phi_\Delta^\xi\left(v\frac{h-j+1}{h+1}\right)\end{array}\right].
\end{aligned}
$$

Denote $F_{X_T}^Q(\cdot)$ as the cumulative probability distribution function of the logarithm of the geometric average at time T, $X_T = \ln G_T$, then we can construct $F_{X_T}^Q(\cdot)$ using the inversion of the characteristic function as

$$
F_{X_T}^Q(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{ivx}\phi_T^x(-v) - e^{-ivx}\phi_T^x(v)}{iv}dv,
$$

and the corresponding probability density function $f_{X_T}^Q(x)$ is

$$
f_{X_T}^Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ivx}\phi_T^x(v)dv.
$$

Accordingly, the probability that the geometric average G_T at time T is below a specific level can be obtained as

$$
F_{X_T}^Q(G_T \le q) = F_{X_T}^Q(X_T \le \ln q) = F_{X_T}^Q(\ln q)
$$

Therefore, $F_{X_T}^Q(G_T \le K_D) = F_{X_T}^Q(\ln K_D)$, $F_{X_T}^Q(G_T > K_U) = 1 - F_{X_T}^Q(\ln K_U)$, and

$$
\begin{aligned}
E_0^Q\left[G_T I_{\{K_D < G_T \le K_U\}}\right] &= E_0^Q\left[X_T I_{\{\ln K_D < \ln G_T \le \ln K_U\}}\right] \\
&= \int_{\ln K_D}^{\ln K_U} x_T f_{X_T}^Q(x)dx,
\end{aligned}
$$

where the integration can be numerically calculated.

Appendix E

When the payoff of the bullish mortality bond depends on the sum of the mortality index observed at each time t_j, $j = 0, 1, \ldots, h$, $\hat{q}_T$ is defined as A_T, $A_T = (\Sigma_{j=0}^h q_{t_j})$. Recall the definition of ξ_j^Δ, we denote $Z_j^\Delta \equiv \ln(q_{t_j}) - \ln(q_{t_{j-1}}) = \xi_j^\Delta + m_j^\Delta\Delta$. Then for $j = 1$, $Z_1^\Delta \equiv \ln(q_{t_1}) - \ln(q_{t_0})$, we have $q_{t_1} = q_{t_0}e^{Z_1^\Delta}$. For $j = y$, we can further obtain that $q_{t_y} = q_{t_0}e^{Z_1^\Delta + Z_2^\Delta + \ldots + Z_y^\Delta}$, therefore

$$
\begin{aligned}
\sum_{j=0}^{h} q_{t_j} &= q_{t_0}(e^{Z_1^\Delta} + e^{Z_1^\Delta + Z_2^\Delta} + \ldots + e^{Z_1^\Delta + Z_2^\Delta + \ldots +}e^{Z_h^\Delta}) \\
&= q_{t_0}\left[e^{Z_1^\Delta}(1 + e^{Z_2^\Delta}(\ldots(1 + e^{Z_h^\Delta})))\right].
\end{aligned}
$$

Next, we define recursively of $L_j^\Delta \equiv e^{Z_j^\Delta}(1 + L_{j+1}^\Delta)$, for $j = h-1, \ldots, 1$, and $L_h^\Delta \equiv e^{Z_h^\Delta}$. We can then arrive that $A_T = \Sigma_{j=0}^h q_{t_j} = q_{t_0}(1 + L_1^\Delta)$. Therefore, finding the distribution of A_T is reduced to working out the distribution of L_1^Δ. We define $B_1^\Delta \equiv \ln(L_1^\Delta)$, then B_j^Δ can be defined recursively by B_{j+1}^Δ as

$$
\begin{aligned}
B_j^\Delta &\equiv \ln(L_j^\Delta) = \ln\left[e^{Z_j^\Delta}(1 + L_{j+1}^\Delta)\right] = Z_j^\Delta + \ln(1 + L_{j+1}^\Delta) \\
&= Z_j^\Delta + \ln(1 + e^{B_{j+1}^\Delta}).
\end{aligned}
$$

Since Z_j^Δ and L_{j+1}^Δ are independent, and $B_j^\Delta = Z_j^\Delta + \ln(1 + e^{B_{j+1}^\Delta})$, the density of B_j^Δ is thus the convolution of the density $f_{Z_j^\Delta}$ and that of $\ln(1 + e^{B_{j+1}^\Delta})$. Therefore the density of $f_{B_j^\Delta}$ satisfies the recursion

$$
\begin{aligned}
f_{B_j^\Delta}(x) &= \int_{-\infty}^{\infty} f_{Z^\Delta}(x - \ln(1 + e^y)) f_{B_{j+1}^\Delta}(y)dy \\
&\text{for } j = h-1, \ldots, 1
\end{aligned}
\qquad (12)
$$

with the initial condition set as $f_{B_h^\Delta} \equiv f_{Z^\Delta}$. Since Z_j^Δ are i.i.d., we drop the index j for Z_j^Δ. The density function of Z_j^Δ, $f_{Z_j^\Delta}$, does not depend on the time index j. The integration

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in Equation (12) can be approximated using an M-point quadrature formula.

Accordingly, the probability that the sum of the mortality index at time T, A_T, is below a specific level can be obtained as

$$F^{Q}(\hat{q}_T \leq q) = F^{Q}_{A_T}(A_T \leq q) = \int_{-\infty}^{\ln\left(\frac{q}{q_{t_0}}-1\right)} f_{B_1^{\Delta}}(x)dx$$

$$\text{Therefore, } F^{Q}_{A_T}(A_T \leq K_D) = \int_{-\infty}^{\ln\left(\frac{K_D}{q_{t_0}}-1\right)} f_{B_1^{\Delta}}(x)dx,$$

$$\text{and } F^{Q}_{A_T}(A_T > K_U) = \int_{\ln\left(\frac{K_U}{q_{t_0}}-1\right)}^{\infty} f_{B_1^{\Delta}}(x)dx,$$

$$\text{and } F^{Q}_{A_T}(K_D < A_T \leq K_U) = \int_{\ln\left(\frac{K_D}{q_{t_0}}-1\right)}^{\ln\left(\frac{K_U}{q_{t_0}}-1\right)} f_{B_1^{\Delta}}(x)dx.$$

Finally, the term

$$E_0^{Q}\left[A_T I_{\{K_D < A_T \leq K_U\}}\right] = \int_{\ln\left(\frac{K_D}{q_{t_0}}-1\right)}^{\ln\left(\frac{K_U}{q_{t_0}}-1\right)} q_{t_0}(1+e^{x}) f_{B_1^{\Delta}}(x)dx,$$

where the integration can be numerically calculated.

ENDNOTES

[1]Due to insufficient demand, the EIB longevity bond was withdrawn for redesign in late 2005. Blake et al. [2006] find that some features of this bond's design might have discouraged investors, especially the basis risk embedded in the bond.

[2]Pelsser [2008] proposes an example to demonstrate that the Wang transform is not consistent with the arbitrage-free pricing approach. He argues that the Wang transform cannot be a universal framework for pricing insurance risks.

[3]The Human Mortality Database (HMD) provides easy access to comparable national mortality data via the Internet. The HMD is accessible at http://www.mortality.org.

[4]For details about the maximum likelihood estimation method, see Lin and Cox [2008].

[5]The spot rates $R(0, T)$ for different maturities T are interpolated from swap rates, which are available in the market. We use the spot rates to find the implied forward rate for a tenor of 0.5 years; that is, $F(0; \mathrm{T}, T+0.5)$. We list the rates in Appendix B.

[6]To relax this assumption, we could choose a specific interest rate model to forecast the future LIBOR rates.

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